Part V

Complex Analysis

SECTION 1
ANALYTIC AND HARMONIC FUNCTIONS

5101

True-False. If the assertion is true, quote a relevant theorem or reason; if false, give a counterexample or other justification.

(a) if $f(z) = u + iv$ is continuous at $z = 0$, and the partials u_x, u_y, v_x, v_y exist at $z = 0$ with $u_x = v_y$ and $u_y = -v_x$ at $z = 0$, then $f'(0)$ exists.

(b) if $f(z)$ is analytic in Ω and has infinitely many zeros in Ω, then $f \equiv 0$.

(c) if f and g are analytic in Ω and $f(z) \cdot g(z) \equiv 0$ in Ω, then either $f \equiv 0$ or $g \equiv 0$.

(d) if $f(z)$ is analytic in $\Omega = \{z; \mathrm{Re} z > 0\}$, continuous on $\overline{\Omega}$ with $|f(iy)| \le 1$ $(-\infty < y < +\infty)$, then $|f(z)| \le 1$ $(z \in \Omega)$.

(e) if $\sum a_n z^n$ has radius of convergence exactly R, then $\sum n^3 a_n z^n$ has radius of convergence exactly R.

(f) $\sin\sqrt{z}$ is an entire function.

(*Indiana–Purdue*)

Solution.

(a) False. A counterexample is $f(x,y) = \sqrt{|xy|}$. f satisfies Cauchy-Riemann equations at $z = 0$, but $f'(0)$ doesn't exist.

(b) False. A counterexample is $f(z) = \sin\frac{1}{1-z}$. f is analytic in $\Omega = \{z : |z| < 1\}$, and has zeros $z = 1 - \frac{1}{n\pi}$, $n = 1, 2, \cdots$. But f is not identically zero in Ω.

(c) True. If neither of f and g is identically zero in Ω, then both f and g have at most countably many zeros in Ω, and the zeros have no limit point in Ω. Then $f(z) \cdot g(z)$ is not identically zero in Ω.

(d) False. A counterexample is $f(z) = e^z$, which is analytic in Ω, and continuous on $\overline{\Omega}$ with $|f(iy)| \equiv 1$. But $f(z)$ is not bounded in Ω.

(e) True. Because $\lim\limits_{n\to\infty} \sqrt[n]{n^3} = 1$, it follows from

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \frac{1}{R}$$

that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{n^3|a_n|} = \frac{1}{R}.$$

(f) False. $\sin\sqrt{z}$ is not analytic at $z = 0$. Actually, $z = 0$ is a branch point of $\sin\sqrt{z}$.

5102

(a) Let $f(z)$ be a complex-valued function of a complex variable. If both $f(z)$ and $zf(z)$ are harmonic in a domain Ω, prove that f is analytic there.

(b) Suppose that f is analytic with $|f(z)| < 1$ in $|z| < 1$ and that $f(\pm a) = 0$ where a is a complex number with $0 < |a| < 1$. Show that $|f(0)| \leq a^2$. What can you conclude if this holds with equality.

(c) Determine all entire function f that $|f'(z)| < |f(z)|$.

(*Stanford*)

Solution.

(a) It is well known that the Laplacian can be written as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}}.$$

Because

$$\frac{\partial^2}{\partial z \partial \overline{z}}(zf(z)) = \frac{\partial}{\partial \overline{z}}f(z) + z\frac{\partial^2}{\partial z \partial \overline{z}}f(z),$$

it follows from

$$\frac{\partial^2}{\partial z \partial \overline{z}}f(z) = 0$$

and

$$\frac{\partial^2}{\partial z \partial \overline{z}}(zf(z)) = 0$$

that

$$\frac{\partial}{\partial \overline{z}}f(z) = 0,$$

which implies that $f(z)$ is analytic in Ω.

(b) Define

$$F(z) = f(z) \cdot \frac{1 - \overline{a}z}{z - a} \cdot \frac{1 + \overline{a}z}{z + a},$$

then $F(z)$ is analytic in $\{|z| < 1\}$. When $|z| = 1$,

$$\left| \frac{1 - \overline{a}z}{z - a} \cdot \frac{1 + \overline{a}z}{z + a} \right| = 1,$$

hence

$$\overline{\lim_{|z| \to 1}} |F(z)| \leq 1,$$

which implies that $|F(z)| \leq 1$ for $|z| < 1$. Take $z = 0$, we obtain

$$|f(0)| \leq |a|^2.$$

When it holds with equality, we have $F(z) \equiv e^{i\theta}$, which is equivalent to

$$f(z) = e^{i\theta} \frac{z-a}{1-\overline{a}z} \cdot \frac{z+a}{1+\overline{a}z}.$$

(c) From $|f'(z)| < |f(z)|$, we know that f has no zero in $\mathbb{C}$, which implies that $\frac{f'(z)}{f(z)}$ is also an entire function. It follows from $\left|\frac{f'(z)}{f(z)}\right| < 1$ that $\frac{f'(z)}{f(z)} = c$, $|c| < 1$. Integrating on both sides, we obtain $\log f(z) = cz + d$. Hence $f(z) = c'e^{cz}$, where c and c' are constants and $|c| < 1$.

5103

Let G be a region in $\mathbb{C}$ and suppose $u : G \to \mathbb{R}$ is a harmonic function.

(a) Show that $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is an analytic function on G.

(b) Show that u has a harmonic conjugate on G if and only if $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ has a primitive (anti-derivative) on G.

(*Indiana*)

Solution.

(a) Let

$$P(x,y) = \frac{\partial u}{\partial x}, \quad Q(x,y) = -\frac{\partial u}{\partial y}.$$

Because u is a harmonic function, we have

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We also have

$$\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y} = 0.$$

So $P(x,y)$ and $Q(x,y)$ satisfy the Cauchy-Riemann equations, hence

$$P + iQ = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

is analytic on G.

(b) If $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ has a primtive, then for any closed curve $c \subset G$, the integral

$$\int_c \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) dz = \int_c \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial x}dy\right) + i\left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right) = 0.$$

It follows that

$$\int_c -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = 0$$

holds for any closed curve $c \subset G$. Hence we can define a single-valued function $v(x, y)$ on G:

$$v(x,y) = \int_{z_0}^{z} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy,$$

where z_0, $z \in G$, and the integral is taken along any curve connecting z_0 and z in G. Because

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x},$$

we know that $v(x, y)$ is a harmonic conjugate of $u(x, y)$ on G.

On the contrary, if u has a harmonic conjugate v on G, then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

For any closed curve $c \subset G$, we have

$$\begin{aligned}
\int_c \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) dz &= \int_c \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + i\left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right) \\
&= \int_c \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}dy\right) \\
&= \int_c d(u + iv) = 0.
\end{aligned}$$

Hence $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ has a primitive $\int_{z_0}^{z} \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) dz$ on G.

5104

Suppose that u and v are real valued harmonic functions on a domain Ω such that u and v satisfy the Cauchy-Riemann equations on a subset S of Ω which has a limit point in Ω. Prove that $u + iv$ must be analytic on Ω.

(*Indiana–Purdue*)

Solution.

Because u and v are harmonic functions, $f_1 = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ and $f_2 = \frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y}$ are analytic functions on Ω. The reason lies on the fact that the real and imaginary parts of f_1 and f_2 satisfy the Cauchy-Riemann equations respectively (see 5103 (a)).

By the assumption of the problem,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

when $z \in S \subset \Omega$. Hence

$$f_1 = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = if_2 = i\left(\frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y}\right)$$

when $z \in S$. Because the subset S has a limit point in Ω, by the uniqueness theorem of analytic functions, we know that $f_1 = if_2$ holds for all $z \in \Omega$. It follows from $f_1 = if_2$ for $z \in \Omega$ that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

for $z \in \Omega$, which implies that $u + iv$ is analytic on Ω.

5105

Let $Q = [0,1] \times [0,1] \subset \mathbb{C}$ be the unit square, and let f be holomorphic in a neighborhood of Q. Suppose that

$$f(z+1) - f(z) \text{ is real and } \geq 0 \quad \text{for } z \in [0, i]$$
$$f(z+i) - f(z) \text{ is real and } \geq 0 \quad \text{for } z \in [0, 1].$$

Show that f is constant.

(*Indiana*)

Solution.

Because f is holomorphic on the closed unit square Q, by Cauchy integral theorem, we have

$$\begin{aligned}\int_{\partial Q} f(z)dz &= \int_0^1 f(x)dx + \int_0^1 f(1+yi)idy - \int_0^1 f(x+i)dx \\ &\quad - \int_0^1 f(yi)idy \\ &= \int_0^1 (f(x) - f(x+i))dx + i\int_0^1 (f(1+yi) - f(yi))dy \\ &= 0.\end{aligned}$$

As

$$f(x) - f(x+i) \leq 0$$

for $0 \leq x \leq 1$ and

$$f(1+yi) - f(yi) \geq 0$$

for $0 \leq y \leq 1$, by comparing the real and imaginary parts in the above identity, we obtain that $f(x+i) = f(x)$ for $0 \leq x \leq 1$ and $f(1+yi) = f(yi)$ for $0 \leq y \leq 1$. Hence $f(z)$ can be analytically extended to a double-periodic function by

$$f(z) = f(z+1) = f(z+i),$$

which is holomorphic in $\mathbb{C}$ and satisfies

$$|f(z)| \leq \max_{z \in Q}\{|f(z)|\} < +\infty.$$

This shows that $f(z)$ must be a constant.

5106

Let f be continuous on the closure $\overline{S}$ of the unit square

$$S = \{z = x + iy \in \mathbb{C} : 0 < x < 1, 0 < y < 1\},$$

and let f be analytic on S. If $\mathbf{R}f = 0$ on $\overline{S} \cap (\{y = 0\} \cup \{y = 1\})$, and if $\mathbf{I}f = 0$ on $\overline{S} \cap (\{x = 0\} \cup \{x = 1\})$, prove that $f = 0$ everywhere on S.

(*Indiana*)

Solution.

Define $F(z) = \int_0^z f(z)dz$, where the integral is taken along any curve in $\overline{S}$ which has endpoints 0 and z. Then $F(z)$ is analytic in S and continuous on $\overline{S}$. For $z \in \partial S$, we choose the integral path on ∂S and consider the differential form $f(z)dz$ in the integral. Let $f = u + iv$, then

$$f(z)dz = (udx - vdy) + i(vdx + udy).$$

On $\overline{S} \cap (\{y = 0\} \cup \{y = 1\})$ we have $u = 0$ and $dy = 0$, and on $\overline{S} \cap (\{x = 0\} \cup \{x = 1\})$ we have $v = 0$ and $dx = 0$. Hence we obtain $\text{Re}(f(z)dz) = 0$ on ∂S which implies $\text{Re}F(z) = 0$ when $z \in \partial S$.

Let $G(z) = e^{F(z)}$. Then $G(z)$ is analytic in S and $|G(z)| = 1$ when $z \in \partial S$. Because $G(z)$ has no zeros in S, so $1/G(z)$ is also analytic in S and $|1/G(z)| = 1$ when $z \in \partial S$. Apply the maximum modulus principle to both $G(z)$ and $1/G(z)$, we obtain $|G(z)| \equiv 1$ for $z \in S$, which implies that $G(z)$ is a constant of modulus 1. It follows from $G(z) = e^{F(z)}$ that $F(z)$ is also a constant. Hence $f(z) = F'(z) \equiv 0$.

5107

(a) Find the constant c such that the function

$$f(z) = \frac{1}{z^4 + z^3 + z^2 + z - 4} - \frac{c}{z-1}$$

is holomorphic in a neighborhood of $z = 1$.

(b) Show that the function f is holomorphic on an open set containing the closed disk $\{z : |z| \le 1\}$.

(*Iowa*)

Solution.

(a) As

$$\begin{aligned} & \lim_{z\to 1}(z-1) \cdot \frac{1}{z^4 + z^3 + z^2 + z - 4} \\ = & \lim_{z\to 1} \frac{1}{(z^4 + z^3 + z^2 + z - 4)'} \\ = & \lim_{z\to 1} \frac{1}{4z^3 + 3z^2 + 2z + 1} \\ = & \frac{1}{10}, \end{aligned}$$

we know that $z = 1$ is a simple pole of $\frac{1}{z^4+z^3+z^2+z-4}$ with residue equal to $\frac{1}{10}$. Hence when $c = \frac{1}{10}$, $f(z)$ is holomorphic in a neighborhood of $z = 1$.

(b) When $|z| \le 1$, we have

$$|z^4 + z^3 + z^2 + z - 4| \ge 4 - |z^4 + z^3 + z^2 + z| \ge 4 - |z|^4 - |z|^3 - |z|^2 - |z| \ge 0,$$

and the equalities hold if and only if $z = 1$, which shows that $z = 1$ is the only zero of $z^4 + z^3 + z^2 + z - 4$ in $\{z : |z| \le 1\}$. By (a), we obtain that $f(z)$ has no singular point in $\{z : |z| \le 1\}$, hence $f(z)$ is holomorphic on an open set containing $\{z : |z| \le 1\}$.

5108

Let $P(z)$ be a polynomial of degree d with simple roots $z_1, z_2, \cdots, z_d$. A "partial fractions" expression of $\frac{1}{P}$ has the form:

$$\frac{1}{P(z)} = \sum_{n=1}^{d} \frac{c_n}{z - z_n}. \qquad (*)$$

(a) Give a direct formula for c_n in terms of P.

(b) Show that $\frac{1}{P(z)}$ really has a representation of the form (*).

(c) Give a formula similar to (*) that works when $z_1 = z_2$ but all other roots are simple.

(Courant Inst.)

Solution.

(a)

$$c_n = \lim_{z \to z_n} \frac{z - z_n}{P(z)} = \frac{1}{P'(z_n)},$$

which is the residue of $\frac{1}{P(z)}$ at $z = z_n$.

(b) Let

$$f(z) = \frac{1}{P(z)} - \sum_{n=1}^{d} \frac{c_n}{z - z_n}.$$

Then $f(z)$ is analytic on $\mathbb{C}$ and $\lim_{z \to \infty} f(z) = 0$. By Liouville's theorem, $f(z)$ is identically equal to zero, hence

$$\frac{1}{P(z)} = \sum_{n=1}^{d} \frac{c_n}{z - z_n}.$$

(c) Denote the Taylor expansion of $P(z)$ at $z = z_1 (= z_2)$ by

$$P(z) = \sum_{n=2}^{\infty} a_n (z - z_1)^n.$$

Then the Laurent expansion of $\frac{1}{P(z)}$ at $z = z_1$ is

$$\frac{1}{P(z)} = \frac{c_1'}{(z - z_1)^2} + \frac{c_2'}{z - z_1} + \sum_{n=0}^{\infty} b_n (z - z_1)^n,$$

where

$$c_1' = \frac{1}{a_2} = \frac{2}{P''(z_1)}, \quad c_2' = -\frac{a_3}{a_2^2} = -\frac{2P'''(z_1)}{3P''(z_1)^2}.$$

Hence $\frac{1}{P(z)}$ has the form:

$$\frac{1}{P(z)} = \frac{c_1'}{(z - z_1)^2} + \frac{c_2'}{z - z_1} + \sum_{n=3}^{d} \frac{c_n}{z - z_n}.$$

5109

Suppose f is meromorphic in a neighborhood of $\overline{D}$ ($D = \{|z| < 1\}$) whose only pole is a simple one at $z = a \in D$. If $f(\partial D) \subseteq \mathbb{R}$, show that there is a complex constant A and a real constant B such that

$$f(z) = \frac{Az^2 + Bz + \overline{A}}{(z-a)(1-\overline{a}z)}.$$

(*Indiana*)

Solution.

Assume that the residue of f at $z = a$ is A_1. Define

$$g(z) = f(z) - \frac{A_1}{z-a} - \frac{\overline{A}_1 z}{1-\overline{a}z}.$$

It is obvious that $g(z)$ is analytic on $\overline{D}$ and $g(\partial D) \subseteq \mathbb{R}$. By the reflection principle, $g(z)$ can be extended to an analytic function on the Riemann sphere $\overline{\mathbb{C}}$, hence $g(z)$ must be a constant. Suppose $g(z) \equiv B_1$, then B_1 is real and

$$\begin{aligned} f(z) &= \frac{A_1}{z-a} + \frac{\overline{A}_1 z}{1-\overline{a}z} + B_1 \\ &= \frac{Az^2 + Bz + \overline{A}}{(z-a)(1-\overline{a}z)}, \end{aligned}$$

where $A = \overline{A}_1 - \overline{a}B_1$, $B = -(\overline{a}A_1 + a\overline{A}_1) + B_1(1 + |a|^2) \in \mathbb{R}$.

5110

Let $K_1, K_2, \cdots, K_n$ be pairwise disjoint disks in $\mathbb{C}$, and let f be an analytic function in $\mathbb{C} \setminus \bigcup\limits_{j=1}^{n} K_j$. Show that there exist functions $f_1, f_2, \cdots, f_n$ such that

(a) f_j is analytic in $\mathbb{C} \setminus K_j$, and

(b) $f(z) = \sum\limits_{j=1}^{n} f_j(z)$ for $z \in \mathbb{C} \setminus \bigcup\limits_{j=1}^{n} K_j$.

(*Indiana*)

Solution.

Assume $K_1 = \{z; |z - z_1| \leq r_1\}$. Choose $\varepsilon_1 > 0$ sufficiently small, such that

$$\Sigma_1 = \{z; r_1 < |z - z_1| < r_1 + \varepsilon_1\} \subset \mathbb{C} \setminus \bigcup_{j=1}^{n} K_j.$$

In Σ_1, $f(z)$ has the Laurent expansion

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k^{(1)}(z - z_1)^k.$$

Set

$$f_1(z) = \sum_{k=-\infty}^{0} a_k^{(1)}(z - z_1)^k.$$

$f_1(z)$ is analytic in $\mathbb{C}\backslash K_1$. Because $f(z) - f_1(z)$ has an analytic continuation to K_1, $f(z) - f_1(z)$ is analytic in $\mathbb{C}\backslash \bigcup_{j=2}^{n} K_j$.

Assume $K_2 = \{z; |z - z_2| \leq r_2\}$. Choose $\varepsilon_2 > 0$ sufficiently small, such that $\Sigma_2 = \{z; r_2 < |z - z_2| < r_2 + \varepsilon_2\} \subset \mathbb{C}\backslash \bigcup_{j=2}^{n} K_j$. In Σ_2, $f(z) - f_1(z)$ has the Laurent expansion

$$f(z) - f_1(z) = \sum_{k=-\infty}^{+\infty} a_k^{(2)}(z - z_2)^k.$$

Set $f_2(z) = \sum_{k=-\infty}^{0} a_k^{(2)}(z - z_2)^k$. $f_2(z)$ is analytic in $\mathbb{C}\backslash K_2$. Because $f(z) - f_1(z) - f_2(z)$ has an analytic continuation to K_2, $f(z) - f_1(z) - f_2(z)$ is analytic in $\mathbb{C}\backslash \bigcup_{j=3}^{n} K_j$.

Repeat the above procedure $n - 1$ times, we get a function $f(z) - f_1(z) - f_2(z) - \cdots - f_{n-1}(z)$, which is analytic in $\mathbb{C}\backslash K_n$. Set

$$f_n(z) = f(z) - f_1(z) - f_2(z) - \cdots - f_{n-1}(z).$$

Then we have

$$f(z) = \sum_{j=1}^{n} f_j(z),$$

where $f_j(z)$ is analytic in $\mathbb{C}\backslash K_j$, and the above identity holds for $z \in \mathbb{C}\backslash \bigcup_{j=1}^{n} K_j$.

5111

Recall that a divisor D_f of a rational function $f(z)$ on $\mathbb{C}$ is a set $\{p \in \mathbb{C} \cup \{\infty\}\}$, consisting of zeros and poles p of $f(z)$ (including the point ∞), counted with their multiplicities $n_p \in Z$. Let f and g be two rational functions with disjoint divisors. Prove that

$$\prod_{p\in D_f} g(p)^{n_p} = \prod_{q\in D_g} f(q)^{n_q}.$$

(*SUNY, Stony Brook*)

Solution.

Let p_i $(i = 1, 2, \cdots, n)$ be all the zeros and poles of $f(z)$ with multiplicities n_{p_i} respectively. It should be noted that p_i is a zero of f when $n_{p_i} > 0$ and a pole of f when $n_{p_i} < 0$. By the property of rational functions, we have $\sum\limits_{i=1}^{n} n_{p_i} = 0$. Similarly, let q_j $(j = 1, 2, \cdots, m)$ be all the zeros and poles of $g(z)$ with multiplicities m_{q_j} respectively, then we have $\sum\limits_{j=1}^{m} m_{q_j} = 0$.

First we assume that the point ∞ is not a zero or a pole of f or g, then f and g can be represented by

$$f(z) = A\prod_{i=1}^{n}(z-p_i)^{n_{p_i}}$$

and

$$g(z) = B\prod_{j=1}^{m}(z-q_j)^{m_{q_j}}.$$

Then

$$\begin{aligned}\prod_{p\in D_f} g(p)^{n_p} &= \prod_{i=1}^{n} g(p_i)^{n_{p_i}} = \prod_{i=1}^{n} B^{n_{p_i}} \cdot \prod_{i=1}^{n}\prod_{j=1}^{m}(p_i-q_j)^{n_{p_i}m_{q_j}} \\ &= \prod_{i=1}^{n}\prod_{j=1}^{m}(p_i-q_j)^{n_{p_i}m_{q_j}},\end{aligned}$$

and

$$\begin{aligned}\prod_{q\in D_g} f(q)^{n_q} &= \prod_{j=1}^{m} f(q_j)^{m_{q_j}} = \prod_{j=1}^{m} A^{m_{q_j}} \prod_{j=1}^{m}\prod_{i=1}^{n}(q_j-p_i)^{n_{p_i}m_{q_j}} \\ &= \prod_{i=1}^{n}\prod_{j=1}^{m}(p_i-q_j)^{n_{p_i}m_{q_j}}.\end{aligned}$$

In case the point ∞ is a zero or a pole of f or g, we may assume $p_n = \infty$ without loss of generality. Then

$$f(z) = A \prod_{i=1}^{n-1} (z - p_i)^{n_{p_i}}$$

and

$$g(z) = B \prod_{j=1}^{m} (z - q_j)^{m_{q_j}}.$$

Since $\sum_{j=1}^{m} m_{q_j} = 0$, we may assume that $g(p_n) = g(\infty) = B$. Hence

$$\begin{aligned}
\prod_{p \in D_f} g(p)^{n_p} &= \prod_{i=1}^{n-1} g(p_i)^{n_{p_i}} \cdot B^{n_{p_n}} \\
&= \left(\prod_{i=1}^{n-1} B^{n_{p_i}}\right) \cdot B^{n_{p_n}} \cdot \prod_{i=1}^{n-1} \prod_{j=1}^{m} (p_i - q_j)^{n_{p_i} m_{q_j}} \\
&= \prod_{i=1}^{n-1} \prod_{j=1}^{m} (p_i - q_j)^{n_{p_i} m_{q_j}},
\end{aligned}$$

and

$$\begin{aligned}
\prod_{q \in D_g} f(q)^{n_q} &= \prod_{j=1}^{m} f(q_j)^{m_{q_j}} = \prod_{j=1}^{m} A^{m_{q_j}} \cdot \prod_{j=1}^{m} \prod_{i=1}^{n-1} (q_j - p_i)^{n_{p_i} m_{q_j}} \\
&= \prod_{i=1}^{n-1} \prod_{j=1}^{m} (p_i - q_j)^{n_{p_i} m_{q_j}},
\end{aligned}$$

which completes the proof of the problem.

5112

Let $f(z)$ be the "branch" of $\log z$ defined off the negative real axis so that $f(1) = 0$.

(a) Find the Taylor polynomial of f of degree 2 at $-4 + 3i$, simplifying the coefficients.

(b) Find the radius of convergence R of the Taylor series $T(z)$ of $f(z)$ at $-4 + 3i$.

(c) Identify on a picture any points z where $T(z)$ converges but $T(z) \neq f(z)$, and describe the relationship between f and T at such points. If there are no such points, is this something special to this example, or a general impossibility? Explain and/or give examples.

(Minnesota)

Solution.

(a) When z is in the neighborhood of $z_0 = -4 + 3i$, we have

$$\begin{aligned} f(z) = \log z &= \log[(-4+3i) + (z+4-3i)] \\ &= \log(-4+3i) + \log\left[1 + \frac{z+4-3i}{-4+3i}\right] \\ &= \log 5 + i(\pi - \arcsin\frac{3}{5}) + \frac{z+4-3i}{-4+3i} \\ &\quad -\frac{1}{2}\left(\frac{z+4-3i}{-4+3i}\right)^2 + \cdots . \end{aligned}$$

Hence the Taylor polynomial of f of degree 2 at $-4+3i$ is

$$c_0 + c_1(z+4-3i) + c_2(z+4-3i)^2,$$

where

$$\begin{aligned} c_0 &= \log 5 + i(\pi - \arcsin\frac{3}{5}), \\ c_1 &= -\frac{4+3i}{25}, \end{aligned}$$

and

$$c_2 = -\frac{25+24i}{1250}.$$

(b) Denote the Taylor series of $f(z)$ at $-4+3i$ by $T(z)$. Because $\log z$ has only $z=0$ and $z=\infty$ as its branch points, and has no other singular point, the radius of convergence R of $T(z)$ is equal to the distance between $z=-4+3i$ and $z=0$. Hence $R=5$.

(c) Denote the shaded domain shown in Fig.5.1 by Ω. When $z \in \Omega = \{z : |z+4-3i| < 5, \mathrm{Im} z < 0\}$, $T(z) \neq f(z)$. It is because $T(z)$ in Ω is the continuation of $\log z$ at $-4+3i$ in the disk $\{z : |z+4-3i| < 5\}$, while $f(z)$ in Ω is the continuation of $\log z$ at $-4+3i$ in the slit plane $\mathbb{C}\backslash(-\infty, 0]$. Hence the difference is $2\pi i$, i.e., $T(z) = f(z) + 2\pi i$.

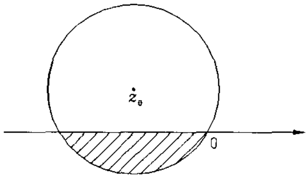

Fig.5.1

5113

Let f be the analytic function defined in the disk $\Delta = \{z : |z-4| < 4\}$ so that $f(z) = z^{\frac{1}{3}}(z+1)^{\frac{1}{2}}$ in Δ and $f(x)$ is positive for $0 < x < 8$. An analytic function g in Δ is obtained from f by analytic continuation along the path starting and ending at $z = 4$ (see Fig.5.2). Express g in terms of f.

(*Indiana*)

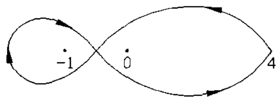

Fig.5.2

Solution.

Denote the closed path in Fig.5.2 by Γ, and denote the change of $\phi(z)$ when z goes along Γ from the start point to the end point by $\Delta_\Gamma \phi(z)$. Then

$$g(z) = |g(z)|e^{i\arg g(z)} = |f(z)|e^{i(\arg f(z) + \Delta_\Gamma \arg f(z))}.$$

We have

$$\begin{aligned} \Delta_\Gamma \arg f(z) &= \frac{1}{3}\Delta_\Gamma \arg z + \frac{1}{2}\Delta_\Gamma \arg(z+1) = \frac{1}{3}(2\pi) + \frac{1}{2}(-2\pi) \\ &= -\frac{\pi}{3}. \end{aligned}$$

Hence

$$g(z) = e^{-\frac{\pi}{3}i} f(z).$$

5114

Define

$$f(z) = \frac{e^{\sqrt{z}} - e^{-\sqrt{z}}}{\sin\sqrt{z}}.$$

(a) Where is f single-valued and analytic?
(b) Classify the singularities of f.
(c) Evaluate $\int_{|z|=25} f(z)dz$.

(*Indiana*)

Solution.

(a) It is known that $z = 0$ and $z = \infty$ are the branch points of function $\sqrt{z}$. Let $\Gamma = \{z : |z| = r\}$, and when z goes along Γ once in the counterclockwise sense, $\sqrt{z}$ is changed to $-\sqrt{z}$, while $f(z)$ is changed to

$$\frac{e^{-\sqrt{z}} - e^{\sqrt{z}}}{\sin(-\sqrt{z})} = \frac{e^{\sqrt{z}} - e^{-\sqrt{z}}}{\sin\sqrt{z}}$$

which is still $f(z)$. Hence $z = 0$ and $z = \infty$ are no longer the branch points of $f(z)$.

When z is in the small neighborhood of $z = 0$, $f(z)$ can be represented by

$$\begin{aligned} f(z) &= \frac{e^{\sqrt{z}} - e^{-\sqrt{z}}}{\sin\sqrt{z}} = \frac{\sum\limits_{n=0}^{\infty} \frac{1}{n!} z^{\frac{n}{2}} - \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n!} z^{\frac{n}{2}}}{\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{n+\frac{1}{2}}} \\ &= \frac{2\sum\limits_{n=0}^{\infty} \frac{1}{(2n+1)!} z^n}{\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^n}, \end{aligned}$$

which implies that $z = 0$ is a removable singular point of $f(z)$. It is obvious that $z = n^2\pi^2$ $(n = 1, 2, \cdots)$ are poles of $f(z)$. Hence $f(z)$ is single-valued and analytic in $\mathbb{C}\backslash\{z = n^2\pi^2 : n = 1, 2, \cdots\}$.

(b) We have

$$\lim_{z \to n^2\pi^2} \frac{e^{\sqrt{z}} - e^{-\sqrt{z}}}{(\sin\sqrt{z})'} = \frac{e^{n\pi} - e^{-n\pi}}{\frac{\cos(n\pi)}{2n\pi}} = 2n\pi(-1)^n(e^{n\pi} - e^{-n\pi}),$$

which shows that $z = n^2\pi^2$ are simple poles of $f(z)$ with residues

$$2n\pi(-1)^n(e^{n\pi} - e^{-n\pi}).$$

As to $z=\infty$, it is the limit point of the poles of $f(z)$, and hence is a non-isolated singular point of $f(z)$.

