5.7. Calculation of Definite Integrals

Cauchy first caught the imagination of the mathematical world with his ability to compute some definite integrals where an explicit antiderivative is not available, and this use of complex variables is still central to many applied areas. Since it it not central to the theoretical concerns that dominate this book, we'll not spend as long as some texts, but we'd be remiss if we didn't say something. The choice of contour is an art as much as a science, so we'll mainly study by example, which we'll break into six parts. All use the residue theorem, Theorem 4.3.1.

5.7.1. Periodic Functions Over a Period. These are the simplest since they essentially come as an integral over a closed contour.

Example 5.7.1. For $a > |b|$ both real, compute

$$\int_0^{2\pi} \frac{1}{a + b\cos\theta} \frac{d\theta}{2\pi} \tag{5.7.1}$$

Let $z = e^{i\theta}$ so $dz = iz\,d\theta$ and the integral in (5.7.1) becomes (with the usual counterclockwise contour on $\partial\mathbb{D}$)

$$\frac{1}{2\pi i} \oint \frac{1}{a + \frac{1}{2}b(z + z^{-1})} \frac{dz}{z} = \frac{1}{2\pi i} \oint \frac{2}{bz^2 + 2az + b}\, dz$$

The poles are at $(\pm\sqrt{a^2 - b^2} - a)/b = z_\pm$. Both are in $(-\infty, 0)$, one inside the circle, one outside. The polynomial is thus $b(z - z_+)(z - z_-)$ and the residue is $1/b(z_+ - z_-) = 1/2(\sqrt{a^2 - b^2})$, that is,

$$\int_0^{2\pi} \frac{1}{a + b\cos\theta} \frac{d\theta}{2\pi} = \frac{1}{\sqrt{a^2 - b^2}} \tag{5.7.2}$$

Taking $a = 1$ and expanding both sides as a power series in b (using the binomial theorem for $(1 - b^2)^{-1/2}$) one gets (one can also get this from the binomial theorem and go backwards to (5.7.2))

$$\int_0^{2\pi} \cos^{2n}(\theta) \frac{d\theta}{2\pi} = \frac{(2n)!}{2^{2n}(n!)^2} \tag{5.7.3}$$

□

5.7.2. Integrals of Rational Functions. Most of the remaining examples are $\int_{-\infty}^{\infty}$ or $\int_0^\infty$ and so are "improper," that is, over infinite intervals. There are three possible meanings of such an integral:

(a) $\int_0^\infty |f(x)|\,dx < \infty$ or $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$. Such integrals are said to be *absolutely convergent.*

(b) $\lim_{R\to\infty} \int_0^R f(x)\,dx$ exists but $\int_0^\infty |f(x)|\,dx = \infty$. Such integrals are said to be *conditionally convergent.*

(c) $\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$ exists but $\lim_{R\to\infty}\int_0^R f(x)\,dx$ does not. Such integrals are called *principal values at infinity* (written pv).

Examples of the three types are $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$ (see Example 5.7.3 below), $\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx$ (see Example 5.7.5 below), and $\mathrm{pv}\int_{-\infty}^{\infty}\frac{x\,dx}{x^2+1}=0$. While on the subject of principal values, if $\int_{|x|>\varepsilon,\,x\in(a,b)}|f(x)|\,dx$ is finite for all $\varepsilon>0$ but $\int_0^{\varepsilon}|f(x)|\,dx=\int_{-\varepsilon}^{0}|f(x)|\,dx$ $(=\infty)$, we define for $a<0<b$,

$$\mathrm{pv}\int_a^b f(x)\equiv\lim_{\varepsilon\downarrow 0}\left[\int_a^{-\varepsilon} f(x)\,dx+\int_{\varepsilon}^{b} f(x)\,dx\right] \tag{5.7.4}$$

if the limit exists. If the singular point is at x_0 rather than 0, we make a similar definition. If γ is a simple contour, C^1, near $\gamma(t_0)$, we can similarly define a principal value as integrating without $(t_0-\varepsilon,t_0+\varepsilon)$. Here is the key fact:

