Mathematics Department
The University of Georgia
Math 8150 Homework Assignment 2

Due during lecture on 2/10/2020 . Late homework will not be accepted

Complex Analysis, by Elias M. Stein and Rami Shakarchi,
2.6: 1, 2, 5, 6, 7, 8, 9, 10, 13, 14, 15

Additional problems

1. Let $a_n \neq 0$ and assume that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$. Show that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

2. Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.

3. Prove the following:

   (a) The power series $\sum_{n=1}^{\infty} nz^n$ does not converge at any point of the unit circle.

   (b) The power series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges at every point of the unit circle.

   (c) The power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges at every point of the unit circle except at $z = 1$.

4. Don't use the Cauchy integral formula. Show that if $|\alpha| < r < |\beta|$, then

$$\int_\gamma \frac{dz}{(z-\alpha)(z-\beta)} = \frac{2\pi i}{\alpha - \beta}$$

   where γ denotes the circle centered at the origin, of radius r, with positive orientation.

5. Assume f is continuous in the region: $x \geq x_0,\ 0 \leq y \leq b$ and the limit

$$\lim_{x\to+\infty} f(x+iy) = A$$

   exists uniformly with respect to y (independent of y). Show that

$$\lim_{x\to+\infty} \int_{\gamma_x} f(z)dz = iAb\ ,$$

   where $\gamma_x := \{z \mid z = x + it,\ 0 \leq t \leq b\}$.

6. Show by example that there exists function $f(z)$ that is holomorphic in $0 < |z| < 1$ and $\int_{|z|=r} f(z)dz = 0$ for all $r < 1$, but f is not holomorphic at $z = 0$.

7. Let f be analytic on a region R. Suppose $f'(z_0) \neq 0$ for some $z_0 \in R$. Show that if C is a circle of sufficiently small radius centered at z_0, then

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)}.$$

(Hint: Use the inverse function theorem).

8. Assume real value functions u, v of two variables have continuous partial derivatives at (x_0, y_0). Show that $f = u + iv$ has derivative $f'(z_0)$ at $z_0 = x_0 + iy_0$ if and only if

$$\lim_{r\to 0} \frac{1}{\pi r^2} \int_{|z-z_0|=r} f(z)\, dz = 0\ .$$

9. (Cauchy's formula for "exterior" region) Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z\to\infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

10. Let $f(z)$ be bounded and analytic in $\mathbb{C}$. Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists

$$\lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that $f(z)$ must be a constant (Liouville's theorem).

11. Suppose that $f(z)$ is entire and $\lim_{z\to\infty} f(z)/z = 0$. Show that $f(z)$ is a constant.

12. Let f be analytic on a domain D and let γ be a closed curve in D. For any z_0 in D not on γ, show that

$$\int_\gamma \frac{f'(z)}{(z-z_0)} dz = \int_\gamma \frac{f(z)}{(z-z_0)^2} dz$$

Give a generalization of this result.

13. Compute $\int_{|z|=1} (z + \frac{1}{z})^{2n} \frac{dz}{z}$ and use it to show that

$$\int_0^{2\pi} \cos^{2n} \theta d\theta = 2\pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$