(c) $f(z)$ has only one pole $z=\pi^2$ in the disk $\{z:|z|<25\}$. Hence

$$\begin{aligned}\int_{|z|<25} f(z)dz &= 2\pi i \mathrm{Res}(f,\pi^2)\\ &= -4\pi^2 i(e^{\pi}-e^{-\pi}).\end{aligned}$$

5115

Let Ω be the plane with the segment $\{-1\le x\le 1, y=0\}$ deleted. For which of the multi-valued functions

(a) $f(z)=\frac{z}{\sqrt{1-z^2}}$,

(b) $g(z)=\frac{1}{\sqrt{1-z^2}}$,

can we choose single-valued branches which are holomorphic in Ω. Which of these branches are (is) the derivative of a single-valued holomorphic function in Ω. Why?

(*Indiana–Purdue*)

Solution.

Let Γ be an arbitrary simple closed curve in Ω, and denote by $\Delta_\Gamma\phi(z)$ the change of $\phi(z)$ when z goes continuously along Γ counterclockwise once. It is known that f and g can be represented by

$$\begin{aligned} f(z) &= \frac{z}{\sqrt{1-z^2}} = e^{\{\log z-\frac{1}{2}\log(1+z)-\frac{1}{2}\log(1-z)\}}\\ &= \left|\frac{z}{\sqrt{1-z^2}}\right| e^{i[\arg z-\frac{1}{2}\arg(1+z)-\frac{1}{2}\arg(1-z)]}\end{aligned}$$

and

$$\begin{aligned} g(z) &= \frac{1}{\sqrt{1-z^2}} = e^{\{-\frac{1}{2}\log(1+z)-\frac{1}{2}\log(1-z)\}}\\ &= \left|\frac{1}{\sqrt{1-z^2}}\right| e^{i[-\frac{1}{2}\arg(1+z)-\frac{1}{2}\arg(1-z)]}.\end{aligned}$$

Because

$$\Delta_\Gamma[\arg z-\frac{1}{2}\arg(1+z)-\frac{1}{2}\arg(1-z)]=0$$

and

$$\Delta_\Gamma[-\frac{1}{2}\arg(1+z)-\frac{1}{2}\arg(1-z)]=\begin{cases}0 & \{-1\le x\le 1, y=0\}\text{ not inside }\Gamma\\ -2\pi & \{-1\le x\le 1, y=0\}\text{ inside }\Gamma\end{cases}$$

we have $\Delta_\Gamma f(z) = 0$ and $\Delta_\Gamma g(z) = 0$. Hence both $f(z)$ and $g(z)$ have single-valued branches which are holomorphic in Ω, and each of f and g has two single-valued branches.

In order to know which of f and g has a single-valued primitive in Ω, we consider the integrals $\int_\Gamma f(z)dz$ and $\int_\Gamma g(z)dz$. If the segment $\{-1 \le x \le 1, y = 0\}$ is not inside Γ, it is obvious that $\int_\Gamma f(z)dz = 0$ and $\int_\Gamma g(z)dz = 0$. If the segment $\{-1 \le x \le 1, y = 0\}$ is inside Γ, we consider the Laurent expansion of f and g about $z = \infty$:

$$\begin{aligned} f(z) &= \pm i(1 - \frac{1}{z^2})^{-\frac{1}{2}} = \pm i \left(1 + \frac{a_2}{z^2} + \frac{a_4}{z^4} + \cdots\right), \\ g(z) &= \pm \frac{i}{z}(1 - \frac{1}{z^2})^{-\frac{1}{2}} = \pm i \left(\frac{1}{z} + \frac{b_3}{z^3} + \frac{b_5}{z^5} + \cdots\right). \end{aligned}$$

It follows that $\int_\Gamma f(z)dz = 0$ and $\int_\Gamma g(z)dz = \pm 2\pi$. Hence we obtain that both of the single-valued branches of f are the derivatives of single-valued holomorphic functions in Ω, and the primitives are $\int_{z_0}^z f(z)dz + c$, where the integral is taken along any curve connecting z_0 and z in Ω. But neither of the branches of g is the derivative of a single-valued holomorphic function in Ω.

5116

(a) Let $D \subset \mathbb{C}$ be the complement of the simply connected closed set $\{e^{\theta + i\theta} \mid \theta \in \mathbb{R}\} \cup \{0\}$. Let log be a branch of the logarithm on D such that $\log e = 1$. Find $\log e^{15}$. Justify your answer.

(b) Let γ denote the unit circle, oriented counterclockwise. By lifting the integration to an appropriate covering space, give a precise meaning to the integral $\int_\gamma (\log z)^2 dz$ and find all possible values which can be assigned to it.

(*Harvard*)

Solution.

(a) The set $\{e^{\theta + i\theta} \mid \theta \in \mathbb{R}\} \cup \{0\}$ is a spiral which intersects the positive real axis at $\{e^{2n\pi} : n = 0, \pm 1, \pm 2, \cdots\}$. The single-valued branch of $\log z$ is defined by $\log e = 1$. Hence $\log e^{15} = \log e + \Delta_\Gamma \log z$, where Γ is a continuous curve connecting $z = e$ and $z = e^{15}$ in D and $\Delta_\Gamma \log z$ is the change of $\log z$ when z goes continuously along Γ from $z = e$ to $z = e^{15}$. It follows that $\Delta_\Gamma \log z = \Delta_\Gamma \log |z| + i\Delta_\Gamma \arg z$, and $\Delta_\Gamma \log |z| = 15 - 1 = 14$. Because $e \in (e^0, e^{2\pi}), e^{15} \in (e^{4\pi}, e^{6\pi})$, we know that when Γ connects e and e^{15} in D, $\Delta_\Gamma \arg z$ must be 4π. Hence $\log e^{15} = 1 + (14 + 4\pi i) = 15 + 4\pi i$.

(b) Define the lift mapping by $w = \log z$ which lifts the unit circle γ one-to-one onto a segment with length 2π on the imaginary axis of w-plane. Because both the starting point of γ and the single-valued branch of $\log z$ on γ can be arbitrarily chosen, the segment on w-plane can be denoted by $[it, i(t+2\pi))$, where t can be any real number. Hence we have

$$\begin{aligned}
\int_\gamma (\log z)^2 dz &= \int_{it}^{i(t+2\pi)} w^2 e^w dw = (w^2 e^w)\Big|_{it}^{i(t+2\pi)} - 2\int_{it}^{i(t+2\pi)} w e^w dw \\
&= e^{it}(-4\pi t - 4\pi^2) - (2we^w)\Big|_{it}^{i(t+2\pi)} + 2\int_{it}^{i(t+2\pi)} e^w dw \\
&= -4\pi(t+\pi+i)e^{it} = 4\pi(t+\pi+i)e^{i(t+\pi)},
\end{aligned}$$

which implies that the set of values being assigned to the integral $\int_\gamma (\log z)^2 dz$ is a spiral $\{4\pi(s+i)e^{is} : s \in \mathbb{R}\}$.

5117

Find the most general harmonic function of the form $f(|z|)$, $z \in \mathbb{C}\backslash 0$. Which of these $f(|z|)$ have a single valued harmonic conjugate?

(*Indiana*)

Solution.

Because $f(|z|)$ is harmonic, we have reason to assume that the function f (with real variable t) has continuous derivatives $f'(t)$ and $f''(t)$. Note that the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}},$$

and

$$\begin{aligned}
\frac{\partial}{\partial z} f(|z|) &= \frac{\partial}{\partial z} f(\sqrt{z\overline{z}}) = \frac{1}{2} f'(|z|) \cdot \sqrt{\frac{\overline{z}}{z}}, \\
\frac{\partial^2}{\partial z \partial \overline{z}} f(|z|) &= \frac{1}{4} f''(|z|) + \frac{1}{4|z|} f'(|z|),
\end{aligned}$$

we obtian

$$f''(t) + \frac{f'(t)}{t} = 0,$$

where $t = |z|$. This differential equation is easy to solve, and the solution is $f(t) = \alpha \log t + \beta$, where α, β are two real constants. Hence the most general harmonic function of the form $f(|z|)$ in $\mathbb{C}\backslash 0$ is $\alpha \log |z| + \beta$.

Since $\log |z|$ has no single-valued harmonic conjugate in $\mathbb{C}\backslash 0$, we know that when $f(|z|)$ has a single-valued harmonic conjugate in $\mathbb{C}\backslash 0$, it must be a constant.

5118

Consider the regular pentagram centered at the origin in the complex plane. Let u be the harmonic function in the interior of the pentagram which has boundary values 1 on the two segments shown and 0 on the rest of the boundary. What is the value of u at the origin? Justify your claim.

(*Stanford*)

Solution.

Denote the interior domain of the pentagram shown in Fig.5.3 by D, and the ten segments of the boundary by $l_1, l_2, \cdots, l_{10}$, put in order of counterclockwise.

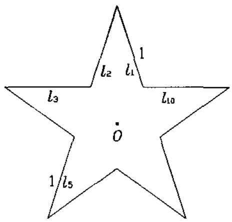

Fig.5.3

Then denote the harmonic function on D with boundary values 1 on l_k and 0 on the rest of the boundary by $u_k(z)$, $k = 1, 2, \cdots, 10$. By the symmetry of domain D, we have

$$\begin{aligned} u_2(z) &= u_1(-\overline{z}), \\ u_3(z) &= u_1(e^{-\frac{2\pi}{5}i}z), \\ u_4(z) &= u_2(e^{-\frac{2\pi}{5}i}z), \\ u_5(z) &= u_1(e^{-\frac{4\pi}{5}i}z), \\ &\cdots \\ u_{10}(z) &= u_2(e^{-\frac{8\pi}{5}i}z). \end{aligned}$$

It follows from

$$u(z) = \sum_{k=1}^{10} u_k(z) \equiv 1$$

and $u_1(0) = u_2(0) = \cdots = u_{10}(0)$ that $u_k(0) = \frac{1}{10}$ for $k = 1, 2, \cdots, 10$. Hence

$$u(0) = u_1(0) + u_5(0) = \frac{1}{5}.$$

5119

Suppose G is a region in $\mathbb{C}$, $[0,1] \subset G$, and $h : G \to \mathbb{R}$ is continuous. $h|_{G\setminus[0,1]}$ is harmonic, does this implies that h is harmonic on G?

(*Iowa*)

Solution.

The answer is No.

A counterexample is $h(z) = \mathrm{Re}\sqrt{z(z-1)}$, where the single-valued branch of $\sqrt{z(z-1)}$ is chosen by $\sqrt{z(z-1)}|_{z=2} = \sqrt{2}$. Since $\sqrt{z(z-1)}$ is analytic in $\mathbb{C}\setminus[0,1]$, $h(z)$ is harmonic there. When $0 \le x \le 1$,

$$\lim_{\substack{z=x+yi\to x\\ y>0}} \sqrt{z(z-1)} = \sqrt{x(1-x)}i,$$

$$\lim_{\substack{z=x+yi\to x\\ y<0}} \sqrt{z(z-1)} = -\sqrt{x(1-x)}i.$$

Hence $h(z) = 0$ when $z = x$, $0 \le x \le 1$, and $h(z)$ is continuous on $\mathbb{C}$. But $h(z)$ is not harmonic on $\mathbb{C}$, because $z = 0$ and $z = 1$ are branch points of $\sqrt{z(z-1)}$.

Remark. If the problem is changed to $h : G \to \mathbb{C}$ is continuous and $h|_{G\setminus[0,1]}$ is holomorphic, then h must be holomorphic on G.

5120

Let γ be an arc of the unit circle. Suppose that u and v are harmonic in $D = \{z : |z| < 1\}$ and continuously differentiable on $D \cup \gamma$. If the boundary values satisfy $u = v$ on γ and the radial derivatives satisfy $\frac{\partial u}{\partial r} = \frac{\partial v}{\partial r}$ on γ, prove that $u = v$ in D.

(*Indiana*)

Solution.

Let u^* be a conjugate harmonic function of u in D and v^* be a conjugate harmonic function of v in D. We know that a variation of Cauchy-Riemann equations for $f = u + iu^*$ and $g = v + iv^*$ are

$$r\frac{\partial u}{\partial r} = \frac{\partial u^*}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial u^*}{\partial r}$$

and

$$r\frac{\partial v}{\partial r} = \frac{\partial v^*}{\partial \theta}, \quad \frac{\partial v}{\partial \theta} = -r\frac{\partial v^*}{\partial r}.$$

It follows from the continuous differentiability of u and v on $D \cup \gamma$ that u^* and v^* can be continuously extended to $D \cup \gamma$ and then are also continuously differentiable on $D \cup \gamma$. Let z_0 be a fixed point on γ, and for $z \in \gamma$ denote the subarc of γ from z_0 to z by γ_z. Without loss of generality, we may assume that $u^*(z_0) = v^*(z_0) = 0$. Then for $z \in \gamma$,

$$\begin{aligned} u^*(z) &= \int_{\gamma_z} \frac{\partial u^*}{\partial r} dr + \frac{\partial u^*}{\partial \theta} d\theta = \int_{\gamma_z} \frac{\partial u^*}{\partial \theta} d\theta = \int_{\gamma_z} \frac{\partial u}{\partial r} d\theta \\ &= \int_{\gamma_z} \frac{\partial v}{\partial r} d\theta = v^*(z). \end{aligned}$$

Hence we obtain two functions $f = u + iu^*$ and $g = v + iv^*$ which are analytic in D and continuous on $D \cup \gamma$, such that $f = g$ on γ. Let $F = f - g$. Then by the reflection principle, F can be analytically extended to an analytic function on $D \cup \gamma \cup D^*$, where $D^* = \{z : |z| > 1\}$. Since $F = 0$ on γ, we obtain $F \equiv 0$ on $D \cup \gamma \cup D^*$, which implies $u = v$ in D.

5121

Use conformal mapping to find a harmonic function $U(z)$ defined on the unit disc $\{|z| < 1\}$ such that

$$\lim_{r \to 1-} U(re^{i\theta}) = \begin{cases} +1 & \text{for } 0 < \theta < \pi \\ -1 & \text{for } \pi < \theta < 2\pi. \end{cases}$$

Give the correct determination of any multiple-valued functions appearing in your answer.

(*Courant Inst.*)

Solution.

It is easy to know that $w = -i\frac{z+1}{z-1}$ is a conformal mapping of the unit disc $D = \{z : |z| < 1\}$ onto the upper half plane $H = \{w : \text{Im}w > 0\}$. The boundary correspondence is that the negative real axis $\{w : -\infty < w < 0\}$ corresponds to the arc $\Gamma_1 = \{z = e^{i\theta} : 0 < \theta < \pi\}$ and the positive real axis $\{w : 0 < w < +\infty\}$ corresponds to the arc $\Gamma_2 = \{z = e^{i\theta} : \pi < \theta < 2\pi\}$.

It is well known that $u(w) = \frac{2}{\pi}\text{arg}w - 1$ is a harmonic function in H and assume $+1$ on the negative real axis and -1 on the positive real axis. Hence

$$U(z) = u(-i\frac{z+1}{z-1}) = \frac{2}{\pi}\text{arg}(\frac{z+1}{z-1}) - 2,$$

where the single-valued branch of $\arg(\frac{z+1}{z-1})$ is defined by $\arg(\frac{z+1}{z-1})|_{z=0} = \pi$, is a harmonic function in $D = \{z : |z| < 1\}$ with the boundary values $+1$ on Γ_1 and -1 on Γ_2.

Remark. This problem can be solved directly from the Poisson formula as follows:

$$\begin{aligned} U(z) &= \frac{1}{2\pi}\int_{|\zeta|=1} U(\zeta)\mathrm{Re}(\frac{\zeta+z}{\zeta-z})\frac{d\zeta}{i\zeta} \\ &= \frac{1}{2\pi}\int_{\Gamma_1}\mathrm{Re}\left(\frac{\zeta+z}{\zeta-z}\right)\frac{d\zeta}{i\zeta} - \frac{1}{2\pi}\int_{\Gamma_2}\mathrm{Re}\left(\frac{\zeta+z}{\zeta-z}\right)\frac{d\zeta}{i\zeta} \\ &= \frac{1}{\pi}\int_{\Gamma_1}\mathrm{Re}\left(\frac{\zeta+z}{\zeta-z}\right)\frac{d\zeta}{i\zeta} - 1 \\ &= \frac{1}{\pi}\mathrm{Re}\left\{\int_{\Gamma_1}\frac{\zeta+z}{i\zeta(\zeta-z)}d\zeta\right\} - 1 \\ &= \frac{1}{\pi}\mathrm{Im}\left\{\int_{\Gamma_1}\left(\frac{2}{\zeta-z}-\frac{1}{\zeta}\right)d\zeta\right\} - 1 \\ &= \frac{1}{\pi}\mathrm{Im}\left\{\int_{\Gamma_1}d(2\log(\zeta-z)-\log\zeta)\right\} - 1 \\ &= \frac{1}{\pi}\Delta_{\Gamma_1}\{2\mathrm{arg}(\zeta-z)-\mathrm{arg}\zeta\} - 1 \\ &= \frac{2}{\pi}\mathrm{arg}\frac{z+1}{z-1} - 2. \end{aligned}$$

5122

Determine all continuous functions on $\{z \in \mathbb{C} : 0 < |z| \le 1\}$ which are harmonic on $\{z : 0 < |z| < 1\}$ and which are identically 0 on $\{z \in \mathbb{C} : |z| = 1\}$.

(*Minnesota*)

Solution.

Suppose $u(z)$ is a continuous function on $\{0 < |z| \le 1\}$ which is harmonic on $\{0 < |z| < 1\}$ and identically zero on $\{|z| = 1\}$. Let ${}^*du = -u_y dx + u_x dy$ and $A = \int_{|z|=r} {}^*du$, where A is a real number not necessarily zero. Denote $v(z) = \int^z {}^*du$, then $v(z)$ is the conjugate harmonic function of $u(z)$, but may be not single-valued. Define

$$f(z) = (u(z) + iv(z)) - \frac{A}{2\pi}\log z,$$

then $f(z)$ is a single-valued analytic function on $\{0 < |z| < 1\}$ and $\mathrm{Re}f(z)$ is identically zero on $\{|z| = 1\}$.

Let $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$ be the Laurent expansion of $f(z)$ on $\{0 < |z| < 1\}$, then $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_{-n}|} = 0$ and $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|} \leq 1$. Define $g(z) = \sum\limits_{n=-\infty}^{\infty} b_n z^n$, satisfying $b_{-n} = -\bar{b}_n$ for $n = 0, 1, 2, \cdots$, and $b_{-n} = a_{-n}$ for $n = 1, 2, \cdots$. Then $g(z)$ is an analytic function on $\{0 < |z| < +\infty\}$. When $|z| = 1$, it follows from $\mathrm{Re}b_0 = 0$ and

$$\mathrm{Re} \sum_{n=-\infty}^{-1} b_n z^n = \mathrm{Re} \sum_{n=1}^{\infty} b_{-n} z^{-n} = \mathrm{Re} \sum_{n=1}^{\infty} -\bar{b}_n z^{-n} = -\mathrm{Re} \sum_{n=1}^{\infty} b_n z^n$$

that $\mathrm{Re}g(z) = 0$. Then $f(z) - g(z) = \sum\limits_{n=0}^{\infty} c_n z^n$ is an analytic function in $\{|z| < 1\}$ and $\mathrm{Re}(f(z) - g(z))$ is identically zero on $\{|z| = 1\}$. Consider $F(z) = e^{f(z)-g(z)}$ which is analytic and does not assume zero in $\{|z| < 1\}$, and $|F(z)| = 1$ on $\{|z| = 1\}$, by the maximum and minimum modulus principles, we have $F(z) \equiv e^{i\alpha}$, hence $f(z) = g(z) + i\alpha$.

From the above discussion, we finally obtain

$$u(z) = \mathrm{Re} \sum_{n=-\infty}^{+\infty} b_n z^n + \frac{A}{2\pi} \log |z|,$$

where $b_{-n} = -\bar{b}_n$ and $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|b_n|} = 0$.

5123

(a) Let $f(z)$ be a holomorphic function in the disc $|z| \leq r$ whose zeros in this disc are given by $a_1, a_2, \cdots, a_n$ counted with multiplicity. Suppose further that $|a_j| < r$ for all $j = 1, 2, \cdots, n$, and $|f(0)| = 1$. Jensen's formula states that

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta = \sum_{j=1}^{n} \log \left(\frac{r}{|a_j|} \right).$$

Prove this.

(b) With the hypotheses and notations of (a), let $n(t)$ be the number of a_j $(j = 1, 2, \cdots, n)$ such that $|a_j| \leq t$. Using Jensen's formula, show that

$$\int_0^r n(t) \frac{dt}{t} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta.$$

(c) For $r < R$ deduce an estimate on $n(r)$ in terms of $\max_{0 \le \theta \le 2\pi} \log|f(Re^{i\theta})|$.

(d) What can be said about the zeros of bounded holomorphic functions in the unit disc?

(*Harvard*)

Solution.

(a) Let

$$F(z) = f(z) \prod_{j=1}^{n} \frac{r^2 - \overline{a}_j z}{r(z - a_j)},$$

then $F(z)$ is holomorphic and has no zero in the disc $\{|z| \le r\}$, which implies that $\log|F(z)|$ is harmonic in $\{|z| \le r\}$. By the mean value theorem of harmonic functions,

$$\log|F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|F(re^{i\theta})| d\theta.$$

Noting that

$$|F(0)| = |f(0)| \prod_{j=1}^{n} \frac{r}{|a_j|} = \prod_{j=1}^{n} \frac{r}{|a_j|}$$

and

$$|F(re^{i\theta})| = |f(re^{i\theta})|,$$

we obtain that

$$\sum_{j=1}^{n} \log(\frac{r}{|a_j|}) = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta.$$

(b) It is obvious that $\log \frac{r}{|a_j|} = \int_{|a_j|}^{r} \frac{dt}{t}$. By the definition of the function $n(t)$ we have

$$\sum_{j=1}^{n} \log \frac{r}{|a_j|} = \sum_{j=1}^{n} \int_{|a_j|}^{r} \frac{dt}{t} = \int_0^r n(t) \frac{dt}{t},$$

which shows that the identity holds.

(c) Apply the identity in (b), we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| d\theta = \int_0^R n(t) \frac{dt}{t} \ge \int_r^R n(t) \frac{dt}{t} \ge n(r) \log \frac{R}{r}.$$

Denote $\max_{0 \le \theta \le 2\pi} \log|f(Re^{i\theta})|$ by $M(R)$, we obtain

$$n(r) \le M(R) / \log \frac{R}{r}.$$

(d) Let $f(z)$ be a bounded holomorphic function in $\{z : |z| < 1\}$. We know that $f(z)$ can have countably many zeros. Suppose $z = 0$ is a zero of $f(z)$ of multiplicity $m \geq 0$ with $\frac{f^{(m)}(0)}{m!} = \alpha$, and let the other zeros be ordered by $0 < |a_1| \leq |a_2| \leq \cdots$. Obviously $|a_n| \to 1$. Apply Jensen's formula in (a) to $F(z) = \frac{f(z)}{\alpha z^m}$ with $0 < r < 1$ such that there is no zero of f on $\{|z| = r\}$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta = \sum_{|a_j|<r} \log(\frac{r}{|a_j|}) + \log(|\alpha|r^m).$$

Since $f(z)$ is bounded, we assume

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta \leq M.$$

For any n, we can choose r such that $r > |a_n|$, and hence

$$\sum_{j=1}^{n} \log(\frac{r}{|a_j|}) \leq \sum_{|a_j|<r} \log(\frac{r}{|a_j|}) \leq M - \log(|\alpha|r^m).$$

Let $r \to 1$, we obtain

$$\prod_{j=1}^{n} |a_j| \geq |\alpha|e^{-M} > 0,$$

which implies that the series $\sum\limits_{j=1}^{\infty}(1 - |a_j|)$ is convergent.

SECTION 2
GEOMETRY OF ANALYTIC FUNCTIONS

5201

Find a one-to-one holomorphic map from the unit disk $\{|z| < 1\}$ ont slit disk $\{|w| < 1\} - \{[0, 1)\}$.

(SUNY, Stony B

Solution.

We construct the map by the following steps:

$$\begin{aligned}
z_1 &= \phi_1(z) = i\frac{z+1}{z-1} : \{z : |z| < 1\} \to \{z_1 : \mathrm{Im} z_1 < 0\};\\
z_2 &= \phi_2(z_1) = \sqrt{z_1^2 - 1} + z_1 \quad (\sqrt{z_1^2 - 1}\Big|_{z_1 = -i} = \sqrt{2}i) :\\
&\quad \{z_1 : \mathrm{Im} z_1 < 0\} \to \{z_2 : |z_2| < 1 \text{ and } \mathrm{Im} z_2 > 0\};\\
w &= \phi_3(z_2) = z_2^2 : \{z_2 : |z_2| < 1 \text{ and } \mathrm{Im} z_2 > 0\} \to\\
&\quad \{w : |w| < 1\} \backslash \{w : \mathrm{Im} w = 0, 0 \le \mathrm{Re} w < 1\}.
\end{aligned}$$

Then $w = \phi_3 \circ \phi_2 \circ \phi_1(z) = f(z)$ is a one-to-one holomorphic map from the unit disk $\{|z| < 1\}$ onto the slit disk $\{|w| < 1\} \backslash \{[0, 1)\}$.

5202

(a) Find a function f that conformally maps the region $\{z : |\arg z| < 1\}$ one-to-one onto the region $\{w : |w| < 1\}$. Show that the function you have found satisfies the required conditions.

(b) Is it possible to require that $f(1) = 0$ and $f(2) = \frac{1}{2}$? If yes, give an explicit map; if No, explain why not.

(Illinois)

Solution.

(a) $\zeta = f_1(z) = z^{\frac{\pi}{2}} = e^{\frac{\pi}{2}\log z}$ $(\log 1 = 0)$ is a conformal map of $\{z : |\arg z| < 1\}$ onto $\{\zeta : \mathrm{Re}\zeta > 0\}$, and $w = f_2(\zeta) = \frac{\zeta - 1}{\zeta + 1}$ is a conformal map of $\{\zeta : \mathrm{Re}\zeta > 0\}$ onto $\{w : |w| < 1\}$. Hence

$$w = f(z) = f_2 \circ f_1(z) = \frac{z^{\frac{\pi}{2}} - 1}{z^{\frac{\pi}{2}} + 1}$$

is a conformal map of $\{z : |\arg z| < 1\}$ onto $\{w : |w| < 1\}$ with $f(1) = 0$ and $f(2) = \frac{2^{\frac{\pi}{2}}-1}{2^{\frac{\pi}{2}}+1}$.

(b) Suppose $\widetilde{w} = \widetilde{f}(z)$ is an arbitrary conformal map of $\{z : |\arg z| < 1\}$ onto $\{\widetilde{w} : |\widetilde{w}| < 1\}$ with $\widetilde{f}(1) = 0$. Then $w = F(\widetilde{w}) = f \circ \widetilde{f}^{-1}(\widetilde{w})$ is a conformal map of $\{\widetilde{w} : |\widetilde{w}| < 1\}$ onto $\{w : |w| < 1\}$ with $F(0) = 0$, and $\widetilde{w} = \widetilde{F}(w) = \widetilde{f} \circ f^{-1}(w)$ is a conformal map of $\{w : |w| < 1\}$ onto $\{\widetilde{w} : |\widetilde{w}| < 1\}$ with $\widetilde{F}(0) = 0$. By Schwarz's lemma, we have both $|F(\widetilde{w})| \le |\widetilde{w}|$ and $|\widetilde{F}(w)| \le |w|$, which implies that $|f(z)| = |\widetilde{f}(z)|$ for every $z \in \{z : |\arg z| < 1\}$. Since

$$f(2) = \frac{2^{\frac{\pi}{2}} - 1}{2^{\frac{\pi}{2}} + 1},$$

we cannot require that $\widetilde{f}(2) = \frac{1}{2}$.

5203

(1) Find one 1-1 onto conformal map f that sends the open quadrant $\{(x, y) : x > 0 \text{ and } y > 0\}$ onto the open lower half disc $\{(x, y) : x^2 + y^2 < 1 \text{ and } y < 0\}$.

(2) Find all such f.

(*Toronto*)

Solution.

(1) Let $\zeta = \phi_1(z) = z^2$. It is a conformal map of $\{z = x + iy : x > 0 \text{ and } y > 0\}$ onto $\{\zeta = \xi + i\eta : \eta > 0\}$.

Let $w = \phi_2(\zeta) = \sqrt{\zeta^2 - 1} + \zeta$, where $\sqrt{\zeta^2 - 1}\Big|_{\zeta = i} = -\sqrt{2}i$. It is a conformal map of $\{\zeta = \xi + i\eta : \eta > 0\}$ onto $\{w = u + iv : u^2 + v^2 < 1 \text{ and } v < 0\}$.

Then $w = \phi_2 \circ \phi_1(z) = \sqrt{z^4 - 1} + z^2$, where $\sqrt{z^4 - 1}\Big|_{z = e^{\frac{\pi}{4}i}} = -\sqrt{2}i$ is a required conformal map.

(2) If f is an arbitrary conformal map satisfying the condition of (1), then $\phi_2^{-1} \circ f \circ \phi_1^{-1}(\zeta)$ is a conformal map of the upper half plane onto itself, which can be represented by $\psi(\zeta) = \frac{a\zeta + b}{c\zeta + d}$, where $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$. Hence f can be written as $\phi_2 \circ \psi \circ \phi_1(z)$.

5204

Map the disk $\{|z| < 1\}$ with slits along the segments $[a, 1]$, $[-1, -b]$ $(0 < a < 1, 0 < b < 1)$ conformally on the full disk $\{|w| < 1\}$ by means of a function

$w = f(z)$ with $f(0) = 0$, $f'(0) > 0$. Compute $f'(0)$ and the lengths of the arcs corresponding to the slits.

(*Harvard*)

Solution.

We construct the conformal mapping by the following steps.

(i) $z_1 = \phi_1(z) = z + \frac{1}{z} : \{|z| < 1\} \backslash \{[a, 1] \cup [-1, -b]\} \to \overline{\mathbb{C}} \backslash \{[-b - \frac{1}{b}, a + \frac{1}{a}]\}$. It has the point correspondences $\phi_1(0) = \infty$, $\phi_1(a) = a + \frac{1}{a}$, $\phi_1(b) = -b - \frac{1}{b}$, $\phi_1(1) = 2$ and $\phi_1(-1) = -2$.