Theorem 5.7.2. *Let γ be a simple closed contour, f meromorphic in a neighborhood of* $\mathrm{ins}(\gamma)$ *so that f has only simple poles on* $\mathrm{Ran}(\gamma)$ *and with γ C^1 near these poles. If γ is oriented so that $n(\gamma,z)=1$ on* $\mathrm{ins}(\gamma)^{\mathrm{int}}$*, then*

$$\frac{1}{2\pi i}\,\mathrm{pv}\oint f(x)\,dz=\sum_{z_0\in\mathrm{ins}(\gamma)^{\mathrm{int}}}\mathrm{Res}(f;z_0)+\tfrac{1}{2}\sum_{z_0\in\mathrm{Ran}(\gamma)}\mathrm{Res}(f;z_0) \tag{5.7.5}$$

Proof. Letting $\tilde\gamma_r$ be the contour

$$\tilde\gamma_r(t)=re^{\pi it},\qquad 0\le t\le 1 \tag{5.7.6}$$

the standard calculation (see Example 2.2.2) shows the integral

$$\int_{\tilde\gamma_r} z^{-1}\,dz=\pi i \tag{5.7.7}$$

Thus, if $\gamma^{(\varepsilon)}$ is the contour γ with symmetric gaps of size 2ε about each point on γ and if $\gamma^{(\varepsilon)\sharp}$ is $\gamma^{(\varepsilon)}$ with semicircles running outside (see Figure 5.7.1)

$$\int_{\gamma^{(\varepsilon)}} f(z)\,dz=\int_{\gamma^{(\varepsilon)\sharp}} f(z)\,dz-\pi i\sum_{z_0\in\mathrm{Ran}(\gamma)}\mathrm{Res}(f;z_0)+O(\varepsilon) \tag{5.7.8}$$

From the residue theorem, we get (5.7.5). □

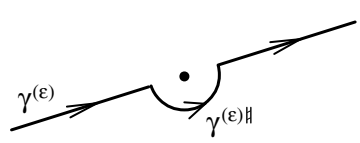

Figure 5.7.1. Resolving a principal value.

If P/Q is a rational function with no poles on $\mathbb{R}$, $\int_{-\infty}^{\infty}(P(x)/Q(x))\,dx$ is absolutely convergent if $\deg(Q)\ge 2+\deg(P)$. If $\deg(Q)=1+\deg(P)$

so $P(x)/Q(x) = c/x + O(1/x^2)$ at infinity, then only the principal value at infinity exists and is easy to accommodate. Thus, we will limit ourselves here to $\deg(Q) \geq 2 + \deg(P)$. Using Theorem 5.7.2, it is easy to accommodate simple poles on $\mathbb{R}$.

Example 5.7.3. Compute for $a > 0$,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} \tag{5.7.9}$$

This is the limit as $R \to \infty$ of $\int_{-R}^{R}$. We now "close the contour," that is, add and subtract the integral counterclockwise along $|z| = R$ from R to $-R$. If $R > a$, the amount subtracted is bounded by $\pi R(R^2 - a^2)^{-1}$, since $\sup_{|z|=R} |1/(z^2 + a^2)| \leq 1/(R^2 - a^2)$. This goes to zero as $R \to \infty$. Thus, if C_R is the closed contour (see Figure 5.7.2), the integral in (5.7.9) is the limit as $R \to \infty$ of the integral over the closed contour (which we'll see is R-independent if $R > a$). Inside the contour, $z^2 + a^2$ has one pole at $z = ia$ and the residue is $1/2ia$. Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = \frac{\pi}{a} \tag{5.7.10}$$

The reader should do the computation if the contour is closed in the lower half-plane.