(ii) $z_2 = \phi_2(z_1) = \frac{z_1 + (b + \frac{1}{b})}{-z_1 + (a + \frac{1}{a})} : \overline{\mathbb{C}} \backslash \{[-b - \frac{1}{b}, a + \frac{1}{a}]\} \to \mathbb{C} \backslash [0, +\infty)$. It has the point correspondences

$$\phi_2(\infty) = -1, \quad \phi_2(-2) = \Big(\frac{\frac{1}{\sqrt{b}} - \sqrt{b}}{\frac{1}{\sqrt{a}} + \sqrt{a}}\Big)^2$$

and

$$\phi_2(2) = \Big(\frac{\frac{1}{\sqrt{b}} + \sqrt{b}}{\frac{1}{\sqrt{a}} - \sqrt{a}}\Big)^2,$$

and it is easy to know that

$$(\phi_2 \circ \phi_1)'(0) = -(a + \frac{1}{a} + b + \frac{1}{b}) < 0.$$

(iii) $z_3 = \phi_3(z_2) = \sqrt{z_2} : \mathbb{C} \backslash [0, +\infty) \to \{z_3 : \mathrm{Im} z_3 > 0\}$. It has the point correspondences

$$\phi_3(-1) = i, \quad \phi_3\Big(\Big(\frac{\frac{1}{\sqrt{b}} - \sqrt{b}}{\frac{1}{\sqrt{a}} + \sqrt{a}}\Big)^2\Big) = \pm\Big(\frac{\frac{1}{\sqrt{b}} - \sqrt{b}}{\frac{1}{\sqrt{a}} + \sqrt{a}}\Big)$$

and

$$\phi_3\Big(\Big(\frac{\frac{1}{\sqrt{b}} + \sqrt{b}}{\frac{1}{\sqrt{a}} - \sqrt{a}}\Big)^2\Big) = \pm\Big(\frac{\frac{1}{\sqrt{b}} + \sqrt{b}}{\frac{1}{\sqrt{a}} - \sqrt{a}}\Big).$$

For the convenience of computation, let

$$A = \frac{\frac{1}{\sqrt{b}} - \sqrt{b}}{\frac{1}{\sqrt{a}} + \sqrt{a}}, \quad B = \frac{\frac{1}{\sqrt{b}} + \sqrt{b}}{\frac{1}{\sqrt{a}} - \sqrt{a}}.$$

We also know that $\phi_3'(-1) = -\frac{i}{2}$.

(iv) $w = \phi_4(z_3) = \frac{z_3 - i}{z_3 + i} : \{z_3 : \mathrm{Im} z_3 > 0\} \to \{w : |w| < 1\}$. It is obvious that $\phi_4(i) = 0$ and $\phi_4'(i) = -\frac{i}{2}$.

Now we define $w = f(z) = \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1(z)$. From the above discussion, we know that f maps the unit disk with slits $[-1, -b]$ and $[a, 1]$ conformally onto the unit disk with $f(0) = 0$ and

$$f'(0) = \phi_4'(i) \cdot \phi_3'(-1) \cdot (\phi_2 \circ \phi_1)'(0) = \frac{1}{4}(a + \frac{1}{a} + b + \frac{1}{b}) > 0.$$

What correspond to the slits are the arc with endpoints $\frac{A-i}{A+i}$ and $\frac{A+i}{A-i}$ containing point $z = -1$ and the arc with endpoints $\frac{B-i}{B+i}$ and $\frac{B+i}{B-i}$ containing point $z = 1$. The lengths of the two arcs are

$$l_1 = \arg\frac{A-i}{A+i} - \arg\frac{A+i}{A-i} = 4\text{arctg}A = 4\text{arctg}\frac{\frac{1}{\sqrt{b}} - \sqrt{b}}{\frac{1}{\sqrt{a}} + \sqrt{a}},$$

and

$$l_2 = \arg\frac{B+i}{B-i} - \arg\frac{B-i}{B+i} = 4\text{arctg}\frac{1}{B} = 4\text{arctg}\frac{\frac{1}{\sqrt{a}} - \sqrt{a}}{\frac{1}{\sqrt{b}} + \sqrt{b}}.$$

5205

Let $0 < \varepsilon < \pi$, let γ_ε denote the arc $\{e^{it} : \varepsilon \leq t \leq 2\pi - \varepsilon\}$ and let Ω_ε be the complement of γ_ε in the Riemann sphere. If f is the conformal map of the unit disk onto Ω_ε, $f(0) = 0$, $f'(0) > 0$, describe the part of the unit disc that f maps onto $\{|z| > 1\}$.

(*Stanford*)

Solution.

We are going to find the map f by the following steps:

$$z_1 = \phi_1(z) = e^{i\varepsilon} \cdot \frac{z - e^{-i\varepsilon}}{z - e^{i\varepsilon}} : \{z : |z| < 1\} \to \{z_1 : \text{Im}z_1 < 0\},$$

with $\phi_1(0) = e^{-i\varepsilon}$, $\arg\phi_1'(0) = -\frac{\pi}{2} - \varepsilon$.

$$z_2 = \phi_2(z_1) = \sqrt{z_1} : \{z_1 : \text{Im}z_1 < 0\} \to \{z_2 : \text{Re}z_2 > 0, \text{Im}z_2 < 0\},$$

with $\phi_2(e^{-i\varepsilon}) = e^{-\frac{\varepsilon}{2}i}$, $\arg\phi_2'(e^{-i\varepsilon}) = -\frac{\varepsilon}{2}$.

$$\zeta = \phi_3(z_2) = e^{\frac{\varepsilon}{2}i} \cdot \frac{z_2 - e^{-\frac{\varepsilon}{2}i}}{z_2 - e^{\frac{\varepsilon}{2}i}} : \{z_2 : \text{Re}z_2 > 0, \text{Im}z_2 < 0\} \to D_1$$

(shown in Fig.5.4), with $\phi_3(e^{-\frac{\varepsilon}{2}i}) = 0$, $\arg\phi_3'(e^{-\frac{\varepsilon}{2}i}) = \frac{\pi}{2} + \frac{\varepsilon}{2}$, where D_1 is a domain bounded by $\{\zeta = e^{i\theta}, \frac{\varepsilon}{2} \le \theta \le 2\pi - \frac{\varepsilon}{2}\}$ and an circular arc l_ε which is orthogonal to $\{|\zeta| = 1\}$ and connects points $e^{\frac{\varepsilon}{2}i}$ and $e^{-\frac{\varepsilon}{2}i}$ in $\{|\zeta| \le 1\}$.

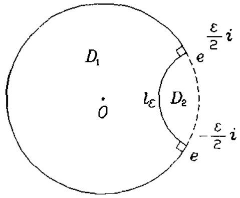

Fig.5.4

Let $\Phi(z) = \phi_3 \circ \phi_2 \circ \phi_1(z)$, then Φ maps $\{z : |z| < 1\}$ conformally onto D_1 with $\Phi(0) = 0$, $\Phi'(0) > 0$. After considering the boundary correspondence, we know that l_ε corresponds to the arc $\{z = e^{it} : |t| < \varepsilon\}$ under the map Φ. Since the symmetric domain of $\{|z| < 1\}$ with respect to arc $\{z = e^{it} : |t| < \varepsilon\}$ is $\{|z| > 1\}$, and the symmetric domain of D_1 with respect to l_ε is $D_2 = \{|\zeta| < 1\}\backslash\overline{D}_1$, by the reflection principle, $\Phi(z)$ can be extended to a conformal map of Ω_ε onto $\{\zeta : |\zeta| < 1\}$. Hence the conformal map f in the problem is nothing but the inverse of Φ, and the domain f maps onto $\{|z| > 1\}$ is D_2, which is bounded by circular arcs l_ε and $\{\zeta = e^{i\theta} : |\theta| \le \frac{\varepsilon}{2}\}$.

5206

Suppose that $w = f(z)$ maps a simply conncted region G one-to-one and conformally onto a circular disk D_r with center $w = 0$, radius r, such that $f(a) = 0$ and $|f'(a)| = 1$ for some point $a \in G$.

(1) Prove that the radius $r = r(G, a)$ of D_r is uniquely determined by G and a.

(2) Determine $r(G, a)$ if G is the region between the hyperbola $xy = 1$ ($x > 0$, $y > 0$) and the positive axes, and if $a = 1 + \frac{i}{2}$.

(*Indiana*)

Solution.

(1)Suppose $\zeta = g(z)$ is another conformal map of G onto a circular disk D_{r_1} with center $\zeta = 0$ and radius r_1, such that $g(a) = 0$ and $|g'(a)| = 1$, then $w = F(\zeta) = f \circ g^{-1}(\zeta)$ is a conformal map of $\{\zeta : |\zeta| < r_1\}$ onto $\{w : |w| < r\}$

with $F(0) = 0$ and $|F'(0)| = \left|\frac{f'(a)}{g'(a)}\right| = 1$. Apply Schwarz's lemma to $F(\zeta)$ and we have $|F'(0)| \le \frac{r}{r_1}$, hence $r_1 \le r$. For the same reason, apply Schwarz's lemma to $F^{-1}(w)$ and we have $r \le r_1$, which implies $r_1 = r$. In other words, r is uniquely determined by G and a.

(2) We construct a conformal map of G onto a circular disk D_r in the following steps:

$$z_1 = \phi_1(z) = z^2 : G \to \{z_1 : 0 < \mathrm{Im} z_1 < 2\},$$

with $\phi_1(1 + \frac{i}{2}) = \frac{3}{4} + i$, $|\phi_1'(1 + \frac{i}{2})| = \sqrt{5}$.

$$z_2 = \phi_2(z_1) = e^{\frac{\pi}{2} z_1} : \{z_1 : 0 < \mathrm{Im} z_1 < 2\} \to \{z_2 : \mathrm{Im} z_2 > 0\},$$

with $\phi_2(\frac{3}{4} + i) = ie^{\frac{3\pi}{8}}$, $|\phi_2'(\frac{3}{4} + i)| = \frac{\pi}{2} e^{\frac{3\pi}{8}}$.

$$w = \phi_3(z_2) = \frac{4}{\sqrt{5}\pi} \cdot \frac{z_2 - ie^{\frac{3}{8}\pi}}{z_2 + ie^{\frac{3}{8}\pi}} : \{z_2 : \mathrm{Im} z_2 > 0\} \to \{w : |w| < \frac{4}{\sqrt{5}\pi}\},$$

with $\phi_3(ie^{\frac{3}{8}\pi}) = 0$, $|\phi_3'(ie^{\frac{3}{8}\pi})| = \frac{2}{\sqrt{5}\pi e^{\frac{3}{8}\pi}}$.

Define $f(z) = \phi_3 \cdot \phi_2 \circ \phi_1(z)$, then $w = f(z) : G \to \{w : |w| < \frac{4}{\sqrt{5}\pi}\}$, with $f(a) = 0$, $|f'(a)| = 1$. Hence $r(G, a) = \frac{4}{\sqrt{5}\pi}$.

5207

Let $T(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

(a) Assume that $z_1, z_2 \in \mathbb{C}$ are two distinct fixed points for T, i.e., $T(z_i) = z_i$, $i = 1, 2$. Show that there exists a constant c such that

$$\frac{T(z) - z_1}{T(z) - z_2} = c \frac{z - z_1}{z - z_2}.$$

(b) Use (a) to find an expression for $T^n(z)$, $n = 1, 2, 3, \cdots$, if

$$T(z) = \frac{1 - 3z}{z - 3}.$$

(*Iowa*)

Solution.

(a) Let $a \in \mathbb{C}$ be a point different from z_1, z_2. Because the cross ratio is invariant under Möbius transformations, we have

$$(T(z), z_1, T(a), z_2) = (z, z_1, a, z_2),$$

which is

$$\frac{T(z)-z_1}{T(z)-z_2}:\frac{T(a)-z_1}{T(a)-z_2}=\frac{z-z_1}{z-z_2}:\frac{a-z_1}{a-z_2}.$$

Denoting

$$\frac{T(a)-z_1}{T(a)-z_2}:\frac{a-z_1}{a-z_2}=c,$$

we obtain

$$\frac{T(z)-z_1}{T(z)-z_2}=c\frac{z-z_1}{z-z_2}.$$

(b) Since $T^n(z)=T(T^{n-1}(z))$, it is easy to have

$$\frac{T^n(z)-z_1}{T^n(z)-z_2}=c\frac{T^{n-1}(z)-z_1}{T^{n-1}(z)-z_2}=c^2\frac{T^{n-2}(z)-z_1}{T^{n-2}(z)-z_2}=\cdots=c^n\frac{z-z_1}{z-z_2}.$$

When $T(z)=\frac{1-3z}{z-3}$, by solving the equation $\frac{1-3z}{z-3}=z$, we obtain that $z=\pm1$ are two fixed points of T. Choose $a=2$, then $T(a)=5$, hence $c=\frac{5-1}{5+1}:\frac{2-1}{2+1}=2$.

It follows from

$$\frac{T^n(z)-1}{T^n(z)+1}=2^n\frac{z-1}{z+1}$$

that

$$T^n(z)=\frac{(2^n+1)z-(2^n-1)}{(2^n+1)-(2^n-1)z}.$$

5208

(a) Justify the statement that "the curves

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

form a family of confocal conics".

(b) Prove that such confocal conics intersect orthogonally, if at all.

(c) Show that the transformation $w=\frac{1}{2}(z+\frac{1}{z})$ carries straight lines through the origin and circles centered at the origin into a family of confocal conics.

(*Harvard*)

Solution.

(a) Without loss of generality, we assume $a>b>0$. When $-a^2<\lambda<-b^2$, the curves form a family of hyperbolas, while when $\lambda>-b^2$, the curves form a family of ellipses. Suppose the focuses of the conics are $(\pm c(\lambda), 0)$. When $-a^2 < \lambda < -b^2$,

$$c(\lambda) = \sqrt{(a^2+\lambda) + [-(b^2+\lambda)]} = \sqrt{a^2-b^2}.$$

When $\lambda > -b^2$,

$$c(\lambda) = \sqrt{(a^2+\lambda) - (b^2+\lambda)} = \sqrt{a^2-b^2}.$$

Hence the curves

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

form a family of confocal conics.

(b) Suppose (x_0, y_0) is the intersection point of

$$L_1 : \frac{x^2}{a^2+\lambda_1} + \frac{y^2}{b^2+\lambda_1} = 1$$

and

$$L_2 : \frac{x^2}{a^2+\lambda_2} + \frac{y^2}{b^2+\lambda_2} = 1,$$

where $\lambda_1 \neq \lambda_2$. It follows from

$$\frac{x_0^2}{a^2+\lambda_1} + \frac{y_0^2}{b^2+\lambda_1} = 1$$

and

$$\frac{x_0^2}{a^2+\lambda_2} + \frac{y_0^2}{b^2+\lambda_2} = 1$$

that

$$\frac{x_0^2}{(a^2+\lambda_1)(a^2+\lambda_2)} + \frac{y_0^2}{(b^2+\lambda_1)(b^2+\lambda_2)} = 0.$$

Noting that the tangent vector of L_1 at (x_0, y_0) is $\vec{\tau}_1 = (\frac{x_0}{a^2+\lambda_1}, \frac{y_0}{b^2+\lambda_1})$, and the tangent vector of L_2 at (x_0, y_0) is $\vec{\tau}_2 = (\frac{x_0}{a^2+\lambda_2}, \frac{y_0}{b^2+\lambda_2})$, we have

$$\vec{\tau}_1 \cdot \vec{\tau}_2 = \frac{x_0^2}{(a^2+\lambda_1)(a^2+\lambda_2)} + \frac{y_0^2}{(b^2+\lambda_1)(b^2+\lambda_2)} = 0,$$

which implies that the confocal conics intersect orthogonally, if at all.

(c) Let $z = re^{i\theta}$, and

$$w = u + iv = \frac{1}{2}(z + \frac{1}{z}) = \frac{1}{2}(r + \frac{1}{r})\cos\theta + \frac{i}{2}(r - \frac{1}{r})\sin\theta.$$

The image of straight lines through the origin is

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 1,$$

which are hyperbolas in w-plane. Because

$$\cos^2\theta + \sin^2\theta = 1,$$

the focuses of the hyperbolas are $(\pm 1, 0)$.

The image of circles centered at the origin is

$$\frac{u^2}{\frac{1}{4}(r+\frac{1}{r})^2} + \frac{v^2}{\frac{1}{4}(r-\frac{1}{r})^2} = 1,$$

which are ellipses in w-plane. Because $\frac{1}{4}(r+\frac{1}{r})^2 - \frac{1}{4}(r-\frac{1}{r})^2 = 1$, the focuses of the ellipses are $(\pm 1, 0)$. Hence the transformation

$$w = \frac{1}{2}(z + \frac{1}{z})$$

carries straight lines through the origin and circles centered at the origin into a family of confocal conics.

5209

If $f : D(0,1) = \{z : |z| < 1\} \to \mathbb{C}$ is an analytic function which satisfies $f(0) = 0$, and if

$$|\text{Re} f(z)| < 1 \text{ for all } z \in D(0,1),$$

prove that

$$|f'(0)| \leq \frac{4}{\pi}.$$

(*Indiana*)

Solution.

It is easy to know that

$$w = g(\zeta) = \frac{e^{\frac{\pi i}{2}\zeta} - 1}{e^{\frac{\pi i}{2}\zeta} + 1}$$

is a conformal mapping of the domain $\{\zeta : |\text{Re}\zeta| < 1\}$ onto the unit disk $\{w : |w| < 1\}$ with $g(0) = 0$. Hence $w = F(z) = g \circ f(z)$ is analytic in

$D(0,1)$ and satisfies $F(0) = 0$ and $|F(z)| < 1$. By Schwarz's lemma, we have $|F'(0)| \leq 1$. Because

$$F'(z) = g'(f(z)) \cdot f'(z) = \frac{\pi i e^{\frac{\pi i}{2} f(z)} \cdot f'(z)}{(e^{\frac{\pi i}{2} f(z)} + 1)^2},$$

it follows from $f(0) = 0$ that

$$|f'(0)| \leq \frac{4}{\pi}.$$

5210

Let $\Omega = \{z \in \mathbb{C}; -1 < \mathrm{Im} z < 1\}$, and let $\mathcal{F}$ be the family of all analytic functions $f : \Omega \to \mathbb{C}$ such that $|f| < 1$ on Ω and $f(0) = 0$. Find

$$\sup_{f \in \mathcal{F}} |f(1)|.$$

(*Indiana*)

Solution.

It is obvious that

$$\zeta = f_0(z) = \frac{e^{\frac{\pi}{2} z} - 1}{e^{\frac{\pi}{2} z} + 1}$$

is a conformal mapping of Ω onto the unit disk with the origin fixed. For any analytic function $w = f(z) : \Omega \to \mathbb{C}$ such that $|f| < 1$ and $f(0) = 0$, we consider the composite function $w = F(\zeta) = f \circ f_0^{-1}(\zeta)$. $F(\zeta)$ is analytic in the unit disk such that $|F(\zeta)| < 1$ and $F(0) = 0$. By Schwarz's lemma,

$$|F(\zeta)| \leq |\zeta|.$$

Choose $\zeta_0 = \frac{e^{\frac{\pi}{2}} - 1}{e^{\frac{\pi}{2}} + 1}$, we have

$$|F(\zeta_0)| = |f(1)| \leq |\zeta_0| = \frac{e^{\frac{\pi}{2}} - 1}{e^{\frac{\pi}{2}} + 1}.$$

The equality holds if and only if $F(\zeta) = e^{i\theta}\zeta$, which implies

$$\sup_{f \in \mathcal{F}} |f(1)| = \frac{e^{\frac{\pi}{2}} - 1}{e^{\frac{\pi}{2}} + 1},$$

and the supremum is attained by $f(z) = e^{i\theta} f_0(z)$, where θ is a real number.

5211

Let f be an analytic function on $D = \{z; |z| < 1\}$ such that $f(0) = -1$, and suppose that $|1 + f(z)| < 1 + |f(z)|$ whenever $|z| < 1$. Prove that $|f'(0)| \leq 4$.

(*Indiana*)

Solution.

Let $\Omega = \mathbb{C} \backslash \{w = u + iv : u \geq 0 \text{ and } v = 0\}$. It follows from $|1 + f(z)| < 1 + |f(z)|$ that $f(D) \subset \Omega$.

Set $g(w) = \frac{\sqrt{w}-i}{\sqrt{w}+i}$, $\left(\sqrt{w}\Big|_{w=-1} = i\right)$. Then $g \circ f(z)$ is an analytic function on D with $g \circ f(0) = 0$ and $|g \circ f(z)| < 1$. By Schwarz's lemma,

$$|(g \circ f)'(0)| \leq 1.$$

Since

$$g'(w) = \frac{i}{\sqrt{w}(\sqrt{w} + i)^2},$$

we have $g'(-1) = -\frac{1}{4}$. From

$$(g \circ f)'(0) = g'(-1)f'(0),$$

we obtain

$$|f'(0)| \leq 4.$$

5212

Let P be the set of holomorphic function f on the open unit disc so that (i) Both the real and imaginary parts of $f(z)$ are positive for $|z| < 1$, (ii) $f(0) = 1 + i$. Let $E = \{f(\frac{1}{2}) : f \in P\}$. Describe E explicitly.

(*Minnesota*)

Solution.

Let $f \in P$ and define

$$\zeta = F(z) = \frac{f^2(z) - 2i}{f^2(z) + 2i}.$$

Then F is a holomorphic function on the unit disc with $F(0) = 0$ and $|F(z)| < 1$. By Schwarz's lemma, we have $|F(z)| \leq |z|$, which implies $|F(\frac{1}{2})| \leq \frac{1}{2}$. It should be noted that when f changes in P, $F(\frac{1}{2})$ can take any value in the disc $\{\zeta : |\zeta| \leq \frac{1}{2}\}$. Because $w = \frac{2i(1+\zeta)}{1-\zeta}$ (that is the inverse of $\zeta = \frac{w-2i}{w+2i}$) is a conformal mapping of $\{\zeta : |\zeta| \le \frac{1}{2}\}$ onto $\{w : |w - \frac{10}{3}i| \le \frac{8}{3}\}$, we obtain that the set $\{f^2(\frac{1}{2}) : f \in P\}$ is equal to

$$\{w : |w - \frac{10}{3}i| \le \frac{8}{3}\} = \{w = \rho e^{i\phi} : |\phi - \frac{\pi}{2}| \le \arcsin\frac{4}{5}, \rho^2 - \frac{20}{3}\rho\sin\phi + 4 \le 0\}.$$

Hence

$$E = \{f(\frac{1}{2}) : f \in P\} = \{re^{i\theta} : |\theta - \frac{\pi}{4}| \le \frac{1}{2}\arcsin\frac{4}{5}, r^4 - \frac{20}{3}r^2\sin 2\theta + 4 \le 0\}.$$

If we denote the two roots of $\rho^2 - \frac{20}{3}\rho\sin\phi + 4 = 0$ by $\rho_1(\phi)$, $\rho_2(\phi)$ where $\rho_1(\phi) \le \rho_2(\phi)$ and $|\phi - \frac{\pi}{2}| \le \arcsin\frac{4}{5}$, the set E can also be represented by

$$\left\{re^{i\theta} : |\theta - \frac{\pi}{4}| \le \frac{1}{2}\arcsin\frac{4}{5}, \sqrt{\rho_1(2\theta)} \le r \le \sqrt{\rho_2(2\theta)}\right\}.$$

5213

Let

$$\Omega = \{w = u + iv : \frac{u^2}{5^2} + \frac{v^2}{3^2} > 1\}.$$

If $\mathcal{F}$ is the family of all analytic function on Ω such that $|f| \le 1$ in Ω and $\lim\limits_{w\to\infty} f(w) = 0$, find $\sup\limits_{f\in\mathcal{F}} |f(8)|$. Your answer should be an explicit number, and you should prove your assertion.

(*Indiana*)

Solution.

Define $w = \phi(z) = 2(\frac{z}{2} + \frac{2}{z})$, it is easy to know that $w = \phi(z)$ is a conformal map of $D = \{z : |z| < 1\}$ onto Ω with $\phi(0) = \infty$ and $\phi(4 - \sqrt{12}) = 8$.

Then $F(z) = f \circ \phi(z) = f(2(\frac{z}{2} + \frac{2}{z}))$ is analytic in D and satisfies $F(0) = 0$ and $|F(z)| \le 1$. By Schwarz's lemma,

$$|F(z)| \le |z|.$$

Hence

$$|f(8)| = |F(4 - \sqrt{12})| \le 4 - \sqrt{12}.$$

This upper bound can be reached if we let $f = \phi^{-1}$ which belongs to family $\mathcal{F}$ and satisfies $\phi^{-1}(8) = 4 - \sqrt{12}$. So we obtain

$$\sup_{f\in\mathcal{F}} |f(8)| = 4 - \sqrt{12}.$$

5214

Let D be the upper-half and let $f \neq id$ be a conformal map of D onto itself such that $f \circ f = id$. Prove that f has a unique fixed point inside D.

(*SUNY, Stony Brook*)

Solution.

Since f is a conformal map of D onto itself, it can be written as $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. Then

$$f \circ f(z) = \frac{(a^2 + bc)z + b(a+d)}{c(a+d)z + d^2 + bc}.$$

It follows from $f \circ f = id$ that $b(a+d) = c(a+d) = 0$ and $a^2 + bc = d^2 + bc \neq 0$.

If $a + d \neq 0$, then $b = c = 0$. Hence $ad - bc > 0$ and $a^2 + bc = d^2 + bc$ impies $f = id$, which contradicts the condition $f \neq id$. Thus we have $a + d = 0$ and the inequality $ad - bc > 0$ can be written as $bc + a^2 < 0$.

Now we consider the equation $f(z) = \frac{az+b}{cz+d} = z$, which is equivalent to $cz^2 + (d-a)z - b = 0$. Since $\Delta = (d-a)^2 + 4bc$ is equal to $4bc + 4a^2 < 0$, we know that $f(z) = z$ has two conjugate roots, one in the upper-half plane and the other in the lower-half plane. So f has a unique fixed point inside D.

5215

Let Ω be a convex, open subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be an analytic function satisfying $\mathrm{Re} f'(z) > 0$, $z \in \Omega$. Prove that f is one-to-one in Ω (i.e., f is injective).

(*Indiana*)

Solution.

Let $z_1 \neq z_2$ be two arbitrary points in Ω. $L : z(t) = z_1 + t(z_2 - z_1)$, $t \in [0,1]$ is the line segment connecting z_1 and z_2. Since Ω is convex, $L \subset \Omega$, we have

$$f(z_2) - f(z_1) = \int_L f'(z)dz = \int_0^1 f'(z(t))(z_2 - z_1)dt.$$

Hence

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} = \int_0^1 f'(z(t))dt.$$

Since $\mathrm{Re} f'(z) > 0$ for $z \in \Omega$, we know that $\int_0^1 f'(z(t))dt \neq 0$, which implies $f(z_1) \neq f(z_2)$ whenever $z_1 \neq z_2$.

5216

Show that if the polynomial $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, $n > 1$, is one-to-one in the unit disk $|z| < 1$ and $a_1 = 1$, then $|na_n| \leq 1$.

(*SUNY, Stony Brook*)

Solution.

It follows from the univalence of $P(z)$ in $\{|z| < 1\}$ that $P'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + 2a_2 z + a_1 \neq 0$ for all $z \in \{|z| < 1\}$. In other words, the roots of $P'(z)$ are all situated outside the open unit disk. Let $z_1, z_2, \cdots, z_{n-1}$ be the roots of $P'(z)$, then $|z_j| \geq 1$ for $j = 1, 2, \cdots, n-1$. Because $P'(z)$ can also be written as $na_n(z - z_1)(z - z_2)\cdots(z - z_{n-1})$, by comparing the constant terms, we have

$$(-1)^{n-1} na_n \prod_{j=1}^{n-1} z_j = a_1.$$

Since $a_1 = 1$, we obtain

$$|na_n| = \frac{|a_1|}{\prod\limits_{j=1}^{n-1} |z_j|} \leq 1.$$

5217

Let $P(z)$ be a polynomial on the complex plane, not identically zero; let $H = \{z : \text{Re}z > 0\}$.

(a) If all roots of $P(z)$ lie in H, show that the same is true for the roots of dP/dz.

(b) For any non-vanishing polynomial $P(z)$, use the result in (a) to show that the convex hull of the roots of $P(z)$ contains the roots of dP/dz.

(*Courant Inst.*)

Solution.

(a) Let $z_1, z_2, \cdots, z_n$ be the zeros of $P(z)$. By assumption,

$$\text{Re}z_j > 0 \quad (j = 1, 2, \cdots, n),$$

and $P(z) = a(z - z_1)(z - z_2)\cdots(z - z_n)$. It follows that

$$(\log P(z))' = \frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n}.$$

When $z \in \{z : \text{Re}z \le 0\}$, then $\frac{\pi}{2} < \arg(z - z_j) < \frac{3\pi}{2}$, or equivalently, $\text{Re}\frac{1}{z-z_j} < 0$. Hence $\text{Re}\sum\limits_{j=1}^{n} \frac{1}{z-z_j} < 0$, which shows $\frac{P'(z)}{P(z)}$ can not be zero on $\{z : \text{Re}z \le 0\}$.

(b) Let $z_1, z_2, \cdots, z_n$ be the zeros of $P(z)$, and l is a directed straight line passing through two zeros z_k and z_l such that the other zeros are on the right side of l (including on l). Denote the intersectional angle from the positive direction of the imaginary axis to l by θ. When z is on the left side of l, we have $\text{Re}\{e^{-i\theta}(z - z_j)\} < 0$. Hence

$$\text{Re}\left\{e^{i\theta}\frac{P'(z)}{P(z)}\right\} = \text{Re}\sum_{j=1}^{n} \frac{e^{i\theta}}{z - z_j} < 0,$$

which shows that the zeros of $P'(z)$ do not lie on the left side of l. After considering all the directed straight lines passing through two of the zeros of $P(z)$ such that the other zeros are on the right side of the line, we obtain that the zeros of $P'(z)$ lie on the convex hull of the zeros of $P(z)$.

5218

Let $f(z)$ be a Laurent series centered at 0, convergent in $\mathbb{C}\backslash\{0\}$, with residue b at $z = 0$.

(a) Show that there exists ζ on $\{z \in \mathbb{C} : |z| = 1\}$ with

$$|f(\zeta) - \zeta^{-1}| \ge |b - 1|,$$

(b) Characterize those functions with

$$\max_{|\zeta|=1} |f(\zeta) - \zeta^{-1}| = |b - 1|.$$

(*Minnesota*)

Solution.

(a) Let $f(z) = \sum\limits_{n=-\infty}^{+\infty} b_n z^n$, then

$$b_{-1} = \frac{1}{2\pi i}\int_{|\zeta|=1} f(\zeta)d\zeta = b.$$

Hence

$$b - 1 = \frac{1}{2\pi i}\int_{|\zeta|=1} (f(\zeta) - \zeta^{-1})d\zeta.$$

If $|f(\zeta)-\zeta^{-1}| < |b-1|$ holds for all ζ with $|\zeta|=1$, then

$$|b-1| \le \frac{1}{2\pi}\max_{|\zeta|=1}|f(\zeta)-\zeta^{-1}|\cdot\int_{|\zeta|=1}|d\zeta| < |b-1|,$$

which is a contradiction. Hence there exists ζ with $|\zeta|=1$ such that

$$|f(\zeta)-\zeta^{-1}| \ge |b-1|.$$

(b) If $\max\limits_{|\zeta|=1}|f(\zeta)-\zeta^{-1}| = |b-1|$, it follows from

$$|b-1| \le \frac{1}{2\pi}\int_{|\zeta|=1}|f(\zeta)-\zeta^{-1}||d\zeta|$$

that

$$|f(\zeta)-\zeta^{-1}| = |b-1|$$

holds for all ζ with $|\zeta|=1$.

Let $f(\zeta)-\zeta^{-1} = (b-1)e^{i\phi(\theta)}$, where $\zeta = e^{i\theta}$ and $\phi(\theta)$ is a continuous real-valued function. It follows from

$$b-1 = \frac{1}{2\pi i}\int_{|\zeta|=1}(f(\zeta)-\zeta^{-1})d\zeta$$

that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(\phi(\theta)+\theta)}d\theta = 1,$$

which implies that $\phi(\theta) = -\theta$, and hence

$$f(\zeta)-\zeta^{-1} = \frac{b-1}{\zeta}$$

holds on $\{\zeta : |\zeta|=1\}$. Apply the discreteness of zeros for analytic functions to $f(z)-\frac{b}{z}$, we obtain $f(z)=\frac{b}{z}$, $z\in\mathbb{C}\backslash\{0\}$.

5219

Assume f is analytic in a neighborhood of $\overline{D}$, f maps D into D, and f maps ∂D into ∂D, where $D=\{z : |z|<1\}$.

(a) Show that $\forall z\in\partial D$, $f'(z)\neq 0$.

(b) Show that $\frac{d}{d\theta}[\arg f(e^{i\theta})] > 0$ for θ in $\mathbb{R}$.

(c) Assume that $f(0) = f'(0) = 0$ and $f|_{\partial D}$ is a two-to-one map from ∂D onto ∂D. Show that $f(z) \neq 0$ whenever $0 < |z| < 1$.

(*Indiana–Purdue*)

Solution.

(a) Assume $f'(z_0) = 0$, where $z_0 \in \partial D$. Let $f(z_0) = w_0 \in \partial D$. Then

$$f(z) - w_0 = (z - z_0)^n g(z),$$

where $n \geq 2$,

$$g(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots,$$

with $b_0 \neq 0$. Let Γ be an arc in $\overline{D}$ defined by $\Gamma = \{z \in \overline{D} : |z - z_0| = r\}$, and denote by $\Delta_\Gamma \phi(z)$ the change of $\phi(z)$ when z goes along the arc Γ in the counterclockwise sense. It is demanded that r is sufficiently small such that $\Delta_\Gamma \arg(z - z_0) > \frac{3\pi}{4}$ and $|g(z) - b_0| < \frac{|b_0|}{2}$ when $z \in \Gamma$. It follows from $f(z) - w_0 = (z - z_0)^n g(z)$ $(n \geq 2)$, that

$$\Delta_\Gamma \arg(f(z) - w_0) = n\Delta_\Gamma \arg(z - z_0) + \Delta_\Gamma \arg g(z) > \frac{3\pi}{2} - \frac{\pi}{3} > \pi,$$

which implies that $f(z)$ assumes values outside the disk $\overline{D}$ when $z \in \Gamma$. It is a contradiction to the fact that f maps D into D. Hence $f'(z) \neq 0$ for all $z \in \partial D$.

(b) Let $z = re^{i\theta}$, and $w = f(z) = Re^{i\psi}$. A variation of the Cauchy-Riemann equations for analytic function $w = f(z)$ is

$$r\frac{\partial R}{\partial r} = R\frac{\partial \psi}{\partial \theta}, \quad \frac{\partial R}{\partial \theta} = -rR\frac{\partial \psi}{\partial r}.$$

Since f maps ∂D into ∂D, we know that $\frac{\partial R}{\partial \theta}(e^{i\theta}) = 0$. If $\frac{\partial \psi}{\partial \theta}(e^{i\theta}) = 0$, then at point $e^{i\theta}$, $\frac{\partial R}{\partial r} = \frac{\partial R}{\partial \theta} = \frac{\partial \psi}{\partial r} = \frac{\partial \psi}{\partial \theta} = 0$, which implies that $\frac{\partial f}{\partial z}(e^{i\theta}) = f'(e^{i\theta}) = 0$. But from (a) it is impossible. If $\frac{\partial \psi}{\partial \theta}(e^{i\theta}) < 0$, it follows from $r\frac{\partial R}{\partial r} = R\frac{\partial \psi}{\partial \theta}$ that $\frac{\partial R}{\partial r}(e^{i\theta}) < 0$. Since $R = 1$ when $r = 1$, $\frac{\partial R}{\partial r}(e^{i\theta}) < 0$ implies that $R > 1$ when $r < 1$. This is also impossible. Hence we obtain

$$\frac{\partial \psi}{\partial \theta}(e^{i\theta}) = \frac{d}{d\theta}[\arg f(e^{i\theta})] > 0.$$

(c) Because $f|_{\partial D}$ is a two-to-one map from ∂D onto ∂D, $\frac{1}{2\pi}\Delta_{|z|=1}\arg f(z) = 2$, which implies that $f(z)$ has two zeros (counted by multiplicity) in D. Since $f(0) = f'(0) = 0$, $z = 0$ is a zero of f of multiplicity $m = 2$. Hence $f(z)$ has no zero in $\{0 < |z| < 1\}$.

SECTION 3
COMPLEX INTEGRATION

5301

Evaluate the integral

$$\int_{|z|=2} e^{e^{\frac{1}{z}}}\,dz.$$

(*Indiana*)

Solution.

Function $e^{e^{\frac{1}{z}}}$ is analytic in $\{z : 0 < |z| < +\infty\}$, and its Laurent expansion around $z = 0$ is:

$$\begin{aligned} e^{e^{\frac{1}{z}}} &= 1 + e^{\frac{1}{z}} + \frac{1}{2!}e^{\frac{2}{z}} + \cdots + \frac{1}{n!}e^{\frac{n}{z}} + \cdots \\ &= 1 + \left\{1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \cdots\right\} + \frac{1}{2!}\left\{1 + \frac{2}{z} + \frac{1}{2!}\left(\frac{2}{z}\right)^2 + \cdots\right\} \\ &\quad + \cdots + \frac{1}{n!}\left\{1 + \frac{n}{z} + \frac{1}{2!}\left(\frac{n}{z}\right)^2 + \cdots\right\} + \cdots. \end{aligned}$$

The coefficient of the term $\frac{1}{z}$ in the above development is

$$1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} + \cdots = e.$$

By the residue theorem, we obtain

$$\int_{|z|=2} e^{e^{\frac{1}{z}}}\,dz = 2\pi i \mathrm{Res}\left(e^{e^{\frac{1}{z}}}, 0\right) = 2\pi e i.$$

5302

Evaluate

$$\int_\gamma \frac{dz}{\sin^3 z},$$

where γ is the positively oriented circle $\{|z| = 1\}$.

(*Indiana*)

Solution.

It is obvious that $\frac{1}{\sin^3 z}$ is analytic in $\{z : 0 < |z| \leq 1\}$, and with $z = 0$ as a pole. The Laurent expansion of $\frac{1}{\sin^3 z}$ around $z = 0$ can be obtained as follows:

$$\begin{aligned}
\frac{1}{\sin^3 z} &= \frac{1}{\left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots\right)^3} \\
&= \frac{1}{z^3\left\{1 - \left(\frac{1}{3!}z^2 - \frac{1}{5!}z^4 + \cdots\right)\right\}^3} \\
&= \frac{1}{z^3}\left\{1 + 3\left(\frac{1}{3!}z^2 - \frac{1}{5!}z^4 + \cdots\right) + 6\left(\frac{1}{3!}z^2 - \frac{1}{5!}z^4 + \cdots\right)^2 + \cdots\right\}.
\end{aligned}$$

Hence the coefficient of the term $\frac{1}{z}$ in the above development is $\frac{1}{2}$. By the residue theorem, we have

$$\int_\gamma \frac{dz}{\sin^3 z} = 2\pi i \text{Res}\left(\frac{1}{\sin^3 z}, 0\right) = \pi i.$$

5303

For what value of a is the function

$$f(z) = \int_1^z \left(\frac{1}{z} + \frac{a}{z^3}\right) \cos z dz$$

single-valued?

(*Indiana*)

Solution.

Function $F(z) = (\frac{1}{z} + \frac{a}{z^3}) \cos z$ is analytic in $\{z : 0 < |z| < +\infty\}$, and its Laurent expansion around $z = 0$ is:

$$\begin{aligned}
F(z) &= \left(\frac{1}{z} + \frac{a}{z^3}\right) \cos z = \left(\frac{1}{z} + \frac{a}{z^3}\right)\left(1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \cdots\right) \\
&= \frac{a}{z^3} + \left(1 - \frac{a}{2}\right)\frac{1}{z} + \left(\frac{a}{24} - \frac{1}{2}\right) z + \cdots.
\end{aligned}$$

The necessary and sufficient condition for $f(z)$ to be single-valued is that the residue of $F(z)$ at $z = 0$ is zero, i.e., the coefficient of the term $\frac{1}{z}$ in the above development is zero. Hence we obtain $a = 2$.

5304

Define

$$h(z) = \int_0^\infty (1 + zte^{-t})^{-1} e^{-t} \cos(t^2) dt.$$

What is the largest possible P so that $h(z)$ is analytic for $|z| < P$?

(*Indiana–Purdue*)

Solution.

When $z = -e$,

$$h(-e) = \int_0^\infty \frac{\cos(t^2)}{e^t - et} dt.$$

It is easy to see that when $t \to 1$,

$$\frac{\cos(t^2)}{e^t - et} \sim \frac{A}{(t-1)^2},$$

where $A = \frac{2}{e}\cos 1$, which implies that the integral is divergent. Hence P can not be larger than e.

For any $r < e$, let $|z| \le r$. Consider the integral

$$h(z) = \int_0^\infty \frac{\cos(t^2)}{e^t + zt} dt.$$

It follows from $|e^t + zt| \ge e^t - rt$ and the convergence of the integral

$$\int_0^\infty \frac{|\cos(t^2)|}{e^t - rt} dt$$

that

$$\int_0^\infty \frac{\cos(t^2)}{e^t + zt} dt$$

is uniformly convergent in any compact subset of $\{z : |z| < e\}$. By Weierstrass theorem, we know that $h(z)$ is analytic in $\{z : |z| < e\}$. Hence the largest possible P is equal to e.

5305

Let $f(z)$ be analytic in $S = \{z \in \mathbb{C}; |z| < 2\}$. Show that

$$\frac{2}{\pi}\int_0^{2\pi} f(e^{it})\cos^2\frac{t}{2}dt = 2f(0)+f'(0).$$

(*Iowa*)

Solution.

It is easy to see that

$$\begin{aligned} f(0) &= \frac{1}{2\pi i}\int_{|z|=1}\frac{f(z)}{z}dz = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})dt, \\ f'(0) &= \frac{1}{2\pi i}\int_{|z|=1}\frac{f(z)}{z^2}dz = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-it}dt. \end{aligned}$$

Note that

$$0 = \frac{1}{2\pi i}\int_{|z|=1} f(z)dz = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{it}dt.$$

It follows from the above three equalities that

$$\begin{aligned} 2f(0)+f'(0) &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})(2+e^{it}+e^{-it})dt \\ &= \frac{2}{\pi}\int_0^{2\pi} f(e^{it})\cos^2\frac{t}{2}dt. \end{aligned}$$

5306

Suppose that the real-valued function u is harmonic in the disk $\{|z|<2\}$, v is its harmonic conjugate and $u(0)=v(0)=0$. Show that

$$\int_\gamma u^2(z)v^2(z)\frac{dz}{z} = \frac{1}{6}\int_\gamma (u^4(z)+v^4(z))\frac{dz}{z},$$

where $\gamma(t)=e^{2\pi it}$, $t\in[0,1]$.

(*SUNY, Stony Brook*)

Solution.

Let $f(z)=u(z)+iv(z)$. Then $f(z)$ is analytic in $\{z:|z|<2\}$, and we have

$$\begin{aligned} \int_\gamma f^4(z)\frac{dz}{z} &= 2\pi i f^4(0) = 0, \\ \int_\gamma \overline{f}^4(z)\frac{dz}{z} &= \overline{\left(\int_\gamma f^4(z)\frac{d\overline{z}}{\overline{z}}\right)} \end{aligned}$$

$$= \overline{\left(\int_\gamma z f^4(z) d\left(\frac{1}{z}\right)\right)}$$
$$= \overline{\left(-\int_\gamma \frac{f^4(z)}{z} dz\right)} = 0.$$

It follows from

$$u(z) = \frac{f(z) + \overline{f(z)}}{2}$$

and

$$v(z) = \frac{f(z) - \overline{f(z)}}{2i}$$

that

$$\begin{aligned}
\int_\gamma u^2(z)v^2(z)\frac{dz}{z} &= -\frac{1}{16}\int_\gamma \left(f^4(z) + \overline{f}^4(z) - 2|f(z)|^4\right)\frac{dz}{z} \\
&= \frac{1}{8}\int_\gamma |f(z)|^4 \frac{dz}{z} \\
&= \frac{1}{8}\int_\gamma (u^4(z) + v^4(z) + 2u^2(z)v^2(z))\frac{dz}{z},
\end{aligned}$$

which implies that

$$\int_\gamma u^2(z)v^2(z)\frac{dz}{z} = \frac{1}{6}\int_\gamma (u^4(z) + v^4(z))\frac{dz}{z}.$$

5307

Let f be an analytic function on an open set containing $\overline{D(0,1)} = \{z; |z| \leq 1\}$.

(a) Prove that

$$\frac{d^n f}{dz^n}(0) = \frac{n!}{\pi}\int_0^{2\pi} e^{-ni\theta}[\text{Re} f(e^{i\theta})]d\theta.$$

(b) If $f(0) = 1$, and if $\text{Re} f(z) > 0$ for all points $z \in D(0,1)$, prove that

$$\left|\frac{d^n f}{dz^n}(0)\right| \leq 2(n!).$$

(*Indiana*)

Solution.

(a) Assume that

$$f(z)=\sum_{k=0}^{\infty} a_k z^n,$$

we have

$$\begin{aligned}\frac{n!}{2\pi}\int_0^{2\pi}\overline{f(e^{i\theta})}e^{-ni\theta}d\theta &= \frac{n!}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{\infty}\bar{a}_k e^{-ki\theta}\right)e^{-ni\theta}d\theta\\ &= \frac{n!}{2\pi}\sum_{k=0}^{\infty}\bar{a}_k\left(\int_0^{2\pi}e^{-(n+k)i\theta}d\theta\right)=0.\end{aligned}$$

By Cauchy Integral Formula,

$$\frac{d^n f}{dz^n}(0)=\frac{n!}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta^{n+1}}d\zeta=\frac{n!}{2\pi}\int_0^{2\pi}f(e^{i\theta})e^{-ni\theta}d\theta.$$

Hence

$$\begin{aligned}\frac{d^n f}{dz^n}(0) &= \frac{n!}{2\pi}\int_0^{2\pi}f(e^{i\theta})e^{-ni\theta}d\theta+\frac{n!}{2\pi}\int_0^{2\pi}\overline{f(e^{i\theta})}e^{-ni\theta}d\theta\\ &= \frac{n!}{\pi}\int_0^{2\pi}e^{-ni\theta}[\mathrm{Re}f(e^{i\theta})]d\theta.\end{aligned}$$

(b) Because $\mathrm{Re}f(z)$ is harmonic on $\overline{D(0,1)}$, by the mean-value formula of harmonic functions,

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}f(e^{i\theta})d\theta=\mathrm{Re}f(0)=1.$$

Noting that $\mathrm{Re}f(e^{i\theta})\geq 0$, we have

$$\begin{aligned}\left|\frac{d^n f}{dz^n}(0)\right| &= \left|\frac{n!}{\pi}\int_0^{2\pi}e^{-ni\theta}[\mathrm{Re}f(e^{i\theta})]d\theta\right|\\ &\leq \frac{n!}{\pi}\int_0^{2\pi}|e^{-ni\theta}|[\mathrm{Re}f(e^{i\theta})]d\theta\\ &= \frac{n!}{\pi}\int_0^{2\pi}\mathrm{Re}f(e^{i\theta})d\theta\\ &= 2(n!).\end{aligned}$$

5308

If f is analytic in the unit disk and its derivative satisfies

$$|f'(z)| \leq (1 - |z|)^{-1},$$

show that the coefficients in the expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

satisfy $|a_n| < e$ for $n \geq 1$, where e is the base of natural logarithms.

(*Stanford*)

Solution.

It follows from

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

that

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1},$$

where

$$n a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f'(z)}{z^n} dz, \quad (0 < r < 1).$$

It is obvious that

$$|a_1| = |f'(0)| \leq 1 < e.$$

For $n > 1$, we choose $r = 1 - \frac{1}{n}$,

$$\begin{aligned} |a_n| &= \frac{1}{2\pi n} \left| \int_{|z|=1-\frac{1}{n}} \frac{f'(z)}{z^n} dz \right| \\ &\leq \frac{1}{2\pi n} \cdot \frac{(\frac{1}{n})^{-1}}{(1-\frac{1}{n})^n} \cdot 2\pi(1 - \frac{1}{n}) \\ &= (1 + \frac{1}{n-1})^{n-1} < e. \end{aligned}$$

5309

Let $f = u + iv$ be an entire function.

(a) Show that if $u^2(z) \geq v^2(z)$ for all $z \in \mathbb{C}$, then f must be a constant.

(b) Show that if $|f(z)| \leq A+B|z|^h$ for all $z \in \mathbb{C}$ with some positive numbers A, B, h, then $f(z)$ is a polynomial of degree bounded by h.

(*Stanford*)

Solution.

(a) Let

$$F(z) = e^{-f^2(z)} = e^{-(u^2(z)-v^2(z))-2iu(z)v(z)}.$$

Then $F(z)$ is an entire function with

$$|F(z)| = e^{-(u^2(z)-v^2(z))} \leq 1.$$

By Liouville's theorem, $F(z)$ must be a constant, which implies that $f(z)$ is a constant.

(b) Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then

$$a_n = \frac{1}{2\pi i}\int_{|z|=R} \frac{f(z)}{z^{n+1}} dz.$$

For any integer $n > h$,

$$\begin{aligned} |a_n| &\leq \frac{1}{2\pi}\int_{|z|=R} \left|\frac{f(z)}{z^{n+1}}\right| \cdot |dz| = \frac{1}{2\pi R^n}\int_0^{2\pi} |f(\mathrm{Re}^{i\theta})| d\theta \\ &\leq \frac{A+BR^h}{R^n}. \end{aligned}$$

Letting $R \to +\infty$, we obtain that $a_n = 0$, which implies that $f(z)$ is a polynomial of degree bounded by h.

5310

Let f be an entire function that satisfies $|\mathrm{Re}\{f(z)\}| \leq |z|^n$ for all z, where n is a positive integer. Show that f is a polynomial of degree at most n.

(*Indiana*)

Solution.

Let R be an arbitrary positive number. Then it follows from Schwarz's theorem that when $|z| < R$,

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R} \mathrm{Re}\{f(\zeta)\} \cdot \frac{\zeta+z}{\zeta-z} \cdot \frac{d\zeta}{i\zeta} + i\mathrm{Im}\{f(0)\}.$$

Especially when $|z| = \frac{R}{2}$,

$$|f(z)| \le \frac{1}{2\pi} \cdot 3R^n \cdot 2\pi + |\mathrm{Im}\{f(0)\}| = 3R^n + |\mathrm{Im}\{f(0)\}|,$$

which implies that there exist constants A, B such that

$$|f(z)| \le A|z|^n + B$$

holds for all $z \in \mathbb{C}$.

Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where

$$a_k = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{k+1}}dz.$$

Hence when $k > n$,

$$|a_k| \le \frac{1}{2\pi}\int_{|z|=r} \frac{|f(z)|}{|z|^{k+1}}|dz| \le \frac{Ar^n + B}{r^k} \to 0 \quad (r \to +\infty),$$

which shows that $f(z)$ is a polynomial of degree at most n.

5311

Compute the double integral

$$\int\int_D \cos z dxdy$$

where D is the disk given by $\{z = x + iy \in \mathbb{C} : x^2 + y^2 < 1\}$.

(*Iowa*)

Solution.

First we have the following complex forms of Green's formula:

$$\begin{aligned}
\int\int_D w_z dxdy &= \int\int_D \frac{1}{2}(w_x - iw_y)dxdy \\
&= -\frac{1}{2i}\int_{\partial D} w(dx - idy) = -\frac{1}{2i}\int_{\partial D} wd\bar{z}, \\
\int\int_D w_{\bar{z}} dxdy &= \int\int_D \frac{1}{2}(w_x + iw_y)dxdy \\
&= \frac{1}{2i}\int_{\partial D} w(dx + idy) = \frac{1}{2i}\int_{\partial D} wdz.
\end{aligned}$$

The problem can be solved directly by either one of the above two forms:

$$\int\int_D \cos z dxdy = \frac{1}{2i}\int_{|z|=1} \overline{z}\cos z dz = \frac{1}{2i}\int_{|z|=1} \frac{\cos z}{z} dz = \pi;$$

or

$$\begin{aligned}\int\int_D \cos z dxdy &= -\frac{1}{2i}\int_{|z|=1} \sin z d\overline{z} = -\frac{1}{2i}\int_{|z|=1} \sin z d(\frac{1}{z}) \\ &= \frac{1}{2i}\int_{|z|=1} \frac{\sin z}{z^2} dz = \pi.\end{aligned}$$

5312

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be analytic in $D = \{|z| < 1\}$ and assume that the integral

$$A = \int\int_D |f'(z)|^2 dxdy$$

is finite.

(a) Express A in terms of the coefficients a_n.

(b) Prove that

$$|f(z) - f(0)| \le \sqrt{\frac{A}{\pi}\log\frac{1}{1-|z|^2}}$$

for $z \in D$.

(Indiana)

Solution.

(a) By

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1},$$

we have

$$\begin{aligned}A &= \int\int_D |f'(z)|^2 dxdy = \int_0^1 rdr \int_0^{2\pi} (f'(re^{i\theta}))\overline{(f'(re^{i\theta}))} d\theta \\ &= \int_0^1 rdr \int_0^{2\pi} \left(\sum_{n=1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta}\right)\left(\sum_{n=1}^{\infty} n\overline{a}_n r^{n-1} e^{-i(n-1)\theta}\right) d\theta.\end{aligned}$$

Noting that

$$\int_0^{2\pi} e^{ik\theta} \cdot e^{-il\theta} d\theta = \begin{cases} 0 & k \neq l, \\ 2\pi & k = l, \end{cases}$$

we obtain that

$$\begin{aligned} A &= \int\int_D |f'(z)|^2 dxdy = \int_0^1 rdr \int_0^{2\pi} \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} d\theta \\ &= 2\pi \int_0^1 \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-1} dr = \pi \sum_{n=1}^{\infty} n|a_n|^2. \end{aligned}$$

(b) By Cauchy's inequality, we have

$$\begin{aligned} |f(z) - f(0)| &= \left|\sum_{n=1}^{\infty} a_n z^n\right| = \left|\sum_{n=1}^{\infty} \left(\sqrt{n} a_n \cdot \frac{1}{\sqrt{n}} z^n\right)\right| \\ &\leq \sqrt{\sum_{n=1}^{\infty} n|a_n|^2 \cdot \sum_{n=1}^{\infty} \frac{1}{n} |z|^{2n}} = \sqrt{\frac{A}{\pi} \log \frac{1}{1 - |z|^2}}. \end{aligned}$$

5313

Let f be analytic in $\{0 < |z| < 1\}$ and in L^2 with respect to planar Lebesque measure. Is 0 a removable singularity? Proof or counterexample.

(*Stanford*)

Solution.

The answer to the problem is Yes.

Let the Laurent expansion of f in $\{z : 0 < |z| < 1\}$ be

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{|z|=r<1} \frac{f(z)}{z^{n+1}} dz, \quad (n = 0, \pm 1, \pm 2, \cdots).$$

From

$$|a_n| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(re^{i\theta})|}{r^n} d\theta,$$

we have

$$|a_n|^2 r^{2n+1} \le \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\sqrt{r}d\theta\right)^2 \le \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 r d\theta.$$

Let $\varepsilon < 1$ be a small positive number, and then

$$\int_\varepsilon^1 |a_n|^2 r^{2n+1} dr \le \frac{1}{2\pi}\int_0^{2\pi}\int_\varepsilon^1 |f(re^{i\theta})|^2 r dr d\theta < \frac{1}{2\pi}\int\int_{0<|z|<1} |f(z)|^2 dxdy.$$

Then a_n must be zero when $n \le -1$. Otherwise, let $\varepsilon \to +0$, the left side of the above inequality will tend to infinity, while the right side of the inequality is finite, which leads to a contradiction. Hence

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which shows that $z = 0$ is a removable singularity of f.

5314

Evaluate the integral

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}, \quad |a| \ne \rho.$$

(*Indiana*)

Solution.

Let $z = \rho e^{i\theta}$, $a = re^{i\phi}$.

$$\begin{aligned}
\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} &= \int_0^{2\pi} \frac{\rho d\theta}{\rho^2 + r^2 - \rho r(e^{i(\theta-\phi)} + e^{i(\phi-\theta)})} \\
&= \int_0^{2\pi} \frac{\rho d\theta}{\rho^2 + r^2 - \rho r(e^{i\theta} + e^{-i\theta})} \\
&= \int_{|z|=\rho} \frac{\rho dz/(iz)}{\rho^2 + r^2 - rz - \rho^2 r/z} \\
&= \int_{|z|=\rho} \frac{\rho i dz}{rz^2 - (\rho^2 + r^2)z + \rho^2 r}
\end{aligned}$$

When $r < \rho$,

$$\begin{aligned}\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} &= \int_{|z|=\rho} \frac{\rho i dz}{r(z-\frac{\rho^2}{r})(z-r)} \\ &= 2\pi i \cdot \frac{\rho i}{r} \mathrm{Res}\left(\frac{1}{(z-\frac{\rho^2}{r})(z-r)}, r\right) \\ &= \frac{2\pi\rho}{\rho^2 - r^2}.\end{aligned}$$

When $r > \rho$,

$$\begin{aligned}\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} &= \int_{|z|=\rho} \frac{\rho i dz}{r(z-\frac{\rho^2}{r})(z-r)} \\ &= 2\pi i \cdot \frac{\rho i}{r} \mathrm{Res}\left(\frac{1}{(z-\frac{\rho^2}{r})(z-r)}, \frac{\rho^2}{r}\right) \\ &= \frac{2\pi\rho}{r^2 - \rho^2}.\end{aligned}$$

5315

Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a + \sin^2 x}, \quad |a| > 1,$$

by the method of residues.

(*Columbia*)

Solution.

Denote

$$I(a) = \int_0^{\frac{\pi}{2}} \frac{dx}{a + \sin^2 x}.$$

It is obvious that $I(a)$ is an analytic function in $\{a : |a| > 1\}$. Then we have

$$\begin{aligned}I(a) &= \int_0^{\frac{\pi}{2}} \frac{dx}{a + \sin^2 x} = \int_0^{\frac{\pi}{2}} \frac{2dx}{2a + 1 - \cos 2x} \\ &= \int_0^{\pi} \frac{dx}{2a + 1 - \cos x} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{dx}{2a + 1 - \cos x}.\end{aligned}$$

Let $z = e^{ix}$, then

$$\begin{aligned} dx &= \frac{dz}{iz}, \\ \cos x &= \frac{z + z^{-1}}{2}, \end{aligned}$$

and

$$I(a) = \int_{|z|=1} \frac{idz}{z^2 - 2(2a+1)z + 1}.$$

Denote the two roots of $z^2 - 2(2a+1)z + 1 = 0$ by z_1 and z_2. Since $z_1 \cdot z_2 = 1$, we may assume that $|z_1| > 1$, $|z_2| < 1$. By the residue theorem we have

$$\begin{aligned} I(a) &= \int_{|z|=1} \frac{idz}{(z - z_1)(z - z_2)} = \frac{2\pi}{z_1 - z_2} \\ &= \frac{2\pi}{\sqrt{(z_1 + z_2)^2 - 4z_1 z_2}} = \frac{\pi}{2\sqrt{a(a+1)}}. \end{aligned}$$

It should be noted that $\frac{\pi}{2\sqrt{a(a+1)}}$ is also analytic in $\{a : |a| > 1\}$, and the branch of $\sqrt{a(a+1)}$ should be chosen by $\arg\sqrt{a(a+1)}\,|_{a>1} = 0$.

5316

Consider the function

$$g(z, \theta) = \frac{1}{1 + z\sin\theta}.$$

(a) Use the residue theorem to find an explicit formula for

$$f(z) = \int_0^{2\pi} g(z, \theta) d\theta$$

when $|z| < 1$.

(b) Integrate the Taylor expansion

$$g(z, \theta) = \sum_{n=0}^{\infty} g_n(\theta) z^n$$

term by term to find the coefficients in the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} f_n z^n.$$

(c) Verify directly that (a) and (b) agree when $|z| < 1$.

(*Courant Inst.*)

Solution.

(a) Let $\zeta = e^{i\theta}$. Then

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{\zeta^2 - 1}{2i\zeta},$$

and

$$f(z) = \int_0^{2\pi} g(z,\theta)d\theta = \int_{|\zeta|=1} \frac{2i\zeta}{z\zeta^2 + 2i\zeta - z} \cdot \frac{d\zeta}{i\zeta} = \int_{|\zeta|=1} \frac{2d\zeta}{z(\zeta - \zeta_1)(\zeta - \zeta_2)},$$

where $\zeta_1 = \frac{i}{z}(\sqrt{1-z^2}-1)$, $\zeta_2 = \frac{i}{z}(-\sqrt{1-z^2}-1)$, and the single-valued branch of $\sqrt{1-z^2}$ in $\{|z| < 1\}$ is defined by $\sqrt{1-z^2}\ |_{z=0} = 1$. Because $|\zeta_1 \cdot \zeta_2| = 1$, we know that $\zeta_1 \in \{|\zeta| < 1\}$ and $\zeta_2 \in \{|\zeta| > 1\}$. Hence

$$f(z) = 2\pi i \cdot \frac{2}{z} \cdot \frac{1}{\zeta_1 - \zeta_2} = \frac{2\pi}{\sqrt{1-z^2}}.$$

(b) It follows from $|\sin\theta| \le 1$ and $|z| < 1$ that

$$g(z,\theta) = \sum_{k=0}^{\infty} (-1)^k \sin^k\theta \cdot z^k, \quad (|z| < 1).$$

Since the series converges uniformly for all $\theta \in [0, 2\pi]$, the integration with respect to θ can be taken term by term, and

$$f(z) = \int_0^{2\pi} \left(\sum_{k=0}^{\infty} (-1)^k \sin^k\theta \cdot z^k \right) d\theta = \sum_{k=0}^{\infty} a_k z^k,$$

where

$$a_k = \int_0^{2\pi} (-1)^k \sin^k\theta d\theta.$$

It is easy to obtain that $a_{2n-1} = 0$ and

$$a_{2n} = 4\int_0^{\frac{\pi}{2}} \sin^{2n}\theta d\theta = \frac{(2n-1)!!}{(2n)!!} \cdot 2\pi.$$

(c) In order to verify that (a) and (b) agree when $|z| < 1$, we develop the function $f(z)$ in (a) into a power series:

$$f(z) = \frac{2\pi}{\sqrt{1-z^2}} = 2\pi(1-z^2)^{-\frac{1}{2}} = 2\pi \sum_{n=0}^{\infty} (-1)^n C_{-\frac{1}{2}}^n z^{2n}.$$

Since

$$(-1)^n C^n_{-\frac{1}{2}} = (-1)^n \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-1}{2})}{n!} = \frac{(2n-1)!!}{(2n)!!},$$

we know that the results in (a) and (b) agree when $|z| < 1$.

5317

If a is real, show that

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-(x+ia)^2} dx$$

exists and is independent of a.

(*UC, Irvine*)

Solution.

First we have

$$\left|e^{-(x+ia)^2}\right| \le e^{a^2} \cdot e^{-x^2}.$$

It follows from the existence of

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-x^2} dx$$

that

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-(x+ia)^2} dx$$

exists.

Define $f(z) = e^{-z^2}$ and choose the contour of integration $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ as shown in Fig.5.5.

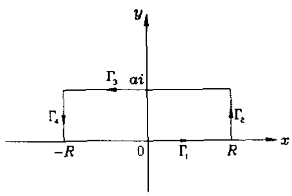

Fig.5.5

As $f(z)$ is analytic inside Γ, by Cauchy integral theorem,

$$\int_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz + \int_{\Gamma_3} f(z)dz + \int_{\Gamma_4} f(z)dz$$

$$\begin{aligned} &= \int_{-R}^{R} e^{-x^2}dx + ie^{-R^2}\int_0^a e^{y^2-2Ryi}dy - \int_{-R}^{R} e^{-(x+ia)^2}dx \\ &\quad -ie^{-R^2}\int_0^a e^{y^2+2Ryi}dy \\ &= 0. \end{aligned}$$

Letting $R \to \infty$, it follows from the facts that $e^{-R^2} \to 0$ $(R \to \infty)$ and

$$\left|\int_0^a e^{y^2 \pm 2Ryi}dy\right| \le \int_0^a e^{y^2}dy$$

that

$$\lim_{R\to\infty}\int_{-R}^{R} e^{-(x+ia)^2}dx = \lim_{R\to\infty}\int_{-R}^{R} e^{-x^2}dx = \sqrt{\pi}.$$

5318

Let $n \ge 2$ be an integer. Compute

$$\int_0^\infty \frac{1}{1+x^n}dx.$$

(*Iowa*)

Solution.

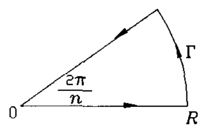

Fig.5.6

Let $f(z) = \frac{1}{1+z^n}$, and select the integral contour Γ as shown in Fig.5.6. $f(z)$ has one simple pole $z = e^{\frac{\pi}{n}i}$ inside Γ. By the residue theorem, we have

$$\int_\Gamma f(z)dz = 2\pi i \mathrm{Res}(f, e^{\frac{\pi}{n}i}).$$

$$\begin{aligned} \int_\Gamma f(z)dz &= \int_0^R \frac{dx}{1+x^n} + \int_0^{\frac{2\pi}{n}} iRe^{i\theta}f(Re^{i\theta})d\theta + \int_R^0 \frac{e^{\frac{2\pi}{n}i}dx}{1+x^n} \\ &= (1-e^{\frac{2\pi}{n}i})\int_0^R \frac{dx}{1+x^n} + \int_0^{\frac{2\pi}{n}} iRe^{i\theta}f(Re^{i\theta})d\theta. \end{aligned}$$

It is obvious that

$$\lim_{R\to\infty}\int_0^{\frac{2\pi}{n}} iRe^{i\theta}f(Re^{i\theta})d\theta = 0,$$

and

$$\mathrm{Res}(f, e^{\frac{\pi}{n}i}) = \left.\frac{1}{(1+z^n)'}\right|_{z=e^{\frac{\pi}{n}i}} = \frac{1}{ne^{\frac{n-1}{n}\pi i}}.$$

Letting $R\to\infty$, we obtain

$$\int_0^\infty \frac{dz}{1+x^n} = \frac{2\pi i}{ne^{\frac{n-1}{n}\pi i}\left(1-e^{\frac{2\pi}{n}i}\right)} = \frac{2\pi i}{n\left(e^{\frac{\pi}{n}i}-e^{-\frac{\pi}{n}i}\right)} = \frac{\pi}{n\sin\frac{\pi}{n}}.$$

5319

Evaluate

$$\int_0^\infty \cos(x^2)dx$$

with full justification.

(*Minnesota*)

Solution.

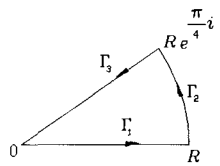

Fig.5.7

Define

$$f(z) = e^{-z^2},$$

and choose the contour of integration $\Gamma = \sum_{j=1}^{3}\Gamma_j$ as shown in Fig.5.7. Because $f(z) = e^{-z^2}$ is analytic on Γ and inside Γ, by Cauchy integral theorem, we have

$$\int_\Gamma f(z)dz = \sum_{j=1}^{3}\int_{\Gamma_j} f(z)dz = 0.$$

For the integral of $f(z)$ on Γ_2, we make a change of variable by $w = z^2$, then

$$\int_{\Gamma_2} f(z)dz = \int_{\gamma_2} e^{-w} \cdot \frac{dw}{2w^{\frac{1}{2}}},$$

where

$$\gamma_2 = \{w : |w| = R^2, 0 \leq \arg w \leq \frac{\pi}{2}\}.$$

By Jordan's lemma, we have

$$\lim_{R\to\infty} \int_{\Gamma_2} f(z)dz = 0.$$

For the integral of $f(z)$ on Γ_3, we have

$$\begin{aligned}\int_{\Gamma_3} f(z)dz &= -\int_0^R e^{-x^2 i} e^{\frac{\pi}{4} i} dx \\ &= -\int_0^R \frac{\sqrt{2}}{2}(\cos x^2 + \sin x^2)dx - i\int_0^R \frac{\sqrt{2}}{2}(\cos x^2 - \sin x^2)dx.\end{aligned}$$

It is well known that

$$\int_{\Gamma_1} f(z)dz = \int_0^R e^{-x^2} dx \to \frac{\sqrt{\pi}}{2}$$

when $R \to \infty$. Hence we obtain by letting $R \to \infty$ that

$$\int_0^\infty (\cos x^2 + \sin x^2)dx + i\int_0^\infty (\cos x^2 - \sin x^2)dx = \frac{\sqrt{2\pi}}{2},$$

which implies

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{\sqrt{2\pi}}{4}.$$

5320

Evaluate

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx.$$

(*Iowa*)

Solution.

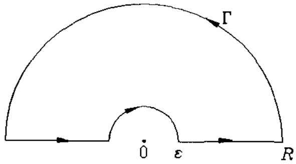

Fig.5.8

Define

$$f(z) = \frac{1 - e^{2iz}}{z^2},$$

and select the integral contour Γ as shown in Fig.5.8. Because $f(z)$ is analytic inside Γ, by Cauchy integral theorem,

$$\int_\Gamma f(z)dz = 0,$$

where

$$\begin{aligned}
\int_\Gamma f(z)dz &= \int_\varepsilon^R \frac{1 - e^{2ix}}{x^2}dx + \int_0^\pi iRe^{i\theta} f(Re^{i\theta})d\theta + \int_{-R}^{-\varepsilon} \frac{1 - e^{2ix}}{x^2}dx \\
&\quad + \int_\pi^0 i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta})d\theta \\
&= \int_\varepsilon^R \frac{2 - e^{2ix} - e^{-2ix}}{x^2}dx + \int_0^\pi iRe^{i\theta} f(Re^{i\theta})d\theta \\
&\quad + \int_\pi^0 i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta})d\theta \\
&= \int_\varepsilon^R \frac{4\sin^2 x}{x^2}dx + \int_0^\pi iRe^{i\theta} f(Re^{i\theta})d\theta + \int_\pi^0 i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta})d\theta.
\end{aligned}$$

It is easy to see that

$$\lim_{R\to\infty} \int_0^\pi iRe^{i\theta} f(Re^{i\theta})d\theta = 0$$

and

$$\lim_{\varepsilon\to 0} \int_\pi^0 i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta})d\theta = -\pi i\mathrm{Res}(f, 0).$$

Since the Laurent expansion of f about $z=0$ is

$$f(z)=\sum_{n=-1}^{\infty}a_n z^n,$$

where $a_{-1}=-2i$, we know that $\mathrm{Res}(f,0)=-2i$.

Letting $\varepsilon\to 0$ and $R\to\infty$, we obtain

$$\int_0^{\infty}\frac{\sin^2 x}{x^2}dx=\frac{\pi}{2}.$$

5321

Let $f(z)$ be holomorphic in the unit disk $|z|\le 1$. Prove that

$$\int_0^1 f(x)dx=\frac{1}{2\pi i}\int_{|z|=1}f(z)\log z dz,$$

where respective integration goes along the straight line from 0 to 1 and along the positively oriented unit circle starting from the point $z=1$. The branch of log is chosen to be real for positive z.

(*SUNY, Stony Brook*)

Solution.

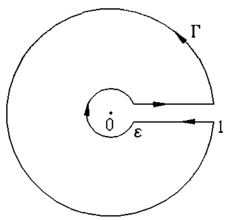

Fig.5.9

Let the contour of integration Γ be shown as in Fig.5.9, and the single-valued branch of $\log z$ be chosen by $\arg z|_{z=-1}=\pi$. Since $f(z)\log z$ is holomorphic inside the contour Γ, by Cauchy integral theorem,

$$\int_{\Gamma}f(z)\log z dz=0,$$

where

$$\begin{aligned}\int_\Gamma f(z)\log z dz &= \int_\varepsilon^1 f(x)\log x dx + \int_{|z|=1} f(z)\log z dz \\ &\quad + \int_1^\varepsilon f(x)(\log x + 2\pi i)dx + \int_{2\pi}^0 f(\varepsilon e^{i\theta})\log(\varepsilon e^{i\theta}) i\varepsilon e^{i\theta} d\theta \\ &= -2\pi i \int_\varepsilon^1 f(x)dx + \int_{|z|=1} f(z)\log z dz \\ &\quad - \int_0^{2\pi} f(\varepsilon e^{i\theta})\log(\varepsilon e^{i\theta}) i\varepsilon e^{i\theta} d\theta.\end{aligned}$$

It is easy to see that

$$\lim_{\varepsilon\to 0}\int_0^{2\pi} f(\varepsilon e^{i\theta})\log(\varepsilon e^{i\theta}) i\varepsilon e^{i\theta} d\theta = 0.$$

Letting $\varepsilon \to 0$, we obtain

$$\int_0^1 f(x)dx = \frac{1}{2\pi i}\int_{|z|=1} f(z)\log z dz,$$

where the integration contour $|z| = 1$ has starting point and end point $z = 1$, and the value of $\log z$ at the starting point $z = 1$ is defined as 0.

5322

Find the value of

$$\int_0^{2\pi} \log\left|a + be^{i\phi}\right| d\phi$$

where a and b are complex constants, not both equal to zero.

(*Harvard*)

Solution.

First we assume $|a| > |b|$, and then the multi-valued analytic function $\log(a + bz)$ has single-valued branch on $\{z : |z| \le 1\}$. Take $e^{i\phi} = z$, then $d\phi = \frac{dz}{iz}$, and

$$\begin{aligned}\int_0^{2\pi} \log|a + be^{i\phi}|d\phi &= \mathrm{Re}\left\{\int_0^{2\pi} \log(a + be^{i\phi})d\phi\right\} \\ &= \mathrm{Re}\left\{\int_{|z|=1} \frac{\log(a+bz)}{iz}dz\right\} \\ &= \mathrm{Re}\{2\pi\log a\} = 2\pi\log|a|.\end{aligned}$$

When $|a| < |b|$, we have

$$\begin{aligned}\int_0^{2\pi} \log|a+be^{i\phi}|d\phi &= \int_0^{2\pi} \log|\bar{b}+\bar{a}e^{i\phi}|d\phi \\ &= 2\pi\log|\bar{b}| = 2\pi\log|b|.\end{aligned}$$

In the case $|a| = |b|$, let $b = ae^{i\alpha}$. Then

$$\begin{aligned}\int_0^{2\pi} \log|a+be^{i\phi}|d\phi &= \int_0^{2\pi} (\log|a| + \log|1+e^{i(\phi+\alpha)}|)d\phi \\ &= 2\pi\log|a| + \int_{-\pi}^{\pi} \log|1+e^{i\phi}|d\phi.\end{aligned}$$

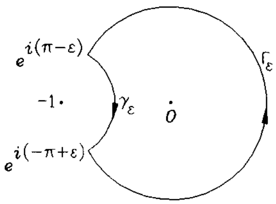

Fig.5.10

In order to evaluate the integral

$$\int_{-\pi}^{\pi} \log|1+e^{i\phi}|d\phi,$$

we define

$$f(z) = \frac{\log(1+z)}{z},$$

where the single-valued branch is defined by $\log(1+z)\ |_{z=0} = 0$. Choose a contour of integration $\Gamma = \Gamma_\varepsilon \cup \gamma_\varepsilon$ as shown in Fig.5.10. Since $f(z)$ is analytic on Γ and inside Γ, by Cauchy integral theorem, $\int_\Gamma f(z)dz = 0$. Because

$$\left|\int_{\gamma_\varepsilon} f(z)dz\right| \le \frac{\log\frac{1}{\varepsilon} + \frac{\pi}{2}}{1-\varepsilon} \cdot \pi\varepsilon \to 0 \quad (\varepsilon \to 0),$$

we have

$$\int_{-\pi}^{\pi} \log|1+e^{i\phi}|d\phi = \mathrm{Re}\int_{-\pi}^{\pi} \log(1+e^{i\phi})d\phi$$

$$= \lim_{\varepsilon \to 0} \text{Re} \left\{ \int_{\Gamma_\varepsilon} \log(1+z) \frac{dz}{iz} \right\}$$
$$= \lim_{\varepsilon \to 0} \text{Re} \left\{ \frac{1}{i} \int_{\Gamma} f(z) dz \right\} = 0.$$

Hence we obtain

$$\int_0^{2\pi} \log |a + be^{i\phi}| d\phi = 2\pi \max\{\log |a|, \log |b|\}.$$

5323

Evaluate

$$\int_0^\infty \frac{\log x}{(1+x)^3} dx.$$

(*Iowa*)

Solution.

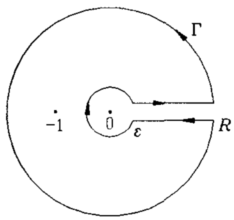

Fig.5.11

Let

$$f(z) = \frac{\log^2 z}{(1+z)^3},$$

and select the integral path Γ as shown in Fig.5.11. The single-valued branch of $\log z$ is chosen by $\arg z|_{z=-1} = \pi$. By the residue theorem, we have

$$\int_\Gamma f(z) dz = 2\pi i \text{Res}(f, -1),$$

where

$$\begin{aligned}
\int_\Gamma f(z)dz &= \int_\varepsilon^R \frac{\log^2 x}{(1+x)^3}dx + \int_0^{2\pi} iRe^{i\theta} f(Re^{i\theta})d\theta + \int_R^\varepsilon \frac{(\log x + 2\pi i)^2}{(1+x)^3}dx \\
&\quad + \int_{2\pi}^0 i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta})d\theta \\
&= \int_\varepsilon^R \frac{-4\pi i \log x + 4\pi^2}{(1+x)^3}dx + \int_0^{2\pi} iRe^{i\theta} f(Re^{i\theta})d\theta \\
&\quad + \int_{2\pi}^0 i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta})d\theta.
\end{aligned}$$

It is obvious that

$$\lim_{R\to\infty} \int_0^{2\pi} iRe^{i\theta} f(Re^{i\theta})d\theta = 0$$

and

$$\lim_{\varepsilon\to 0} \int_{2\pi}^0 i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta})d\theta = 0.$$

In order to find $\text{Res}(f, -1)$, we consider the Laurent expansion of f about $z = -1$:

$$\begin{aligned}
f(z) &= \frac{\log^2[(z+1)-1]}{(z+1)^3} = \frac{(\pi i + \log[1-(z+1)])^2}{(z+1)^3} \\
&= \frac{(\pi i - (z+1) - \frac{1}{2}(z+1)^2 - \cdots)^2}{(z+1)^3} \\
&= \sum_{n=-3}^{\infty} a_n (z+1)^n,
\end{aligned}$$

where $a_{-1} = 1 - \pi i$. Hence

$$2\pi i \text{Res}(f, -1) = 2\pi i + 2\pi^2.$$

As $\varepsilon \to 0$ and $R \to \infty$, it turns out that

$$\int_0^\infty \frac{-4\pi i \log x + 4\pi^2}{(1+x)^3}dx = 2\pi i + 2\pi^2.$$

Comparing the imaginary parts on the two sides of the above identity, we obtain

$$\int_0^\infty \frac{\log x}{(1+x)^3}dx = -\frac{1}{2}.$$

5324

Evaluate the following integrals:

(a) $\int_{-i\infty}^{+i\infty} \frac{dz}{(z^2-4)\log(z+1)}$ (the integration is over the imaginary axis),

(b) $\int_0^\infty \frac{x^\alpha}{x^3+1}dx$ for α in the range $-1 < \alpha < 2$.

(*Courant Inst.*)

Solution.

(a)

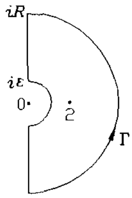

Fig.5.12

Define

$$f(z) = \frac{1}{(z^2-4)\log(z+1)}.$$

The single-valued branch for $\log(z+1)$ is chosen by $\log(z+1)|_{z=0} = 0$, and the contour Γ of integration is shown in Fig.5.12. As $f(z)$ is analytic on and inside Γ except a simple pole at $z = 2$, we have

$$\int_\Gamma f(z)dz = 2\pi i \text{Res}(f, 2),$$

where

$$\begin{aligned}\int_\Gamma f(z)dz &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(Re^{i\theta})iRe^{i\theta}d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\varepsilon e^{i\theta})i\varepsilon e^{i\theta}d\theta \\ &\quad - \int_{-iR}^{-i\varepsilon} f(z)dz - \int_{i\varepsilon}^{+iR} f(z)dz,\end{aligned}$$

and

$$\text{Res}(f, 2) = \lim_{z\to 2} \frac{(z-2)}{(z^2-4)\log(z+1)} = \frac{1}{4\log 3}.$$

Because

$$\lim_{R\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(Re^{i\theta})iRe^{i\theta}d\theta = 0$$

and

$$\lim_{\varepsilon\to 0}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\varepsilon e^{i\theta})i\varepsilon e^{i\theta}d\theta = \pi i\text{Res}(f,0) = -\frac{\pi i}{4},$$

by letting $\varepsilon \to 0$ and $R \to \infty$, we obtain

$$\int_{-i\infty}^{+i\infty} \frac{dz}{(z^2-4)\log(z+1)} = \frac{\pi i}{4}(1-\frac{2}{\log 3}).$$

(b)

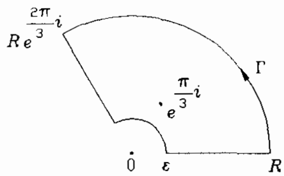

Fig.5.13

Define

$$f(z) = \frac{z^\alpha}{z^3+1}.$$

The single-valued branch for z^α is chosen by $\arg z|_{z=x>0} = 0$, and the contour Γ of integration is shown in Fig.5.13. As $f(z)$ is analytic on and inside Γ except a simple pole at $z = e^{\frac{\pi}{3}i}$, we have

$$\int_\Gamma f(z)dz = 2\pi i\text{Res}(f, e^{\frac{\pi}{3}i}),$$

where

$$\begin{aligned}\int_\Gamma f(z)dz \quad &= \quad \int_\varepsilon^R f(x)dx + \int_0^{\frac{2\pi}{3}} f(Re^{i\theta})iRe^{i\theta}d\theta - \int_\varepsilon^R e^{\frac{2\pi\alpha}{3}i}f(x)\cdot e^{\frac{2\pi}{3}i}dx \\ &\quad -\int_0^{\frac{2\pi}{3}} f(\varepsilon e^{i\theta})i\varepsilon e^{i\theta}d\theta,\end{aligned}$$

and

$$\begin{aligned}\text{Res}(f,e^{\frac{\pi}{3}i}) \quad &= \quad \lim_{z\to e^{\frac{\pi}{3}i}}(z-e^{\frac{\pi}{3}i})f(z) \\ &= \frac{e^{\frac{\pi\alpha}{3}i}}{3e^{\frac{2\pi}{3}i}} \\ &= \frac{1}{3e^{\frac{\pi}{3}(2-\alpha)i}}.\end{aligned}$$

Because

$$\lim_{R\to\infty}\int_0^{\frac{2\pi}{3}} f(Re^{i\theta})iRe^{i\theta}d\theta = 0$$

when $\alpha < 2$ and

$$\lim_{\varepsilon\to 0}\int_0^{\frac{2\pi}{3}} f(\varepsilon e^{i\theta})i\varepsilon e^{i\theta}d\theta = 0$$

when $\alpha > -1$, by letting $\varepsilon \to 0$ and $R \to \infty$, we obtain

$$\int_0^\infty \frac{x^\alpha}{x^3+1}dx = \frac{\pi}{3\sin(\frac{2-\alpha}{3}\pi)}.$$

5325

Show that

$$\int_0^\infty \frac{x^\alpha}{(1+x^2)^2}dx = \frac{\pi(1-\alpha)}{4\cos(\frac{\pi\alpha}{2})},$$

for $-1 < \alpha < 3$, $\alpha \neq 1$. What happens if $\alpha = 1$?

(*Harvard*)

Solution.

Let

$$f(z) = \frac{z^\alpha}{(1+z^2)^2},$$

where $(\arg z^\alpha)_{z=x>0} = 0$, and select the integral path Γ as shown in Fig.5.14. By the residue theorem, we have

$$\int_\Gamma f(z)dz = 2\pi i \mathrm{Res}(f(z), i),$$

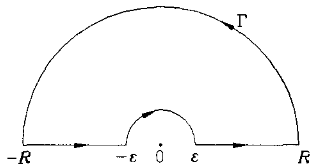

Fig.5.14

where

$$\int_\Gamma f(z)dz \quad = \quad \int_\varepsilon^R \frac{x^\alpha}{(1+x^2)^2}dx + \int_0^\pi iRe^{i\theta}f(Re^{i\theta})d\theta + \int_{-R}^{-\varepsilon} \frac{(-x)^\alpha e^{i\pi\alpha}}{(1+x^2)^2}dx$$

$$+\int_{\pi}^{0} i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta}) d\theta$$

$$= (1+e^{i\pi\alpha})\int_{\varepsilon}^{R} \frac{x^{\alpha}}{(1+x^2)^2}dx + \int_0^{\pi} iRe^{i\theta} f(Re^{i\theta}) d\theta + \int_{\pi}^{0} i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta}) d\theta,$$

and

$$\operatorname{Res}(f(z), i) = \lim_{z\to i}\left[\frac{z^{\alpha}}{(z+i)^2}\right]' = \frac{1-\alpha}{4i} e^{i\frac{\pi\alpha}{2}}.$$

It follows from $\alpha < 3$ that

$$\lim_{R\to\infty}\int_0^{\pi} iRe^{i\theta} f(Re^{i\theta}) d\theta = 0,$$

and from $\alpha > -1$ that

$$\lim_{\varepsilon\to 0}\int_{\pi}^{0} i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta}) d\theta = 0.$$

Letting $\varepsilon \to 0$ and $R \to \infty$, we obtain

$$(1+e^{i\pi\alpha})\int_0^{\infty} \frac{x^{\alpha}}{(1+x^2)^2}dx = \frac{\pi(1-\alpha)}{2} e^{i\frac{\pi\alpha}{2}}.$$

When $\alpha \neq 1$,

$$\int_0^{\infty} \frac{x^{\alpha}}{(1+x^2)^2}dx = \frac{\pi(1-\alpha)}{4\cdot\left(\frac{e^{i\frac{\pi\alpha}{2}}+e^{-i\frac{\pi\alpha}{2}}}{2}\right)} = \frac{\pi(1-\alpha)}{4\cos(\frac{\pi\alpha}{2})}.$$

when $\alpha = 1$.

$$\int_0^{\infty} \frac{x}{(1+x^2)^2}dx = \lim_{\alpha\to 1}\frac{\pi(1-\alpha)}{4\cos(\frac{\pi\alpha}{2})} = \frac{1}{2}.$$

5326

(a) Prove that

$$\int_0^{\infty} e^{ix} x^{-\alpha} dx$$

converges if $0 < \alpha < 1$.

(b) Use complex integration to show that

$$\int_0^\infty x^{-\alpha} \cos x dx = \sin \frac{\pi\alpha}{2} \cdot \Gamma(-\alpha + 1).$$

(*Harvard*)

Solution.

(a)

$$\int_0^\infty e^{ix} x^{-\alpha} dx = \left(\int_0^1 x^{-\alpha} \cos x dx + \int_1^\infty x^{-\alpha} \cos x dx \right) + i \int_0^\infty x^{-\alpha} \sin x dx.$$

It follows from $\alpha < 1$ that

$$\int_0^1 x^{-\alpha} \cos x dx$$

is convergent. It is also obvious that

$$\left| \int_1^A \cos x dx \right| \leq 2, \quad \left| \int_0^A \sin x dx \right| \leq 2,$$

$x^{-\alpha}$ is monotonic decreasing and

$$\lim_{x \to +\infty} x^{-\alpha} = 0 \quad \text{for } \alpha > 0.$$

By Dirichlet's criterion, we know that $\int_1^\infty x^{-\alpha} \cos x dx$ and $\int_0^\infty x^{-\alpha} \sin x dx$ are also convergent. Hence $\int_0^\infty e^{ix} x^{-\alpha} dx$ is convergent when $0 < \alpha < 1$.

(b)

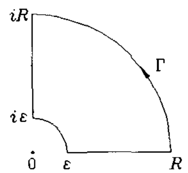

Fig.5.15

Let $f(z) = z^{-\alpha} e^{-z}$, and the contour of integration Γ is chosen as shown in Fig.5.15. The single-valued branch of $f(z)$ on Γ is definde by $z^{-\alpha}|_{z=x>0} > 0$.

By Cauchy integral theorem,

$$\begin{aligned}\int_\Gamma f(z)dz &= \int_\varepsilon^R x^{-\alpha}e^{-x}dx + \int_0^{\frac{\pi}{2}} iRe^{i\theta}f(Re^{i\theta})d\theta + \int_R^\varepsilon x^{-\alpha}e^{-\frac{\pi\alpha}{2}i}e^{-ix}idx \\ &\quad + \int_{\frac{\pi}{2}}^0 i\varepsilon e^{i\theta}f(\varepsilon e^{i\theta})d\theta = 0.\end{aligned}$$

It follows from $\alpha < 1$ that

$$\lim_{\varepsilon\to 0}\int_{\frac{\pi}{2}}^0 i\varepsilon e^{i\theta}f(\varepsilon e^{i\theta})d\theta = 0,$$

and from $\alpha > 0$ and Jordan's lemma that

$$\lim_{R\to\infty}\int_0^{\frac{\pi}{2}} iRe^{i\theta}f(Re^{i\theta})d\theta = 0.$$

Letting $\varepsilon \to 0$ and $R \to \infty$, we have

$$\Gamma(-\alpha+1) = \int_0^\infty x^{-\alpha}e^{-x}dx = ie^{-\frac{\pi\alpha}{2}i}\int_0^\infty x^{-\alpha}e^{-ix}dx.$$

Multiplying both sides by $e^{\frac{\pi\alpha}{2}i}$, and comparing the imaginary parts, we obtain

$$\int_0^\infty x^{-\alpha}\cos x dx = \sin\frac{\pi\alpha}{2}\Gamma(-\alpha+1).$$

5327

Use a change of contour to show that

$$\int_0^\infty \frac{\cos(\alpha x)}{x+\beta}dx = \int_0^\infty \frac{te^{-\alpha\beta t}}{t^2+1}dt,$$

provided that α and β are positive. Define the left side as a limit of proper integral and show that the limit exists.

(*Courant Inst.*)

Solution.

Since

$$\left|\int_0^A \cos(\alpha x)dx\right| \le \frac{2}{\alpha},$$

$\frac{1}{x+\beta}$ is monotonic with respect to x and

$$\lim_{x\to+\infty}\frac{1}{x+\beta}=0,$$

the convergence of the integral

$$\int_0^\infty \frac{\cos(\alpha x)}{x+\beta}dx$$

follows from Dirichlet's criterion.

Define

$$f(z)=\frac{e^{-\alpha z}}{z+\beta i},$$

and choose the contour of integration $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3$ as shown in Fig.5.16.

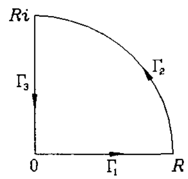

Fig.5.16

By Cauchy integral theorem, we have

$$\begin{aligned}\int_\Gamma f(z)dz &= \int_0^R \frac{e^{-\alpha x}}{x+\beta i}dx+\int_{\Gamma_2} f(z)dz-\int_0^R\frac{e^{-\alpha x i}}{x+\beta}dx\\ &= \int_0^R\frac{e^{-\alpha x}(x-\beta i)}{x^2+\beta^2}dx+\int_{\Gamma_2}f(z)dz-\int_0^R\frac{e^{-\alpha x i}}{x+\beta}dx=0.\end{aligned}$$

It follows from Jordan's lemma that

$$\lim_{R\to\infty}\int_{\Gamma_2} f(z)dz=0.$$

Letting $R\to\infty$ and considering the real part in the above identity, we obtain

$$\int_0^\infty\frac{\cos(\alpha x)}{x+\beta}dx=\int_0^\infty\frac{xe^{-\alpha x}}{x^2+\beta^2}dx=\int_0^\infty\frac{te^{-\alpha\beta t}}{t^2+1}dt.$$

5328

(a) Let c be the unit circle in the complex plane, and let f be a continuous $\mathbb{C}$-valued function on c. Show that

$$F_f(z) = \int_c \frac{f(\zeta)}{\zeta - z} d\zeta$$

is a holomorphic function of z in the interior of the unit disk.

(b) Find a continuous f on c which is not identically zero, but so that the associated function F is identically zero.

(*Minnesota*)

Solution.

(a) Let z_0 be an arbitrary point in the unit disk. Then $1 - |z_0| = \rho > 0$. Choosing $\delta > 0$ such that $\delta < \rho$, we prove that

$$F_f(z) = \int_c \frac{f(\zeta)}{\zeta - z} d\zeta$$

has a power series expansion in $\{|z - z_0| \leq \delta\}$.

It is clear that

$$\left|\frac{z - z_0}{\zeta - z}\right| \leq \frac{\delta}{\rho} < 1$$

when $|z - z_0| \leq \delta$ and $\zeta \in c$. We can also assume $|f(\zeta)| \leq M$ because f is continuous on c. Thus

$$\begin{aligned}
\frac{f(\zeta)}{\zeta - z} &= \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)} = \frac{f(\zeta)}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\
&= \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{f(\zeta)}{\zeta - z_0} \cdot \left(\frac{z - z_0}{\zeta - z_0}\right)^n .
\end{aligned}$$

As

$$\left|\frac{f(\zeta)}{\zeta - z_0}\left(\frac{z - z_0}{\zeta - z_0}\right)^n\right| \leq \frac{M}{\rho} \cdot \left(\frac{\delta}{\rho}\right)^n,$$

and $\sum_{n=0}^{\infty} \frac{M}{\rho}\left(\frac{\delta}{\rho}\right)^n$ is convergent, the series $\sum_{n=0}^{\infty} \frac{f(\zeta)}{\zeta - z_0}\left(\frac{z - z_0}{\zeta - z_0}\right)^n$ converges uniformly for all $\zeta \in c$. Hence termwise integration is permissible, and we obtain

$$F_f(z) = \int_c \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where $|z-z_0| \leq \delta$ and

$$a_n = \int_c \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

Since z_0 is arbitrarily chosen in the unit disk, $F_f(z)$ is holomorphic in $\{|z|<1\}$.

(b) Take $f(\zeta) = \frac{1}{\zeta}$ $(|\zeta|=1)$. Then

$$F_f(z) = \int_c \frac{1}{\zeta(\zeta - z)} d\zeta = \int_c \frac{1}{z}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta}\right) d\zeta = \frac{1}{z}(2\pi i - 2\pi i) = 0.$$

In fact, $f(\zeta)$ can be taken as $\frac{1}{(\zeta - z_0)^n}$ for any positive integer n and fixed $z_0 \in \{z : |z| < 1\}$. When $\zeta \in c$,

$$\overline{\int_c \frac{f(\zeta)}{\zeta - z} d\zeta} = \int_c \frac{1}{(\frac{1}{\zeta} - \overline{z})(\frac{1}{\zeta} - \overline{z}_0)^n} d(\frac{1}{\zeta}) = \int_c \frac{-\zeta^{n-1}}{(1-\overline{z}\zeta)(1-\overline{z}_0\zeta)^n} d\zeta = 0.$$

5329

Let $[a,b]$ be a finite interval in $\mathbb{R}$ and define, for z in $D = \mathbb{C} - [a,b]$,

$$f(z) = \int_a^b \frac{dt}{t-z}.$$

Show that $f(z)$ is analytic in D. Given c, $a<c<b$, calculate the limit of $f(z)$ as z tends to c from the upper half plane and as z tends to c from the lower half plane.

(UC, Irvine)

Solution.

For any $z_0 \in D$, choose $\delta > 0$ sufficiently small such that $\{z : |z-z_0| \leq \delta\} \cap \{z = x+iy : y=0, a \leq x \leq b\} = \emptyset$. When $|z-z_0|<\delta$, $a \leq t \leq b$, we have

$$\frac{1}{t-z} = \frac{1}{(t-z_0)-(z-z_0)} = \frac{1}{t-z_0} \cdot \frac{1}{1-\frac{z-z_0}{t-z_0}} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}},$$

and the series converges uniformly for t with $a \leq t \leq b$. Hence

$$\begin{aligned} f(z) &= \int_a^b \frac{dt}{t-z} = \int_a^b \left(\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}}\right) dt \\ &= \sum_{n=0}^{\infty} \left(\int_a^b \frac{dt}{(t-z_0)^{n+1}}\right)(z-z_0)^n \end{aligned}$$

holds for $z \in \{z : |z - z_0| < \delta\}$, which implies $f(z)$ is analytic in $\{z : |z - z_0| < \delta\}$. Since z_0 is an arbitrary point in D, we obtain that $f(z)$ is analytic in D.

For $z \in D$, $f(z)$ can also be represented explicitly by

$$f(z) = \int_a^b \frac{dt}{t-z} = \int_a^b d\log(t-z) = \log\frac{z-b}{z-a},$$

where the single-valued branch is defined by $\arg\left(\frac{z-b}{z-a}\right)|_{z=x_0>b} = 0$. Let Γ_1 and Γ_2 be two continuous curves connecting $z = x_0 > b$ and $z = c$ in the upper half plane and the lower half plane respectively. Then the limit of $f(z)$ as z tends to c from the upper half plane is

$$\log\left|\frac{c-b}{c-a}\right| + i\Delta_{\Gamma_1}\arg\frac{z-b}{z-a} = \log\left|\frac{c-b}{c-a}\right| + \pi i,$$

while the limit of $f(z)$ as z tends to c from the lower half plane is

$$\log\left|\frac{c-b}{c-a}\right| + i\Delta_{\Gamma_2}\arg\frac{z-b}{z-a} = \log\left|\frac{c-b}{c-a}\right| - \pi i.$$

5330

For each $z \in U = \{z : \mathrm{Im} z > 0\}$ define

$$g(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\sin^2 t}{t-z}dt.$$

Determine which points $a \in \mathbb{R}$ have the following property: there exist $\varepsilon > 0$ and an analytic function f on $D(a,\varepsilon)$ such that $f(z) = g(z)$ for all $z \in U \cap D(a,\varepsilon)$.

(*Indiana*)

Solution.

Let Γ be the half unit circle in the lower half plane whose direction is defined from point $z = -1$ to point $z = 1$, and define a function

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{\sin^2 t}{t-z}dt.$$

It follows from the Cauchy integral theorem that when $z \in U$, $f(z) \equiv g(z)$. With a similar reason as in problem 5328, $f(z)$ is analytic in the complement of Γ. Hence we obtain that for any $a \in \mathbb{R}$, $a \neq \pm 1$, there exists $\varepsilon > 0$

($\varepsilon < \min\{|a-1|, |a+1|\}$) such that $f(z)$ is analytic in $D(a,\varepsilon) = \{z : |z-a| < \varepsilon\}$ and $f(z) = g(z)$ for all $z \in U \cap D(a,\varepsilon)$.

When $a = \pm 1$, such a $f(z)$ does not exist. The reason is as follows: As

$$\frac{\sin^2 t}{t-z} = \frac{\sin^2 t - \sin^2 z}{t-z} + \frac{\sin^2 z}{t-z},$$

where $\frac{\sin^2 t - \sin^2 z}{t-z}$ is an analytic function of two variables for $(t,z) \in \mathbb{C} \times \mathbb{C}$, we know that

$$h(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\sin^2 t - \sin^2 z}{t-z} dt$$

is analytic for $z \in \mathbb{C}$. But

$$\frac{1}{2\pi i}\int_{-1}^{1} \frac{\sin^2 z}{t-z} dt = \frac{\sin^2 z}{2\pi i}\int_{-1}^{1} d\log(t-z) = \frac{\sin^2 z}{2\pi i}\log\frac{z-1}{z+1},$$

which has branch points $z = \pm 1$, hence $g(z)$ can not be analytically continued to $D(\pm 1, \varepsilon)$.

SECTION 4
THE MAXIMUM MODULUS AND ARGUMENT PRINCIPLES

5401

Let $a \in \mathbb{C}$, $|a| \le 1$, and consider the polynomial

$$P(z) = \frac{a}{2} + (1 - |a|^2)z - \frac{\overline{a}}{2}z^2.$$

Show that $|P(z)| \le 1$ whenever $|z| \le 1$.

(*Indiana*)

Solution.

$$\begin{aligned} P(z) &= \frac{a}{2} + (1 - |a|^2)z - \frac{\overline{a}}{2}z^2 \\ &= z[(1 - |a|^2) + \frac{1}{2}(\frac{a}{z} - \overline{a}z)]. \end{aligned}$$

When $|z| = 1$,

$$\begin{aligned} &\mathrm{Re}(\frac{a}{z} - \overline{a}z) = \mathrm{Re}[\frac{a}{z} - \overline{(\overline{a}z)}] = \mathrm{Re}[\frac{a}{z} - \frac{a}{z}] = 0, \\ &\left|\mathrm{Im}(\frac{a}{z} - \overline{a}z)\right| \le 2|a|. \end{aligned}$$

Hence when $|z| = 1$,

$$\begin{aligned} |P(z)|^2 &= (1 - |a|^2)^2 + (\mathrm{Im}[\frac{1}{2}(\frac{a}{z} - \overline{a}z)])^2 \\ &\le (1 - 2|a|^2 + |a|^4) + |a|^2 = 1 - |a|^2 + |a|^4 \le 1. \end{aligned}$$

By the maximum modulus principle, $|P(z)| \le 1$ whenever $|z| \le 1$.

5402

Let f be holomorphic in the unit disk $\{|z| < 1\}$, continuous in $\{|z| \le 1\}$ and $|f(z)| = 1$ whenever $|z| = 1$. Prove that f is a rational function.

(*SUNY, Stony Brook*)

Solution.

If $f(z)$ has infinite many zeros, by the isolatedness of the zeros of holomorphic functions, the zeros must have limit points on the boundary of the unit disk. But it will violate the fact that f is continuous in $\{|z| \le 1\}$ and $|f(z)| = 1$ whenever $|z| = 1$. Hence f has only finite zeros in the unit disk. Denote all these zeros by $z_1, z_2, \cdots, z_n$, multiple zeros being repeated, and define

$$F(z) = f(z) / \prod_{k=1}^{n} \left(\frac{z - z_k}{1 - \overline{z}_k z} \right).$$

Then $F(z)$ is holomorphic in $\{|z| < 1\}$. continuous in $\{|z| \le 1\}$ and $|F(z)| = 1$ when $|z| = 1$. By the maximum modulus principle, $|F(z)| \le 1$ in $\{|z| \le 1\}$. Since $F(z)$ has no zero in $\{|z| \le 1\}$, $\frac{1}{F(z)}$ is also holomorphic in $\{|z| < 1\}$, continuous in $\{|z| \le 1\}$ and $\left|\frac{1}{F(z)}\right| = 1$ when $|z| = 1$. Application of the maximum modulus principle to $\frac{1}{F(z)}$ yields $|F(z)| \ge 1$ in $\{|z| \le 1\}$. Hence $|F(z)| = 1$ holds in $\{|z| \le 1\}$, which implies $F(z) = e^{i\alpha}$ with α a real number. So we obtain

$$f(z) = e^{i\alpha} \prod_{k=1}^{n} \left(\frac{z - z_k}{1 - \overline{z}_k z} \right).$$

5403

Let f be a continuous function on $\overline{U} = \{z : |z| \le 1\}$ such that f is analytic in U. If $f = 1$ on the half-circle $\gamma = \{e^{i\theta} : 0 \le \theta \le \pi\}$, prove that $f = 1$ everywhere in $\overline{U}$.

(*Indiana*)

Solution.

Define $F(z) = (f(z) - 1)(f(-z) - 1)$, then $F(z)$ is also continuous on $\overline{U}$ and analytic in U. When $z \in \partial U$, we have either $f(z) - 1 = 0$ or $f(-z) - 1 = 0$. Hence $F(z) = 0$ holds for all $z \in \overline{U}$, which implies either $f(z) - 1 \equiv 0$ or $f(-z) - 1 \equiv 0$. Since $f(z) - 1 \equiv 0$ is equivalent to $f(-z) - 1 \equiv 0$, we obtain $f(z) \equiv 1$ for all $z \in \overline{U}$.

Remark. The condition that "$f = 1$ on the half-circle γ" can be weakened to that "$f = 1$ on an arc $\gamma = \{e^{i\theta} : 0 \le \theta \le \frac{\pi}{n}\}$, where n is a natural number". In this case, the proof is the same except that $F(z)$ is defined by

$$F(z) = (f(z) - 1)(f(ze^{\frac{\pi}{n}i}) - 1)(f(ze^{\frac{2\pi}{n}i}) - 1) \cdots (f(ze^{\frac{2n-1}{n}\pi i}) - 1).$$

5404

Let S denote the sector in the complex plane given by $S = \{z : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$. Let $\overline{S}$ denote the closure of S. Let f be a continuous complex function on $\overline{S}$ which is holomorphic in S. Suppose further

(1) $|f(z)| \leq 1$ for all z in the boundary of S;

(2) $|f(x+iy)| \leq e^{\sqrt{x}}$ for all $x + iy \in S$.

Prove that $|f(z)| \leq 1$ for all $z \in S$.

(SUNY, Stony Brook)

Solution.

Let $F(z) = e^{-\varepsilon z} f(z)$, where $\varepsilon > 0$ is an arbitrary fixed number. Then $F(z)$ is also continuous on $\overline{S}$ and analytic in S. When z is on the boundary of S, $|F(z)| = e^{-\varepsilon x}|f(x)| \leq 1$. When $|z| \to +\infty$ $(-\frac{\pi}{4} < \arg z < \frac{\pi}{4})$, $|F(z)| \leq e^{-\varepsilon x} \cdot e^{\sqrt{x}} \to 0$. By the maximum modulus principle, we have $|F(z)| \leq 1$ for all $z \in S$, which implies $|f(z)| \leq |e^{\varepsilon z}| = e^{\varepsilon x}$ for all $z \in S$. Because $\varepsilon > 0$ can be arbitrarily chosen, letting $\varepsilon \to 0$, we obtain $|f(z)| \leq 1$ for all $z \in S$.

5405

Let K be a compact, connected subset of $\mathbb{C}$ containing more than one point and let f be a one-to-one conformal map of $\overline{\mathbb{C}} \backslash K$ onto $\Delta = \{z : |z| < 1\}$ with $f(\infty) = 0$. If p is a polynomial of degree n for which $|p(z)| \leq 1$ for $z \in K$, prove that

$$|p(z)| \leq |f(z)|^{-n} \quad \text{for } z \in \mathbb{C} \backslash K.$$

(Indiana)

Solution.

Because f is a one-to-one conformal map of $\overline{\mathbb{C}} \backslash K$ onto Δ with $f(\infty) = 0$, it has a simple zero at $z = \infty$. Since p is a polynomial of degree n, it has a pole of order n at $z = \infty$. Hence the function $F(z) = p(z) f^n(z)$ is analytic in $\overline{\mathbb{C}} \backslash K$ which contains point $z = \infty$. As $f(z)$ maps $\overline{\mathbb{C}} \backslash K$ onto $\Delta = \{z : |z| < 1\}$, we have $\lim\limits_{z \to K} |f(z)| = 1$. Together with $|p(z)| \leq 1$ for $z \in K$, we know that the limit of $|F(z)|$ when z tends to K can not be larger than 1. Apply the maximum modulus principle to $F(z)$ on $\overline{\mathbb{C}} \backslash K$, we obtain $|F(z)| \leq 1$ for $z \in \overline{\mathbb{C}} \backslash K$, which implies $|p(z)| \leq |f(z)|^{-n}$ for all $z \in \mathbb{C} \backslash K$.

5406

Suppose f and g (non-constant functions) are analytic in a region G and continuous on the closure $\overline{G}$ of the region. Assume that $\overline{G}$ is compact. Prove that $|f| + |g|$ achieves its maximum value on the boundary of G.

(*Iowa*)

Solution.

Assume that $|f| + |g|$ achieves its maximum value c ($c > 0$) at $z_0 \in \overline{G}$, we prove that if $z_0 \in G$, then f and g must be constants.

Let

$$|f(z_0)| = f(z_0)e^{i\phi_1}, \quad |g(z_0)| = g(z_0)e^{i\phi_2}.$$

Then for fixed ϕ_1 and ϕ_2,

$$F(z) = f(z)e^{i\phi_1} + g(z)e^{i\phi_2}$$

is analytic in G and continuous on $\overline{G}$. It follows from

$$\begin{aligned} |F(z)| &\le |f(z)| + |g(z)| \le c, \\ F(z_0) &= f(z_0)e^{i\phi_1} + g(z_0)e^{i\phi_2} = |f(z_0)| + |g(z_0)| = c \end{aligned}$$

and $z_0 \in G$ that

$$F(z) = f(z)e^{i\phi_1} + g(z)e^{i\phi_2}$$

must be the constant c.

Without loss of generality, we assume that f is not a constant, and try to lead to a contradiction. Since the image of an open set $\{z : |z - z_0| < \delta\} \subset G$ under f is an open set which contains point $f(z_0)$, $f(z)$ assumes all the values $f(z) = f(z_0) + \varepsilon e^{i\phi}$ for small $\varepsilon > 0$ and $0 \le \phi < 2\pi$ in $\{z : |z - z_0| < \delta\}$. Then when $\phi + \phi_1 \ne 0, \pi$, we have

$$\begin{aligned} |f(z)| + |g(z)| &= |f(z)| + |c - f(z)e^{i\phi_1}| \\ &= |f(z_0) + \varepsilon e^{i\phi}| + |c - f(z_0)e^{i\phi_1} - \varepsilon e^{i(\phi+\phi_1)}| \\ &= |\varepsilon e^{i(\phi+\phi_1)} + f(z_0)e^{i\phi_1}| + |\varepsilon e^{i(\phi+\phi_1)} - g(z_0)e^{i\phi_2}| \\ &> f(z_0)e^{i\phi_1} + g(z_0)e^{i\phi_2} = c, \end{aligned}$$

which contradicts that z_0 is a maximum value point of $|f| + |g|$. Hence f must be a constant, which also implies g is a constant too.

5407

Suppose $f(z)$ is an entire function with

$$|f(z)| \le \frac{1}{|\mathrm{Re}z|}, \qquad \text{all } z.$$

Show that $f(z)$ is identically 0.

(*Iowa*)

Solution.

For any $R > 0$, consider function

$$g(z) = (z - Ri)(z + Ri)f(z).$$

When $|z| = R$, and $\mathrm{Im}z \ge 0$, denote by θ the angle between the line perpendicular to the imaginary axis and the line passing through z and Ri. Then $0 \le \theta \le \frac{\pi}{4}$, and

$$\left|\frac{z - Ri}{\mathrm{Re}z}\right| = \sec\theta \le \sqrt{2}.$$

When $|z| = R$, and $\mathrm{Im}z < 0$, denote by θ the angle between the line perpendicular to the imaginary axis and the line passing through z and $-Ri$. Then $0 \le \theta < \frac{\pi}{4}$, and

$$\left|\frac{z + Ri}{\mathrm{Re}z}\right| = \sec\theta < \sqrt{2}.$$

It follows from the above discussion that when $|z| = R$,

$$|g(z)| = |(z - Ri)(z + Ri)f(z)| \le \left|\frac{(z - Ri)(z + Ri)}{\mathrm{Re}z}\right| \le 2\sqrt{2}R.$$

By the maximum modulus principle, when $|z| < R$,

$$|f(z)| = \left|\frac{g(z)}{(z - Ri)(z + Ri)}\right| \le \frac{2\sqrt{2}R}{R^2 - |z|^2}.$$

Now fixing z, and letting $R \to +\infty$, we obtain $f(z) = 0$. Since R can be arbitrarily large, we have $f(z) = 0$ for all $z \in \mathbb{C}$.

5408

Suppose f is analytic on $\{z; 0 < |z| < 1\}$ and

$$|f(z)| \leq \log \frac{1}{|z|}.$$

Show that $f \equiv 0$.

(Indiana)

Solution.

Denote the Laurent expansion of f on $\{z; 0 < |z| < 1\}$ by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{|z|=r<1} \frac{f(z)}{z^{n+1}} dz.$$

It follows that

$$|a_n| \leq \frac{1}{2\pi} \int_{|z|=r} \left| \frac{f(z)}{z^{n+1}} \right| \cdot |dz| \leq \frac{1}{r^n} \log \frac{1}{r}.$$

When $n < 0$, letting $r \to 0$, we have

$$a_n = 0 \quad (n = -1, -2, \cdots),$$

which implies $z = 0$ is a removable singularity of f. In other words, f can be extended to an analytic function of the unit disk.

Since $\log \frac{1}{|z|} = 0$ when $|z| = 1$. By the maximum modulus principle, we obtain

$$f \equiv 0.$$

5409

Let f be an analytic function on $D = \{z : |z| < 1\}$, $f(D) \subseteq D$ and $f(0) = 0$.

(a) Prove that $|f(z) + f(-z)| \leq 2|z|^2$ for all z in D and if equality occurs for some non-zero z in D, then $f(z) = e^{i\alpha} z^2$.

(b) Prove that

$$\left| \int_{-1}^{1} f(x)dx \right| \le \frac{2}{3}.$$

(*Indiana*)

Solution.

(a) Let $F(z) = f(z) + f(-z)$, then $F(0) = 0$,

$$F'(0) = \lim_{z \to 0} \frac{F(z)}{z} = \lim_{z \to 0} \left(\frac{f(z)}{z} - \frac{f(-z)}{-z} \right) = 0.$$

Hence $\frac{F(z)}{z^2}$ is analytic in D, and when z tends to ∂D, the limit of $\left|\frac{F(z)}{z^2}\right|$ can not be larger than 2. By the maximum modulus principle, $|f(z) + f(-z)| \le 2|z|^2$ holds for all $z \in D$.

If equality occurs for some non-zero z in D, we have

$$f(z) + f(-z) = 2e^{i\alpha} z^2,$$

where α is a real constant.

Let

$$f(z) = \sum_{n=1}^{\infty} a_n z^n,$$

it follows from

$$f(z) + f(-z) = 2e^{i\alpha} z^2$$

that

$$a_2 = e^{i\alpha}, \quad a_4 = a_6 = \cdots = 0.$$

Because $|f(z)| < 1$ for $z \in D$, we have

$$\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=1}^{\infty} |a_n|^2 \le 1.$$

Since $a_2 = e^{i\alpha}$, the other coefficients must be zero, which implies $f(z) = e^{i\alpha} z^2$.

(b)

$$\begin{aligned} \left| \int_{-1}^{1} f(x)dx \right| &= \left| \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx \right| \\ &= \left| \int_{0}^{1} (f(x) + f(-x))dx \right| \le \int_0^1 2x^2 dx = \frac{2}{3}. \end{aligned}$$

5410

If f is analytic and $|f(z)| < 1$ on $\{z : |z| \leq 1\}$, prove that $f(z)$ has a fixed point.

(*Rutgers*)

Solution.

Let $F(z) = f(z) - z$ and $G(z) = -z$.

When $|z| = 1$,

$$|F(z) - G(z)| = |f(z)| < 1 = |G(z)|.$$

By Rouché's theorem, $F(z)$ and $G(z)$ have the same number of zeros in $\{z : |z| < 1\}$. Since $G(z)$ has only one simple zero in $\{z : |z| < 1\}$, we conclude that $f(z) - z$ has one zero in $\{z : |z| < 1\}$, which implies that $f(z)$ has a fixed point in $\{z : |z| < 1\}$.

5411

Let $f(z) = z + e^{-z}$, $\lambda > 1$. Prove or disprove: $f(z)$ takes the value λ exactly once in the right half-plane. If the answer is yes, is the point necessarily real? Justify.

(*Iowa*)

Solution.

Let R be a sufficiently large real number such that $R > 2\lambda$. Take a closed curve Γ on the right half-plane, where

$$\Gamma = \{z = x + iy : x = 0, -R \leq y \leq R\} \cup \{z : |z| = R, -\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2}\}.$$

Define

$$F(z) = \lambda - z - e^{-z}$$

and

$$G(z) = \lambda - z.$$

When $z \in \Gamma$,

$$|F(z) - G(z)| = |e^{-z}| \leq 1 < |G(z)|.$$

Since $G(z)$ has exactly one zero inside Γ, it follows from Rouché's theorem that $F(z)$ has exactly one zero inside Γ. Because R can be arbitrarily large, $F(z)$ has exactly one zero in the right half-plane. Hence $f(z)$ takes value λ exactly once in the right half plane.

Take $z = x \geq 0$. We have

$$F(x) = \lambda - x - e^{-x},$$

which is a real-valued function of real variable x. Since $F(x)$ is continuous and $F(0) > 0$,

$$\lim_{x \to +\infty} F(x) = -\infty,$$

there must exist x_0, $0 < x_0 < +\infty$, such that $F(x_0) = 0$. In other words, the point z in the right half-plane such that $f(z) = \lambda$ is necessarily real.

5412

Suppose f is analytic in a region which contains the closed unit disc $\{z : |z| \leq 1\}$. Assume f is non-zero on the unit circle $\{z : |z| = 1\}$. Let C denote the unit circle traversed in the counterclockwise sense. Suppose that

$$(1) \quad \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = 2,$$

$$(2) \quad \frac{1}{2\pi i} \int_C z \frac{f'(z)}{f(z)} dz = 0,$$

and

$$(3) \quad \frac{1}{2\pi i} \int_C z^2 \frac{f'(z)}{f(z)} dz = \frac{1}{2}.$$

Find the location of the zeros of f in the open unit disc $\{z : |z| < 1\}$.

(*Iowa*)

Solution.

Assume $z_1, z_2, \cdots, z_n$ are the zeros of $f(z)$ in $\{z : |z| < 1\}$, multiple zeros being repeated. Then

$$f(z) = g(z) \prod_{j=1}^{n} (z - z_j),$$

where $g(z)$ is analytic and has no zero in $\{z : |z| \leq 1\}$. We have

$$\begin{aligned}
\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz &= \frac{1}{2\pi i} \int_C d \log f(z) \\
&= \frac{1}{2\pi i} \int_C \left(\frac{g'(z)}{g(z)} + \sum_{j=1}^{n} \frac{1}{z - z_j} \right) dz \\
&= n.
\end{aligned}$$

It follows from (1) that $n = 2$, i.e., $f(z)$ has two zeros in the unit disk. Then for $f(z) = (z - z_1)(z - z_2)g(z)$,

$$\begin{aligned} \frac{1}{2\pi i}\int_C z\frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_C \left(\frac{z}{z-z_1} + \frac{z}{z-z_2} + \frac{zg'(z)}{g(z)}\right) dz \\ &= z_1 + z_2 = 0 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2\pi i}\int_C z^2\frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_C \left(\frac{z^2}{z-z_1} + \frac{z^2}{z-z_2} + \frac{z^2g'(z)}{g(z)}\right) dz \\ &= z_1^2 + z_2^2 = \frac{1}{2}, \end{aligned}$$

which show that $z_{1,2} = \pm\frac{1}{2}$. Hence $z = \pm\frac{1}{2}$ are the only zeros of $f(z)$ in the unit disc.

5413

(a) How many roots does this equation

$$z^4 + z + 5 = 0$$

have in the first quadrant?

(b) How many of them have argument between $\frac{\pi}{4}$ and $\frac{\pi}{2}$?

(*Indiana–Purdue*)

Solution.

(a) Let R be sufficiently large such that when $|z| = R$,

$$|z^4 + 5| > |z|.$$

Set

$$f(z) = z^4 + z + 5$$

and

$$g(z) = z^4 + 5.$$

Choose a closed curve

$$\begin{aligned} \Gamma &= \{z = x + iy; 0 \leq x \leq R, y = 0\} \cup \{z : |z| = R, 0 \leq \arg z \leq \frac{\pi}{2}\} \\ &\quad \cup \{z = x + iy : x = 0, 0 \leq y \leq R\}. \end{aligned}$$

It is obvious that

$$|f(z) - g(z)| < |g(z)|$$

holds when $z \in \Gamma$. By Rouché's theorem, the numbers of the zeros of f and g inside Γ are equal. Since g has only one zero inside Γ, f has also one zero inside Γ. Noting that R can be arbitrarily large, we know that

$$z^4 + z + 5 = 0$$

has one root in the first quadrant.

(b) Let R be sufficiently large such that when $|z| = R$, $\frac{z+5}{z^4}$ is approximately zero. Set

$$f(z) = z^4 + z + 5$$

and

$$\begin{aligned}\Gamma_1 &= \{z = x + iy : x = 0, 0 \leq y \leq R\}, \\ \Gamma_2 &= \{z = re^{\frac{\pi}{4}i} : 0 \leq r \leq R\},\end{aligned}$$

and

$$\Gamma_3 = \{z : |z| = R, \frac{\pi}{4} \leq \arg z \leq \frac{\pi}{2}\}.$$

It is easy to see that $\mathrm{Im} f(z) > 0$ when

$$z \in (\Gamma_1 \cup \Gamma_2) \backslash \{z = 0\},$$

$f(0) = 5$, and

$$f(Ri) \in \{w : 0 < \arg w < \varepsilon\}, \quad f\left(Re^{\frac{\pi}{4}i}\right) \in \{w : \pi - \varepsilon < \arg w < \pi\}$$

where $\varepsilon > 0$ is very small. We also know that

$$\Delta_{\Gamma_3} \arg f(z) = \Delta_{\Gamma_3} \arg z^4 + \Delta_{\Gamma_3} \arg \left(1 + \frac{z+5}{z^4}\right),$$

where $\Delta_{\Gamma_3} \arg f(z)$ denotes the change of $\arg f(z)$ when z goes continuously from $Re^{\frac{\pi}{4}i}$ to Ri along Γ_3. It is obvious that

$$\Delta_{\Gamma_3} \arg z^4 = \pi,$$

while

$$\Delta_{\Gamma_3} \arg \left(1 + \frac{z+5}{z^4}\right)$$

is very small. Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is taken once counterclockwise, it follows from the above discussion that

$$\Delta_\Gamma \arg f(z) = 2\pi.$$

By the argument principle, the number of the roots of $f(z) = 0$ inside Γ is equal to

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\Delta_\Gamma \log f(z) = \frac{1}{2\pi}\Delta_\Gamma \arg f(z) = 1.$$

Hence

$$f(z) = z^4 + z + 5 = 0$$

has exactly one root in the domain

$$\{z : \frac{\pi}{4} < \arg z < \frac{\pi}{2}\}.$$

5414

Prove that the equation $\sin z = z$ has infinitely many solutions in $\mathbb{C}$.

(*Indiana*)

Solution.

Let

$$f(z) = \sin z - z$$

and $z = x + iy$, then $f(z)$ can be written as

$$f(z) = \frac{e^{iz} - e^{-iz}}{2i} - z = \frac{i}{2}\left(e^{y-xi} - e^{-y+xi}\right) - (x + iy).$$

For any fixed natural number n, choose a positive number $t \gg \log n$ and a closed contour $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ in the counterclockwise sense, where

$$\begin{aligned}
\Gamma_1 &= \{z = x + iy : 2n\pi \le x \le 2(n+1)\pi, y = 0\}, \\
\Gamma_2 &= \{z = x + iy : x = 2(n+1)\pi, 0 \le y \le t\}, \\
\Gamma_3 &= \{z = x + iy : 2n\pi \le x \le 2(n+1)\pi, y = t\},
\end{aligned}$$

and

$$\Gamma_4 = \{z = x + iy : x = 2n\pi, 0 \le y \le t\}.$$

Then we consider the image of Γ under $w = f(z)$:

$$f(\Gamma_1) = \{w = u + iv : -2(n+1)\pi \le u \le -2n\pi, v = 0\}$$

with the direction from the right to the left;

$$f(\Gamma_2) = \{w = u + iv : u = -2(n+1)\pi, 0 \le v \le \frac{1}{2}\left(e^t - e^{-t}\right) - t\}$$

with the direction upwards; $f(\Gamma_3)$ lies in the annulus

$$\left\{w : \frac{1}{2}e^t - \left(\frac{1}{2}e^{-t} + t + 2(n+1)\pi\right) \le |w| \le \frac{1}{2}e^t + \left(\frac{1}{2}e^{-t} + t + 2(n+1)\pi\right)\right\}$$

starting from

$$w = -2(n+1)\pi + i\left(\frac{1}{2}e^t - \frac{1}{2}e^{-t} - t\right)$$

and ending at

$$w = -2n\pi + i\left(\frac{1}{2}e^t - \frac{1}{2}e^{-t} - t\right)$$

in the counterclockwise sense;

$$f(\Gamma_4) = \{w = u + iv : u = -2n\pi, 0 \le v \le \frac{1}{2}\left(e^t - e^{-t}\right) - t\}$$

with the direction downwards.

Hence the winding number of $f(\Gamma)$ around $w = 0$ is 1. By the argument principle, $f(z) = \sin z - z$ has one zero inside the contour Γ. Since n is arbitrarily chosen, we conclude that $\sin z = z$ has infinitely many solutions in $\mathbb{C}$.

Remark. This problem can also be proved by Hadamard's theorem.

Assume that

$$f(z) = \sin z - z$$

has only finite zeros in $\mathbb{C}$, and denote all the zeros by $z_1, z_2, \cdots, z_n$, multiple zeros being repeated. By Hadamard's theorem, $f(z)$ can be written as

$$f(z) = e^{g(z)}p(z),$$

where

$$p(z) = \prod_{k=1}^{n}(z - z_k)$$

and $g(z)$ is a polynomial.

It is obvious that $f(z)$ is an entire function of order $\lambda = 1$, where

$$\lambda = \overline{\lim_{r\to\infty}} \frac{\log\log\left\{\max\limits_{|z|=r}|f(z)|\right\}}{\log r},$$

which implies that $g(z)$ must be a polynomial of degree 1. Hence we have

$$\sin z - z = e^{az+b}p(z).$$

Let $z = x+iy$ and x be fixed. By letting $y \to +\infty$ and $y \to -\infty$ respectively, and comparing the increasing order on both sides, we obtain that $\mathrm{Im}a < 0$ in the former case and that $\mathrm{Im}a > 0$ in the latter case. This contradiction implies that $\sin z = z$ has infinite many solutions in $\mathbb{C}$.

5415

(a) Let f be a non-constant analytic function in the annulus $\{1 < |z| < 2\}$ and suppose that $|f| = 5$ on the boundary. Show that f has at least two zeros.

(b) If f is meromorphic in the annulus, is the statement in part (a) still true?

(*Stanford*)

Solution.

(a) Let $D = \{z : 1 < |z| < 2\}$ and $\partial D = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{z : |z| = 2\}$ is in the counterclockwise sense, and $\Gamma_2 = \{z : |z| = 1\}$ is in the clockwise sense. Because f is non-constant analytic in D and $|f| = 5$ when $z \in \partial D$, we know that both $f(\Gamma_1)$ and $f(\Gamma_2)$ must be $\{w : |w| = 5\}$ in the counterclockwise sense. Hence $\frac{1}{2\pi}\Delta_{\Gamma_1}\arg f(z) \geq 1$ and $\frac{1}{2\pi}\Delta_{\Gamma_2}\arg f(z) \geq 1$. In other words,

$$\frac{1}{2\pi}\Delta_{\partial D}\arg f(z) \geq 2,$$

which shows by the argument principle that f has at least two zeros in D.

(b) If f is meromorphic in D, the statement in (a) is not true. It might occur that $f(\Gamma_1)$ and $f(\Gamma_2)$ are two subarcs of $\{w : |w| = 5\}$, or both $f(\Gamma_1)$ and $f(\Gamma_2)$ are $\{w : |w| = 5\}$ in the clockwise sense. In the latter case, f has no zero in D. The following is a counterexample. Let $g(\zeta)$ be a conformal map of

$$\left\{\zeta = \xi + i\eta : \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} < 1, \text{ where } a = \frac{1}{2}\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right), b = \frac{1}{2}\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)\right\}$$

onto $\{w : |w| > 5\}$ with the normalization $g(0) = \infty$, $g'(0) > 0$. Then

$$f(z) = g\left(\frac{1}{2}\left(\frac{z}{\sqrt{2}} + \frac{\sqrt{2}}{z}\right)\right)$$

is a non-constant meromorphic function in D with $|f| = 5$ when $z \in \partial D$. But f has no zero in D.

5416

Let n be a positive integer, and let P be a polynomial of exact degree $2n$:

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{2n} z^{2n},$$

where each $a_j \in \mathbb{C}$, and $a_{2n} \neq 0$. Suppose that there is no real number x such that $P(x) = 0$, and suppose that

$$\lim_{r\to\infty} \int_{-r}^{r} \frac{P'(x)}{P(x)} dx = 0.$$

Prove that P has exactly n roots (counted with multiplicity) in the open upper half plane $\{z \in \mathbb{C} : \text{Im} z > 0\}$.

(*Indiana*)

Solution.

Let $r > 0$ be sufficiently large such that when $|z| = r$,

$$\left|a_{2n} z^{2n}\right| > \left|a_0 + a_1 z + \cdots + a_{2n-1} z^{2n-1}\right|.$$

Take a closed contour $\Gamma = \Gamma_1 \cup \Gamma_2$ in the counterclockwise sense, where

$$\Gamma_1 = \{z = re^{i\theta} : 0 \leq \theta \leq \pi\}$$

and

$$\Gamma_2 = \{z = x + iy : -r \leq x \leq r, y = 0\}.$$

Then the number of zeros of $P(z)$ inside Γ is equal to

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{P'(z)}{P(z)} dz = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{P'(z)}{P(z)} dz + \frac{1}{2\pi i} \int_{\Gamma_2} \frac{P'(z)}{P(z)} dz.$$

It is already known that

$$\lim_{r\to\infty} \int_{\Gamma_2} \frac{P'(z)}{P(z)} dz = \lim_{r\to\infty} \int_{-r}^{r} \frac{P'(x)}{P(x)} dx = 0.$$

We also have

$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{P'(z)}{P(z)}dz = \frac{1}{2\pi i}\int_{\Gamma_1} d\log P(z) = \frac{1}{2\pi}\Delta_{\Gamma_1}\arg P(z)$$

$$= \frac{1}{2\pi}\Delta_{\Gamma_1}\arg\left(a_{2n}z^{2n}\right) + \frac{1}{2\pi}\Delta_{\Gamma_1}\arg\left(1+\frac{a_0+a_1z+\cdots+a_{2n-1}z^{2n-1}}{a_{2n}z^{2n}}\right).$$

Note that

$$\frac{1}{2\pi}\Delta_{\Gamma_1}\arg\left(a_{2n}z^{2n}\right) = n$$

and

$$\lim_{r\to\infty}\frac{1}{2\pi}\Delta_{\Gamma_1}\arg\left(1+\frac{a_0+a_1z+\cdots+a_{2n-1}z^{2n-1}}{a_{2n}z^{2n}}\right) = 0,$$

we obtain that P has exactly n roots (counted with multiplicity) in the open upper half plane.

5417

Consider the function

$$f(z) = 1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\cdots+\frac{1}{n!}\frac{1}{z^n}.$$

(a) What does the integral

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{f'(z)}{f(z)}dz$$

count?

(b) What is the value of the integral for large n and fixed r?

(c) What does this tell you about the zeros of $f(z)$ for large n?

(*Courant Inst.*)

Solution.

(a) Let

$$F(\zeta) = f\left(\frac{1}{\zeta}\right) = 1+\zeta+\frac{1}{2!}\zeta^2+\frac{1}{3!}\zeta^3+\cdots+\frac{1}{n!}\zeta^n.$$

From

$$\begin{aligned}\frac{1}{2\pi i}\int_{|z|=r}\frac{f'(z)}{f(z)}dz &= -\frac{1}{2\pi i}\int_{|\zeta|=\frac{1}{r}}\frac{f'(\frac{1}{\zeta})}{f(\frac{1}{\zeta})}\cdot\frac{d\zeta}{-\zeta^2} \\ &= -\frac{1}{2\pi i}\int_{|\zeta|=\frac{1}{r}}\frac{F'(\zeta)}{F(\zeta)}d\zeta,\end{aligned}$$

we know that the negative of

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{f'(z)}{f(z)}dz$$

represents the number of zeros of $F(\zeta)$ in $\{|\zeta|<\frac{1}{r}\}$, which is just the number of zeros of $f(z)$ in $\{|z|>r\}$.

(b) When $n\to\infty$, $F(\zeta)$ converges to e^ζ uniformly in any compact subset of $\mathbb{C}$. Let

$$\min_{|\zeta|=\frac{1}{r}}|e^\zeta|=m,$$

then $m>0$.

When n is sufficiently large,

$$|F(\zeta)-e^\zeta|<m\leq|e^\zeta|$$

for $|\zeta|=\frac{1}{r}$, which implies the numbers of zeros for $F(\zeta)$ and e^ζ in $\{|\zeta|<\frac{1}{r}\}$ are equal. Since e^ζ has no zero in $\mathbb{C}$, we obtain

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{f'(z)}{f(z)}dz=0$$

for fixed r and large n.

(c) From the above discussion, we conclude that for any fixed $r>0$, when n is sufficiently large, there is no zero of $f(z)$ in $\{|z|>r\}$. In other words, all the n zeros of $f(z)$ are in $\{|z|\leq r\}$.

5418

(a) Suppose that $f(z)$ is analytic in the closed disk $|z|\leq R$, and that there is a unique, simple solution z_1 of the equation $f(z)=w$ in $\{|z|<R\}$. Show that this solution is given by the formula

$$z_1=\frac{1}{2\pi i}\int_{|z|=R}\frac{zf'(z)}{f(z)-w}dz.$$

(b) Show that, if the integer n is sufficiently large, the equation

$$z=1+\left(\frac{z}{2}\right)^n$$

has exactly one solution with $|z|<2$.

(c) If z_1 is the solution in (b), show that

$$\lim_{n\to\infty}(z_1-1)^{\frac{1}{n}}=\frac{1}{2}.$$

(*Courant Inst.*)

Solution.

(a) Let

$$f(z)-w=(z-z_1)Q(z),$$

where $Q(z)$ is analytic and has no zero in $\{|z|<R\}$. Then

$$\frac{f'(z)}{f(z)-w}=[\log(f(z)-w)]'=[\log(z-z_1)+\log Q(z)]'=\frac{1}{z-z_1}+\frac{Q'(z)}{Q(z)}.$$

Hence

$$\begin{aligned}\frac{1}{2\pi i}\int_{|z|=R}\frac{zf'(z)}{f(z)-w}dz &= \frac{1}{2\pi i}\int_{|z|=R}\frac{z}{z-z_1}dz+\frac{1}{2\pi i}\int_{|z|=R}\frac{zQ'(z)}{Q(z)}dz\\ &= \frac{1}{2\pi i}\int_{|z|=R}\frac{z}{z-z_1}dz=z_1.\end{aligned}$$

(b) Let

$$f_n(z)=z-1-\left(\frac{z}{2}\right)^n,\quad g(z)=z-1,$$

and

$$\Gamma_\varepsilon=\{|z|=2-\varepsilon\}.$$

For fixed large n, we choose $\varepsilon>0$ sufficiently small such that when $z\in\Gamma_\varepsilon$,

$$|f_n(z)-g(z)|=\left|\frac{z}{2}\right|^n=(1-\frac{\varepsilon}{2})^n<1-\varepsilon\le|g(z)|.$$

Hence $f_n(z)$ and $g(z)$ have the same number of zeros in $\{|z|<2-\varepsilon\}$, and the number is 1. Since ε can be arbitrarily small, the equation $z=1+\left(\frac{z}{2}\right)^n$ has exactly one solution (denoted by $z_1^{(n)}$) in $\{|z|<2\}$.

(c) $f_n(x)$ is a continuous real-valued function for $1\le x\le\frac{3}{2}$. When n is sufficiently large, we have $f_n(1)<0$ and $f_n\left(\frac{3}{2}\right)>0$.

Hence we have $z_1^{(n)}\in(1,\frac{3}{2})$. It follows from

$$\left|z_1^{(n)}-1\right|=\left|\frac{z_1^{(n)}}{2}\right|^n\le\left(\frac{3}{4}\right)^n\to 0$$

that

$$\lim_{n\to\infty}z_1^{(n)}=1,$$

which implies

$$\lim_{n\to\infty}\left(z_1^{(n)}-1\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{z_1^{(n)}}{2}=\frac{1}{2}.$$

5419

Let

$$\Omega=D(0,1)\backslash\{\frac{1}{2},-\frac{1}{2}\}.$$

Find all analytic functions $f:\Omega\to\Omega$ with the following property: if γ is any cycle in Ω which is not homologous to zero (mod Ω), then $f*\gamma$ is not homologous to zero (mod Ω).

(*Indiana*)

Solution.

Since f is analytic in Ω and bounded by $|f(z)|<1$, the points $z=\pm\frac{1}{2}$ must be the removable singularities of f. Let

$$\gamma_1=\{|z-\frac{1}{2}|=\varepsilon\},\quad \gamma_2=\{|z+\frac{1}{2}|=\varepsilon\},$$

where $\varepsilon>0$ is small, and the directions of γ_1 and γ_2 are both in the counter-clockwise sense. Since γ_1, γ_2 are not homologous to zero (mod Ω), $f*\gamma_1$ and $f*\gamma_2$ are also not homologous to zero (mod Ω). As ε tends to zero, $f(\gamma_1)$ and $f(\gamma_2)$ will tend to either $w=\frac{1}{2}$ or $w=-\frac{1}{2}$, because otherwise, $f*\gamma_1$ or $f*\gamma_2$ will be homologous to zero (mod Ω). Hence we obtain

$$f(\pm\frac{1}{2})=\pm\frac{1}{2}.$$

Now we claim that the case that $f(\frac{1}{2})=f(-\frac{1}{2})$ will not happen. If, for example,

$$f(\frac{1}{2})=f(-\frac{1}{2})=\frac{1}{2},$$

we assume that $z=\frac{1}{2}$ is a zero of $f(z)-\frac{1}{2}$ of order n and $z=-\frac{1}{2}$ is a zero of $f(z)-\frac{1}{2}$ of order m, then

$$f*(m\gamma_1-n\gamma_2)$$

is homologous to zero (mod Ω), while $m\gamma_1-n\gamma_2$ is not homologous to zero (mod Ω), which is a contradiction. Thus we obtain either

$$f(\frac{1}{2})=\frac{1}{2},\quad f(-\frac{1}{2})=-\frac{1}{2}$$

or

$$f(\frac{1}{2}) = -\frac{1}{2}, \quad f(-\frac{1}{2}) = \frac{1}{2}.$$

In the case of

$$f(\frac{1}{2}) = \frac{1}{2}, \quad f(-\frac{1}{2}) = -\frac{1}{2},$$

we consider the function

$$F(z) = \frac{f(z) - \frac{1}{2}}{1 - \frac{1}{2}f(z)} : \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$$

which is analytic in $D(0,1)$ and satisfies $|F(z)| \leq 1$. It follows from $F(-\frac{1}{2}) = 1$ that $F(z) \equiv 1$, which implies that $f(z) = z$.

In the case of

$$f(\frac{1}{2}) = -\frac{1}{2}, \quad f(-\frac{1}{2}) = \frac{1}{2},$$

we consider the function

$$G(z) = \frac{f(z) + \frac{1}{2}}{1 + \frac{1}{2}f(z)} : \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$$

which is also analytic in $D(0,1)$ and satisfies $|G(z)| \leq 1$. It follows from $G(-\frac{1}{2}) = -1$ that $G(z) \equiv -1$, which implies that $f(z) = -z$. Thus we conclude that the functions which satisfy the requirements of the problem are $f(z) = z$ and $f(z) = -z$.

SECTION 5
SERIES AND NORMAL FAMILIES

5501

Let

$$\sum_{n=0}^{\infty} a_n z^n$$

have a radius of convergence r and let the function $f(z)$ to which it converges have exactly one singular point z_0, on $|z| = r$, which is a simple pole. Prove that

$$\lim_{n\to\infty} a_n/a_{n+1} = z_0.$$

(*Indiana*)

Solution.

Assume that the residue of $f(z)$ at z_0 is A, and define

$$F(z) = f(z) - \frac{A}{z - z_0}.$$

Then $F(z)$ is analytic on $\{z : |z| \le r\}$. In other words, the Taylor expansion of $F(z)$ at $z = 0$ has a radius of convergence larger than r. Hence the power series

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty} a_n z^n - \frac{A}{z - z_0} \\ &= \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} A \frac{z^n}{z_0^{n+1}} \\ &= \sum_{n=0}^{\infty} \left(a_n + \frac{A}{z_0^{n+1}} \right) z^n \end{aligned}$$

is convergent at $z = z_0$, which implies

$$\lim_{n\to\infty} \left(a_n + \frac{A}{z_0^{n+1}} \right) z_0^n = 0.$$

It follows that

$$\lim_{n\to\infty} a_n z_0^n = -\frac{A}{z_0} \ne 0$$

and

$$\lim_{n\to\infty} a_{n+1}z_0^{n+1} = -\frac{A}{z_0} \neq 0,$$

and we obtain

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

5502

(1) Show that the series

$$-\sum_{n\geq 1} \alpha^n/n$$

is convergent for $1 \neq \alpha \in \mathbb{C}$ with $|\alpha| = 1$.

(2) Show that this series converges to $\log(1-\alpha)$ for such α.

(*Minnesota*)

Solution.

(1) Let $\alpha = e^{it}$, $t \in (0, 2\pi)$, then

$$-\sum_{n\geq 1} \frac{\alpha^n}{n} = -\sum_{n\geq 1} \frac{\cos nt + i\sin nt}{n}.$$

For $t \in (0, 2\pi)$ we have

$$\left|\sum_{k=1}^{n} \cos kt\right| = \left|\frac{\sin\frac{t}{2} - \sin\frac{2n+1}{2}t}{2\sin\frac{t}{2}}\right| \leq \frac{1}{\sin\frac{t}{2}},$$

and

$$\left|\sum_{k=1}^{n} \sin kt\right| = \left|\frac{\cos\frac{t}{2} - \cos\frac{2n+1}{2}t}{2\sin\frac{t}{2}}\right| \leq \frac{1}{\sin\frac{t}{2}}.$$

Because $\frac{1}{n}$ tends to zero monotonically, by Dirichlet's criterion we know that both $\sum\limits_{n\geq 1} \frac{\cos nt}{n}$ and $\sum\limits_{n\geq 1} \frac{\sin nt}{n}$ converge, which shows that $-\sum\limits_{n\geq 1} \frac{\alpha^n}{n}$ is convergent for $1 \neq \alpha \in \mathbb{C}$ with $|\alpha| = 1$.

(2) Let

$$f(z) = -\sum_{n\geq 1} \frac{z^n}{n} \quad (|z| < 1).$$

Differentiating term by term, we have

$$f'(z) = -\sum_{n\geq 1} z^{n-1} = \frac{-1}{1-z}.$$

Integrating both sides on the above identity, we obtain $f(z) = \log(1-z)$, for $|z| < 1$.

Let $\alpha = e^{it}$, $z = re^{it}$ where $0 < r < 1$, $0 < t < 2\pi$. It follows from Abel's limit theorem that

$$\begin{aligned} -\sum_{n\geq 1} \frac{\alpha^n}{n} &= -\sum_{n\geq 1} \frac{e^{int}}{n} = \lim_{r\to 1^-} -\sum_{n\geq 1} \frac{(re^{it})^n}{n} \\ &= \lim_{r\to 1^-} \log(1 - re^{it}) = \log(1 - e^{it}) = \log(1-\alpha). \end{aligned}$$

5503

Consider a power series

$$\sum_{n=1}^{\infty} \frac{1}{n} z^{n!}.$$

Show that the series converges to a holomorphic function on the open unit disk centered at origin. Prove that the boundary of the disk is the natural boundary of the function.

(*Columbia*)

Solution.

First of all, we prove the following proposition: If the radius of convergence of

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is equal to 1 and $a_n \geq 0$ for all n, then $z = 1$ is a singular point of $f(z)$. Assume the proposition is false, i.e., $z = 1$ is a regular point of f, then for fixed $x \in (0,1)$ there exists a small real number $\delta > 0$ such that the power series expansion of f at point x is convergent at $z = 1 + \delta$. Suppose the series is

$$\sum_{k=0}^{\infty} b_k (z-x)^k,$$

where

$$b_k = \frac{f^{(k)}(x)}{k!} = \frac{1}{k!} \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n x^{n-k}.$$

Thus

$$\sum_{k=0}^{\infty} b_k(z-x)^k = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{n(n-1)\cdots(n-k+1)}{k!} a_n(z-x)^k x^{n-k}$$

is convergent at $z = 1+\delta$. Noting that when $z = 1+\delta$ the right side in the above identity is a convergent double series with positive terms, and hence the order of summation can be changed, we assert that when $z = 1+\delta$,

$$\begin{aligned}
\sum_{k=0}^{\infty} b_k(z-x)^k &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{n(n-1)\cdots(n-k+1)}{k!} a_n(z-x)^k x^{n-k} \\
&= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!} (z-x)^k x^{n-k} \\
&= \sum_{n=0}^{\infty} a_n z^n,
\end{aligned}$$

which contradicts the statement that the radius of convergence of

$$\sum_{n=0}^{\infty} a_n z^n$$

is equal to 1.

Now we return to the power series

$$F(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^{n!}.$$

It follows from

$$\overline{\lim_{n\to\infty}} \sqrt[n!]{\frac{1}{n}} = 1$$

that the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n} z^{n!}$$

is equal to 1. By the above proposition, $z = 1$ is a singular point of $F(z)$. For any natural numbers p and q,

$$F(ze^{\frac{2q}{p}\pi i}) = \sum_{n=1}^{p-1} \frac{1}{n}(ze^{\frac{2q}{p}\pi i})^{n!} + \sum_{n=p}^{\infty} \frac{1}{n} z^{n!}.$$

Since $z = 1$ is a singular point of

$$\sum_{n=p}^{\infty} \frac{1}{n} z^{n!},$$

it is also a singular point of $F(ze^{\frac{2q}{p}\pi i})$. In other words, $z = e^{-\frac{2q}{p}\pi i}$ is a singular point of $F(z)$. Since the set $\{e^{-\frac{2q}{p}\pi i} : p, q = 1, 2, \cdots\}$ is dense on $\{|z| = 1\}$, we conclude that the unit circle $\{|z| = 1\}$ is the natural boundary of $F(z)$.

Remark. By the above discussion, the boundary of the unit disk is also the natural boundary of the function $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} z^{n!}$ although the series is absolutely and uniformly convergent on the closure of the unit disk.

5504

Suppose f is analytic in $U = \{|z| < 1\}$ with $f(0) = 0$ and $|f(z)| < 1$ for all $z \in U$. If the sequence $\{f_n\}$ is defined by composition

$$f_n(z) = \underbrace{f(f(\cdots f(z))\cdots)}_{n}$$

and

$$f_n(z) \to g(z)$$

for all $z \in U$, prove that either $g(z) = 0$ or $g(z) = z$.

(*Indiana–Purdue*)

Solution.

By Schwarz's lemma, it follows from $f(0) = 0$ and $|f(z)| < 1$ that $|f(z)| \leq |z|$ for all $z \in U$, and if $|f(z)| = |z|$ for some $z \neq 0$, then $f(z) = e^{i\alpha} z$ where α is a real number.

In the case when $f(z) = e^{i\alpha} z$. $f_n(z) = e^{in\alpha} z$. Since $f_n(z)$ is convergent, we obtain $\alpha = 0$, which implies that $f(z) = z$ and $g(z) = z$.

In other cases, we have

$$\left| \frac{f(z)}{z} \right| < 1$$

for all $z \in U$. Let $0 < r < 1$. Then

$$\max_{|z| \leq r} \left| \frac{f(z)}{z} \right| = \lambda < 1.$$

For all $z \in \{|z| \le r\}$, we have

$$\begin{aligned}
|f(z)| &\le \lambda|z|, \\
|f_2(z)| &= |f(f(z))| \le \lambda|f(z)| \le \lambda^2|z|, \\
&\cdots \\
|f_n(z)| &= |f(f_{n-1}(z))| \le \lambda|f_{n-1}(z)| \le \lambda^n|z|, \\
&\cdots
\end{aligned}$$

Hence $f_n(z)$ converges to zero uniformly in $\{|z| \le r\}$. Since $0 < r < 1$ is arbitrarily chosen, we obtain $g(z) = 0$ for all $z \in U$.

5505

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of analytic functions in a domain D which converges uniformaly on compact subsets of D to a function f on D.

(a) Prove that if $f_n(z) \neq 0$ for all $n \ge 1$ and $z \in D$, then either f is identically zero in D or $f(z) \neq 0$ for all $z \in D$.

(b) If each f_n is one-to-one on D, show that f is either constant or one-to-one on D.

(*UC, Irvine*)

Solution.

(a) First of all, we know from Weierstrass' theorem that f is analytic on D. Suppose f is not identically zero, but has a zero point $z_0 \in D$. Since the zeros of a non-zero analytic function are isolated, there exists $r > 0$, such that $f(z) \neq 0$ when

$$z \in \{z : 0 < |z - z_0| \le r\} \subset D.$$

Let m be the minimum value of $|f(z)|$ on

$$\{z : |z - z_0| = r\}.$$

Then $m > 0$. As $\{f_n\}$ converges to $f(z)$ uniformly on compact subsets of D, we know that for sufficiently large n,

$$|f_n(z) - f(z)| < m \le |f(z)|$$

holds on $\{z : |z - z_0| = r\}$. It follows from Rouché's theorem that f_n and f have the same number of zeros in $\{z : |z - z_0| < r\}$. Since z_0 is a zero of f, f_n must have a zero in $\{z : |z - z_0| < r\}$. which is a contradiction to the assumption that $f_n(z) \neq 0$ for all $z \in D$.

(b) Suppose f is not a constant, and is not one-to-one on D. Then there exist $z_1, z_2 \in D$ ($z_1 \neq z_2$), such that $f(z_1) = f(z_2)$ (denote it by a). Choose $r > 0$ sufficiently small, such that

$$\{z : |z - z_1| \leq r\} \cap \{z : |z - z_2| \leq r\} = \emptyset,$$
$$\{z : |z - z_1| \leq r\} \cup \{z : |z - z_2| \leq r\} \subset D,$$

and $f(z) - a \neq 0$ in $\{z : 0 < |z - z_1| \leq r\} \cup \{z : 0 < |z - z_2| \leq r\}$. Let m be the minimum value of $|f(z) - a|$ on $\{z : |z - z_1| = r \text{ or } |z - z_2| = r\}$. Then $m > 0$. With the same reason as in (a), when n is sufficiently large,

$$|(f_n(z) - a) - (f(z) - a)| = |f_n(z) - f(z)| < m \leq |f(z) - a|$$

holds on $\{z : |z - z_1| = r \text{ or } |z - z_2| = r\}$. It follows from Rouché's theorem that $f_n(z) - a$ and $f(z) - a$ have the same number of zeros in $\{z : |z - z_1| < r\}$ and $\{z : |z - z_2| < r\}$ respectively. In other words, there exists

$$z_1' \in \{z : |z - z_1| < r\}$$

and

$$z_2' \in \{z : |z - z_2| < r\},$$

such that $f_n(z_1') - a = 0$ and $f_n(z_2') - a = 0$, which implies $f_n(z_1') = f_n(z_2')$ ($z_1' \neq z_2'$). This is a contradiction to the assumption that f_n is one-to-one on D.

5506

Let $D \subset \mathbb{C}$ be a bounded domain, and let $\{f_n\}$ be a sequence of analytic automorphisms of D such that

$$\lim_{n \to \infty} f_n(a) = b \in \partial D$$

for some point $a \in D$. Prove that

$$\lim_{n \to \infty} f_n(z) = b$$

for every $z \in D$.

(*Indiana*)

Solution.

Take $a_0 \in D$, $a_0 \neq a$. If $\{f_n(a_0)\}$ does not converge to b, there exists a subsequence of $\{f_n(a_0)\}$ converging to $b_0 \neq b$. Without loss of generality, we assume

$$\lim_{n\to\infty} f_n(a) = b \in \partial D,$$
$$\lim_{n\to\infty} f_n(a_0) = b_0 \neq b.$$

Since $\{f_n(z)\}$ is a normal family, there is a subsequence $\{f_{n_k}(z)\}$ converging uniformly on compact subsets of D to $f(z)$. Because $f(a) \neq f(a_0)$, $f(z)$ is a non-constant analytic function of D.

Let r be sufficiently small such that $f(z)-b$ has no zero in $\{z : 0 < |z-a| \leq r\} \subset D$, then $m = \min\{|f(z) - b| : |z - a| = r\} > 0$. Since $\{f_{n_k}\}$ converges uniformly to f on $\{z : |z - a| = r\}$, when k is sufficiently large,

$$|f_{n_k}(z) - f(z)| = |(f_{n_k}(z) - b) - (f(z) - b)| < m \leq |f(z) - b|$$

on $\{z : |z - a| = r\}$. By Rouche's theorem, $f_{n_k}(z) - b$ has zero(s) in $\{z : |z-a| < r\}$, which is a contradiction to the fact that f_{n_k} does not assume the value $b \in \partial D$ in D because f_{n_k} is an automorphism of D.

5507

Which of the following families are normal, and which is compact? Justify your answers.

(a) $\mathcal{F} = \{f : f \text{ is analytic in } D, f(0) = 0, \text{ diam } f(D) \leq 2\}$

(b) $\mathcal{G} = \{g : g \text{ is analytic in } D, g(0) = 1, \text{Re}\{g\} > 0, \text{ diam } g(D) \geq 1\}$. Here the diameter of a set S is diam $S = \sup\{|z - \zeta| : z, \zeta \in S\}$.

(*Indiana*)

Solution.

(a) For any $f \in \mathcal{F}$, it follows from $f(0) = 0$ and diam $f(D) \leq 2$ that $|f(z)| \leq 2$, which shows that $\mathcal{F}$ is normal.

Let $\{f_n\}$ be a sequence of functions in $\mathcal{F}$. Then there exists a subsequence $\{f_{n_k}\}$ converging uniformly in compact subsets of D to $f(z)$, which obviously satisfies the conditions that $f(z)$ is analytic in D and $f(0) = 0$. For any two fixed points $z, \zeta \in D$, we have

$$|f_{n_k}(z) - f_{n_k}(\zeta)| \leq 2$$

because diam $f_{n_k}(D) \leq 2$. We choose a compact subset $K \subset D$ such that $z, \zeta \in K$. It follows from the uniform convergence of $\{f_{n_k}\}$ on K that

$$|f(z) - f(\zeta)| \leq 2.$$

Since $z, \zeta \in D$ can be arbitrarily chosen, we obtain diam $f(D) \leq 2$, hence $f(z) \in \mathcal{F}$, which shows that $\mathcal{F}$ is also compact.

(b) Let $\{g_n\}$ be any sequence of functions in $\mathcal{G}$. Then for $G_n(z) = e^{-g_n(z)}$, we have $|G_n(z)| < 1$. Hence there exists a subsequence $\{G_{n_k}\}$ converging uniformly in compact subsets of D to a function $G(z)$ which is either a constant or a non-constant analytic function in D. If $G(z)$ is a constant, then the constant is e^{-1} because

$$G(0) = \lim_{n\to\infty} e^{-g_n(0)} = e^{-1};$$

if $G(z)$ is non-constant analytic, since $G_n(z) \neq 0$ for all $z \in D$, by Hurwitz's theorem, we have $G(z) \neq 0$ for all $z \in D$. Hence we can define an analytic function $g(z) = -\log G(z)$, where the single-valued branch is chosen by $g(0) = -\log G(0) = 1$, and we conclude that

$$g_{n_k}(z) = -\log G_{n_k}(z)$$

converges uniformly in compact subsets of D to $g(z)$, which shows that family $\mathcal{G}$ is normal. But family $\mathcal{G}$ is not compact. First we can choose a sequence of functions $g_n(z)$ in $\mathcal{G}$ as follows: $g_n(z)$ is a conformal mapping of D onto

$$\Omega_n = \{w : |w-1| < \frac{1}{4}\} \cup \{w : |w-3| < 1\} \cup \{w : |\mathrm{Im} w| < \frac{1}{n}, 1 < \mathrm{Re} w < 3\}$$

satisfying $g_n(0) = 1$, $g_n'(0) > 0$. By the Riemann mapping theorem, such a mapping g_n exists and is unique, and it is obvious that g_n satisfies all the conditions required by the family $\mathcal{G}$. Because the domain sequence $\{\Omega_n\}$ converges to $\Omega = \{w : |w-1| < \frac{1}{4}\}$ which is called the kernel of $\{\Omega_n\}$ with respect to $w = 1$, by Caratheodory's theorem, $\{g_n(z)\}$ converges uniformly in compact subsets of D to $g(z)$ which is a conformal mapping of D onto Ω. Since diam $g(D) = \frac{1}{2}$, $g(z)$ does not belong to the family $\mathcal{G}$, which shows that $\mathcal{G}$ is not compact.

5508

Suppose that $1 \leq p < \infty$ and $c \geq 0$ is a real number. Let $\mathcal{F}$ be the set of all analytic functions f on $\{|z| < 1\}$ such that

$$\sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \le c.$$

Show that $\mathcal{F}$ is a normal family.

(*Illinois*)

Solution.

It suffices to prove that the functions in $\mathcal{F}$ are uniformly bounded on every compact set of $\{z| < 1\}$. We prove the assertion by contradiction. If it is not the case, then there exist $z_n \in D$, $f_n \in \mathcal{F}$ such that $z_n \to z_0 \in D$ and $f_n(z_n) \to \infty$.

Let $1 - |z_0| = 3r$. Then when n is sufficiently large, $|z_n - z_0| < r$. By Cauchy integral formula,

$$f_n(z_n) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{f_n(\zeta)}{\zeta - z_n} d\zeta, \quad (2r < \rho < 3r).$$

Hence

$$\begin{aligned} |f_n(z_n)| &\le \frac{3}{2\pi} \int_0^{2\pi} |f_n(z_0 + \rho e^{i\theta})| d\theta \\ &\le \frac{3}{2\pi} \left(\int_0^{2\pi} |f_n(z_0 + \rho e^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \cdot \left(\int_0^{2\pi} d\theta \right)^{\frac{1}{q}} \\ &= \frac{3}{(2\pi)^{\frac{1}{p}}} \left(\int_0^{2\pi} |f_n(z_0 + \rho e^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \end{aligned}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$\begin{aligned} \frac{2\pi}{3^p} |f_n(z_n)|^p \int_{2r}^{3r} \rho d\rho &\le \int_{2r}^{3r} \int_0^{2\pi} |f_n(z_0 + \rho e^{i\theta})|^p \rho d\rho d\theta \\ &\le \int_0^1 \int_0^{2\pi} |f(\rho e^{i\theta})|^p \rho d\rho d\theta \le \frac{c}{2}. \end{aligned}$$

As $n \to \infty$, the left side of the above inequality tends to infinity, while the right side of the inequality is a constant. The contradiction implies that $\mathcal{F}$ is a normal family.

5509

(a) Let f be holomorphic for $|z| < R$ and satisfy $f(0) = 0$, $f'(0) \neq 0$, $f(z) \neq 0$ for $0 < |z| < r \leq R$. Let C be the circle $|z| = \rho$ where $\rho < r$. Show that

$$g(w) = \frac{1}{2\pi i}\int_C \frac{tf'(t)dt}{f(t) - w}$$

define a holomorphic function of w for

$$|w| < m = \min_\theta |f(\rho e^{i\theta})|,$$

and that $z = g(w)$ is the unique solution of

$$f(z) = w$$

that tends to zero with w.

(b) Find the Taylor's expansion of $g(w)$, and apply this to find the explicit series expansion of the root of the equation

$$z^3 + 3z - w = 0$$

that tends to zero with w.

(*Harvard*)

Solution.

(a) It follows from

$$|w| < m = \min_\theta |f(\rho e^{i\theta})|$$

that when $t \in C$,

$$\frac{1}{f(t) - w} = \frac{1}{f(t)\left(1 - \frac{w}{f(t)}\right)} = \sum_{n=0}^{\infty} \frac{w^n}{f(t)^{n+1}}.$$

Hence

$$g(w) = \frac{1}{2\pi i}\int_C \frac{tf'(t)dt}{f(t) - w} = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{tf'(t)}{f(t)^{n+1}}dt\right) w^n,$$

which implies that $g(w)$ is holomorphic in $\{w : |w| < m\}$.

Let Γ be the image of C under f where C is the circle $\{z : |z| = \rho\}$ taken once counterclockwise. Because

$$|w| < m = \min_{\theta} |f(\rho e^{i\theta})|,$$

the winding number

$$n(\Gamma, 0) = n(\Gamma, w),$$

which shows that $f(z)$ and $f(z) - w$ have the same number of zeros in $\{z : |z| < \rho\}$. Since $z = 0$ is the only simple zero of f in $\{z : |z| < \rho\}$, we know that $f(z) = w$ has a unique solution in $\{z : |z| < \rho\}$. Denote the unique solution by z_1, then

$$f(t) - w = (t - z_1)Q(t)$$

where $Q(t)$ is analytic and has no zero in $\{t : |t| < \rho\}$, and

$$\frac{f'(t)}{f(t) - w} = [\log(f(t) - w)]' = [\log(t - z_1) + \log Q(t)]' = \frac{1}{t - z_1} + \frac{Q'(t)}{Q(t)}.$$

Hence

$$\begin{aligned} g(w) &= \frac{1}{2\pi i}\int_C \frac{tf'(t)dt}{f(t) - w} = \frac{1}{2\pi i}\int_C \frac{t}{t - z_1}dt + \frac{1}{2\pi i}\int_C \frac{tQ'(t)}{Q(t)}dt \\ &= z_1, \end{aligned}$$

which shows that $g(w)$ is just the unique solution of $f(z) = w$. As the constant term in the Taylor expansion of $g(w)$ is

$$\frac{1}{2\pi i}\int_C \frac{tf'(t)}{f(t)}dt$$

which is obviously zero, we assert that the unique solution $g(w)$ tends to zero together with w.

(b) Let

$$f(z) = z^3 + 3z,$$

then

$$g(w) = \frac{1}{2\pi i}\int_C \frac{tf'(t)dt}{f(t) - w} = \sum_{n=1}^{\infty} a_n w^n,$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{tf'(t)}{f(t)^{n+1}}dt = \frac{1}{2\pi i}\int_C \frac{t^2 + 1}{3^n t^n}\left(1 + \frac{t^2}{3}\right)^{-n-1} dt.$$

After some computation, we obtain $a_{2k} = 0$ and

$$\begin{aligned} a_{2k-1} &= \frac{1}{2\pi i}\int_C \frac{1}{3^{2k-1}t^{2k-1}}\left[t^2 C_{-2k}^{k-2}\left(\frac{t^2}{3}\right)^{k-2} + C_{-2k}^{k-1}\left(\frac{t^2}{3}\right)^{k-1}\right] dt \\ &= \frac{1}{3^{3k-2}}\left(3C_{-2k}^{k-2} + C_{-2k}^{k-1}\right). \end{aligned}$$

5510

Find an explicit formula for a meromorphic function f whose only singularities are simple poles at $-1, -2, -3, \cdots$ with residue n at $z = -n$. Prove in detail that your function has all the required properties.

(*Illinois*)

Solution.

By Mittag-Leffler's theorem, we construct

$$f(z) = \sum_{n=1}^{\infty}\left(\frac{n}{z+n} - 1 + \frac{z}{n}\right) = \sum_{n=1}^{\infty}\frac{z^2}{n(n+z)}.$$

For any natural number N, when $|z| \le N$, $n \ge 2N$,

$$\left|\frac{z^2}{n(n+z)}\right| \le \frac{2N^2}{n^2}.$$

Hence

$$\sum_{n=2N}^{\infty}\frac{z^2}{n(n+z)}$$

converges uniformly in $\{|z| \le N\}$ to a function which is analytic in $\{|z| < N\}$. In addition,

$$\sum_{n=1}^{2N-1}\frac{z^2}{n(n+z)}$$

is a meromorphic function whose only singularities in $\{|z| < N\}$ are simple poles at $z = -1, -2, \cdots, -N+1$ with residue n at $z = -n$. So $f(z)$ is analytic in $\{|z| < N\}\backslash\{-1, -2, \cdots, -N+1\}$, and $z = -1, -2, \cdots, -N+1$ are its simple poles with residue n at $z = -n$.

Because N can be chosen arbitrarily large, it is obvious that $f(z)$ has all the required properties of the problem.

5511

(a) Does there exist a sequence of polynomials $\{P_n\}$ such that $P_n(z) \to \frac{1}{z^2}$ uniformly on the annulus $1 < |z| < 2$? If Yes, give an explicit formula for the P_n; if No, explain why not.

(b) Does there exist an entire function g whose zero-set is $\{\sqrt{n}(1+i) : n = 0, 1, 2, 3, \cdots\}$? If Yes, give an explicit formula for g; if No, explain why not.

(*Illinois*)

Solution.

(a) No. If there exists a sequence of polynomials $\{P_n\}$ such that $P_n(z) \to \frac{1}{z^2}$ uniformly on $\{1 < |z| < 2\}$, then for any $\varepsilon \in (0, \frac{1}{4})$, there exists $N > 0$ such that when $n > N$, $\left|P_n(z) - \frac{1}{z^2}\right| < \varepsilon$ holds for all $z \in \{1 < |z| < 2\}$. Multiply both sides by $|z|^2$, we have

$$|z^2 P_n(z) - 1| < \varepsilon|z|^2 < 4\varepsilon < 1 \quad \text{for } z \in \{1 < |z| < 2\}.$$

Because $z^2 P_n(z) - 1$ is an analytic function in $\{|z| < 2\}$, it follows from the maximum modulus principle that

$$|z^2 P_n(z) - 1| < 1$$

holds for all $z \in \{|z| < 2\}$. The contradiction follows by taking $z = 0$ in the inequality.

(b) Yes. The function g can be chosen as

$$g(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2},$$

where $a_n = \sqrt{n}(1+i)$.

For any $R > 0$, let $|z| \le R$ and choose $N > R^2$. Then when $n \ge N$,

$$\begin{aligned} \log[(1 - \frac{z}{a_n})e^{\frac{z}{a_n} + \frac{1}{2}(\frac{z}{a_n})^2}] &= \log(1 - \frac{z}{a_n}) + \frac{z}{a_n} + \frac{1}{2}(\frac{z}{a_n})^2 \\ &= -\frac{1}{3}(\frac{z}{a_n})^3 - \cdots - \frac{1}{m}(\frac{z}{a_n})^m - \cdots. \end{aligned}$$

It is easy to see that

$$\left|\log\left[(1 - \frac{z}{a_n})e^{\frac{z}{a_n} + \frac{1}{2}(\frac{z}{a_n})^2}\right]\right| < \frac{R^3}{n^{3/2}}.$$

Because $\sum_{n=N}^{\infty} \frac{R^3}{n^{3/2}}$ converges, we know that

$$\sum_{n=N}^{\infty} \log\left[(1-\frac{z}{a_n})e^{\frac{z}{a_n}+\frac{1}{2}(\frac{z}{a_n})^2}\right]$$

is analytic in $\{z : |z| < R\}$, which implies

$$\prod_{n=N}^{\infty} (1-\frac{z}{a_n})e^{\frac{z}{a_n}+\frac{1}{2}(\frac{z}{a_n})^2}$$

is analytic in $\{z : |z| < R\}$. Hence $g(z)$ is analytic in $\{z : |z| < R\}$, and its zeros in $\{z : |z| < R\}$ are $0, a_1, a_2, \cdots, a_k$ $(\frac{R^2}{2} - 1 \leq k < \frac{R^2}{2})$. Since R can be arbitrarily large, we see that $g(z)$ is an entire function with the required zero-set.

5512

State whether the following statement is True or False, and prove your assertion.

For each positive integer n there exists an entire function f_n such that

$$\max_{1\leq|z|\leq 2} |\text{Re}f_n(z) - \log|z|| < \frac{1}{n}.$$

(*Indiana*)

Solution.

False.

We prove the assertion by contradiction.

If for each positive number n there exists an entire function f_n such that

$$\max_{1\leq|z|\leq 2} |\text{Re}f_n(z) - \log|z|| < \frac{1}{n},$$

then for $1 \leq |z| \leq 2$, we have

$$-1 \leq \text{Re}f_n(z) \leq 1 + \log 2.$$

Define $F_n(z) = e^{f_n(z)}$. Then $F_n(z)$ are entire functions with no zeros, and

$$\frac{1}{e} \leq |F_n(z)| = e^{\text{Re}f_n(z)} \leq 2e$$

for $1 \le |z| \le 2$. By the maximum modulus principle, $|F_n(z)| \le 2e$ for $|z| \le 2$. Hence $\{F_n(z)\}$ is a normal family in $\{z : |z| < 2\}$, and there exists a subsequence $\{F_{n_k}(z)\}$ converging locally uniformly to an analytic function $F(z)$ in $\{z : |z| < 2\}$. Since $|F_n(z)| \ge \frac{1}{e}$ for $1 \le |z| \le 2$, $F(z)$ cannot be identically zero, and by Hurwitz's theorem $F(z)$ has no zero in $\{z : |z| < 2\}$. But we have for $1 \le |z| < 2$,

$$|F(z)| = \lim_{k\to\infty} |F_{n_k}(z)| = \lim_{k\to\infty} e^{\mathrm{Re} f_{n_k}(z)} = e^{\log|z|} = |z|,$$

which implies that $F(z) = \alpha z$ with $|\alpha| = 1$ in $\{z : |z| < 2\}$. This is a contradiction to the fact that $F(z)$ has no zero in $\{z : |z| < 2\}$.

5513

Let $G = D\backslash(-1, 0]$, where $D = \{z : |z| < 1\}$.

(a) Give a single-valued definition for z^i in G.

(b) Why should there exist a sequence of polynomials P_n such that

$$\lim_{n\to\infty} P_n(z) = z^i$$

for all z in G?

(c) Can the polynomials be chosen so that there exists a constant M with $|P_n(z)| \le M$ for all $z \in G$ and all n? Justify your answer.

(*Indiana*)

Solution.

(a) z^i is defined by $e^{i\log z}$. In domain G, single-valued branch of $\log z$ can be chosen. For example, a single-valued branch of z^i in G can be defined by $\arg z|_{0<z<1} = 0$.

(b) Choose

$$K_n = \{z : \frac{1}{n} \le |z| \le 1 - \frac{1}{n}, \quad -\pi + \frac{1}{n} \le \arg z \le \pi - \frac{1}{n}\}$$

where $n > 2$. Then $K_n \subset K_{n+1}$, and

$$\lim_{n\to\infty} K_n = G.$$

Because the complement of K_n is connected and contains $z = \infty$, we know by Runge's theorem that there exists a sequence of polynomials which converges uniformly on K_n to z^i. In other words, we can find a polynomial $P_n(z)$ such that

$$|P_n(z) - z^i| < \frac{1}{n}$$

for all $z \in K_n$. Hence $\{P_n(z); n = 1, 2, \cdots\}$ converges to z^i uniformly on compact subsets of G.

(c) No. If there exists M with $|P_n(z)| \leq M$ for all $z \in G$ and all n, then because $P_n(z)$ are continuous on D, $|P_n(z)| \leq M$ for all $z \in D$ and all n. It follows that $\{P_n(z)\}$ is a normal family in D, and there exists a subsequence $P_{n_k}(z)$ which converges uniformly on compact subsets of D to an analytic function $f(z)$ in D. Since $P_n(z)$ converges to z^i in G, hence $z^i = f(z)$ for $z \in G$, which implies that z^i can be extended to a single-valued analytic function in D. It is obvious impossible, so the contradiction is obtained.

5514

(a) Prove that

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

(b) Use this to show that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Justify your steps.

(c) Develop $\pi \cot \pi z$ in a Laurent series about the origin directly and by use of (b), with enough terms to find the values of $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$.

(*Harvard*)

Solution.

(a) Let

$$f(z) = \frac{\pi^2}{\sin^2 \pi z}.$$

The singular part of f at $z = n$ $(n = 0, \pm 1, \pm 2, \cdots)$ is $\frac{1}{(z-n)^2}$. Now we consider the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

For any natural number N and $|z| \leq N$,

$$\left|\frac{1}{(z-n)^2}\right| \leq \frac{4}{n^2}$$

holds for $n \geq 2N$ and $n \leq -2N$. It follows from the convergence of $\sum\limits_{n=2N}^{\infty} \frac{4}{n^2}$ and $\sum\limits_{-\infty}^{n=-2N} \frac{4}{n^2}$ that $\sum\limits_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ is analytic in

$$\{|z| < N\} \backslash \{z = 0, \pm 1, \cdots, \pm(N-1)\}.$$

Because N can be arbitrarily large, we obtain the result that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

is a meromorphic function which has the same singularities as $f(z)$.

Let

$$g(z) = f(z) - \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

Then $g(z)$ is an entire function.

As $f(z)$ and

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

are both periodic functions with period equal to 1, we restrict z in the strip

$$\{z : 0 < \text{Re} z \leq 1\}.$$

It is obvious that

$$\lim_{\text{Im} z \to \pm\infty} f(z) = 0.$$

As the convergence of

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

is uniform for

$$|\text{Im} z| \geq 1,$$

the limit of the series for $\text{Im} y \to \pm\infty$ can be obtained by taking the limit in each term and the limit is also zero. Hence $g(z)$ is a bounded entire function, which implies that $g(z)$ is a constant. It is obvious that the constant must be zero. Thus we obtain the identity

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}. \tag{1}$$

(b) Let

$$F(z) = \pi \mathrm{ctg} \pi z.$$

The singular part of F at $z = n$ $(n = 0, \pm 1, \pm 2, \cdots)$ is $\frac{1}{z-n}$. Now we consider the series

$$\frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

With similar discussion to that in (a), we know that

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

is a meromorphic function which has the same singularities as $F(z)$.

Let

$$F(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} + G(z).$$

Then $G(z)$ is an entire function. Differentiating both sides of the above identity, we obtain

$$\frac{\pi^2}{\sin^2 \pi z} = \frac{1}{z^2} + \sum_{n=1}^{\infty} \left[\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2} \right] - G'(z)$$

Comparing this identity with (1), we have $G'(z) = 0$ which implies that $G = c$ (c is a constant). For

$$F(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} + c,$$

it follows from the fact that $F(z)$ and

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

are both odd functions that $c = 0$. Hence we obtain

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \tag{2}$$

(c) The Laurent expansion of $\pi \cot \pi z$ about the origin is

$$\pi \cot \pi z = \frac{1}{z} - \frac{\pi^2}{3} z - \frac{\pi^4}{45} z^3 - \cdots . \tag{3}$$

It follows from (2) and (3) that around the origin,

$$\sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} = -\frac{\pi^2}{6} - \frac{\pi^4}{90} z^2 - \cdots . \tag{4}$$

Take $z = 0$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} .$$

After differentiating (4) on both sides, we can also obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} .$$

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Let $z_1, \cdots, z_n$ be distinct complex numbers. Let f and g be polynomials, f of degree $\leq n-2$ and

$$g(z) = (z - z_1) \cdots (z - z_n).$$

(a) Show that

$$\sum_{j=1}^{n} \frac{f(z_j)}{g'(z_j)} = 0.$$

(b) Show that there exists a polynomial of degree $\leq n-2$ with $f(z_j) = a_j$ if and only if

$$\sum_{j=1}^{n} \frac{a_j}{g'(z_j)} = 0.$$

(c) Given a sequence of complex numbers $z_1, z_2, \cdots$ such that $|z_n| \to \infty$, does there exist an entire function f with $f(z_j) = a_j$? Can you write this function down?

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Solution.

(a) Take R sufficiently large such that

$$z_1, z_2, \cdots, z_n \in \{|z| < R\}.$$

Because

$$g(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$$

is of degree n, while $f(z)$ is of degree $\leq n - 2$,

$$\int_{|z|=R} \frac{f(z)}{g(z)} dz = \lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{g(z)} dz = 0.$$

Since

$$\int_{|z|=R} \frac{f(z)}{g(z)} dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}\left(\frac{f(z)}{g(z)}, z_j\right) = 2\pi i \sum_{j=1}^{n} \frac{f(z_j)}{g'(z_j)},$$

we obtain

$$\sum_{j=1}^{n} \frac{f(z_j)}{g'(z_j)} = 0.$$

(b) If $f(z)$ is a polynomial of degree $\leq n - 2$ with $f(z_j) = a_j$ ($j = 1, 2, \cdots, n$), then by (a), we have

$$\sum_{j=1}^{n} \frac{f(z_j)}{g'(z_j)} = \sum_{j=1}^{n} \frac{a_j}{g'(z_j)} = 0.$$

If $a_1, a_2, \cdots, a_n$ are n complex numbers such that

$$\sum_{j=1}^{n} \frac{a_j}{g'(z_j)} = 0,$$

we construct the function $f(z)$ by

$$f(z) = \sum_{j=1}^{n} \frac{a_j}{g'(z_j)} \cdot \frac{g(z)}{(z - z_j)}.$$

For each j,

$$\frac{g(z)}{z - z_j} = (z - z_1) \cdots (z - z_{j-1})(z - z_{j+1}) \cdots (z - z_n)$$

is a polynomial of degree $n - 1$, and the coefficient of z^{n-1} is 1. Since

$$\sum_{j=1}^{n} \frac{a_j}{g'(z_j)} = 0,$$

the coefficient of z^{n-1} of $f(z)$ is zero. In other words, $f(z)$ is a polynomial of degree $\leq n-2$.

Because

$$\lim_{z\to z_j} \frac{a_j}{g'(z_j)} \cdot \frac{g(z)}{(z-z_j)} = a_j,$$

while for $k \neq j$,

$$\frac{a_k}{g'(z_k)} \cdot \frac{g(z)}{z-z_k}\bigg|_{z=z_j} = 0,$$

$f(z)$ satisfies the condition $f(z_j) = a_j$ $(j = 1, 2, \cdots, n)$.

(c) For the given sequence $z_1, z_2, \cdots$, such that $|z_n| \to \infty$, by the Weierstrass theorem about the canonical product of entire functions, we can construct an entire function $g(z)$ with simple zeros $z_1, z_2, \cdots$. Then we define

$$f(z) = \sum_{n=1}^{\infty} u_n(z) = \sum_{n=1}^{\infty} e^{\gamma_n(z-z_n)} \frac{g(z)}{z-z_n} \cdot \frac{a_n}{g'(z_n)},$$

where γ_n is chosen such that when $|z| \leq \frac{|z_n|}{2}$,

$$|u_n(z)| = \left| e^{\gamma_n(z-z_n)} \frac{g(z)}{z-z_n} \cdot \frac{a_n}{g'(z_n)} \right| \leq \frac{1}{n^2}.$$

Because $|z_n| \to \infty$, for any $R > 0$, there exists $N > 0$ such that $|z_n| > 2R$ when $n \geq N$. Hence

$$|u_n(z)| \leq \frac{1}{n^2}$$

holds for all $|z| \leq R$ when $n \geq N$. In other words, $\sum\limits_{n=1}^{\infty} u_n(z)$ converges uniformly for all $|z| \leq R$, so that $f(z)$ is analytic in $\{|z| < R\}$. Since R can be arbitrarily large, $f(z)$ is an entire function.

It is easy to see that

$$\lim_{z\to z_n} u_n(z) = \lim_{z\to z_n} e^{\gamma_n(z-z_n)} \frac{g(z)}{z-z_n} \cdot \frac{a_n}{g'(z_n)} = a_n,$$

while for $k \neq n$,

$$u_k(z_n) = 0,$$

which implies that $f(z)$ is an entire function satisfying the required condition.