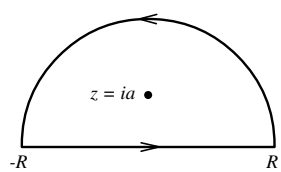

Figure 5.7.2. Contour for Example 5.7.3.

Since $(x^2 + a^2)^{-1} = a^{-1} \frac{d}{dx} \arctan(\frac{x}{a})$, this can also be computed by standard real variable methods. Indeed, by the method of partial fractions, every P/Q which is real on $\mathbb{R}$ with all poles in $\mathbb{C} \setminus \mathbb{R}$ can be reduced to sums of the form (5.7.9) and powers of that integrand. $\square$

5.7.3. Trigonometric Times Rational and Exponential Functions. This class illustrates the power of complex variable methods since the integrand in (5.7.11) is not the derivative of any elementary function.

Example 5.7.4. Compute for $a > 0$ and b real with $b \neq 0$,

$$\int_{-\infty}^{\infty} \frac{\cos bx}{x^2 + a^2}\, dx \tag{5.7.11}$$

Without loss, suppose $b > 0$. Since $\mathrm{Re}(e^{ibx}) = \cos bx$, we need only compute the integral with e^{ibx} replacing $\cos bx$. We can close the contour in the upper

but not lower half-plane (since $|e^{ib(x+iy)}| = e^{-by}$). Computing the residue at $x = ia$ yields

$$\int_{-\infty}^{\infty} \frac{\cos bx}{x^2 + a^2} = \frac{\pi e^{-|b|a}}{a} \tag{5.7.12}$$

□

Example 5.7.5. Compute for $a > 0$,

$$\int_{-\infty}^{\infty} \frac{\sin ax}{x}\, dx \tag{5.7.13}$$

This has four subtleties compared to the last example. First, it is not absolutely convergent, but by general principles (see Problem 3), it is conditionally convergent.

Second, we cannot close the contour naively in either half-plane because the function $\sin az/z$ grows badly in each half-plane. The obvious solution is to write $\sin az/z = (e^{iaz} - e^{-iaz})/2iz$, and for each term close in different half-planes. The problem is that while $\sin az/z$ is regular at $z = 0$, $e^{\pm iaz}/z$ are not. Here, what we do is replace $\int_{-\infty}^{\infty}$ by $\int_{\varepsilon}^{\infty} + \int_{-\infty}^{-\varepsilon} + \int_{C_\varepsilon}$ where C_ε lies in the lower half-plane as the semicircle from $-\varepsilon$ to ε. Since $\sin az/z$ is bounded on that circle uniformly in ε and the contour is $O(\varepsilon)$, we are sure to recover the given integral as $\varepsilon \downarrow 0$ (and, indeed, by the CIT, this sum is ε-independent!).

Third, we cannot close trivially with a large semicircle (although with extra work, one can (see Problem 4(b)). Instead, we close the contour for $e^{iaz}/2iz$ by going from $-R$ to R along $\mathbb{R}$ (with the ε-change above), go upwards to $R + i\sqrt{R}$, horizontally and backward to $-R + i\sqrt{R}$, and then down (see Figure 5.7.3). A simple estimate (Problem 4(a)) shows that as $R \to \infty$, this extra contour contributes zero.

There results two integrals, one involving $e^{iaz}/2iz$ in the upper half-plane and the other $e^{-iaz}/2iz$ in the lower half-plane. The second encloses no poles (with our choice of deformation near zero) and the other only a

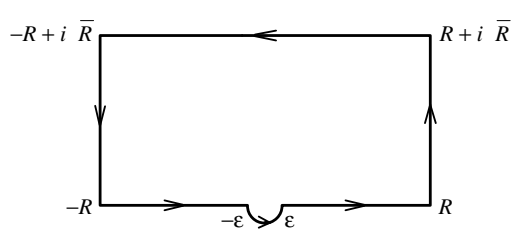

Figure 5.7.3. Contour for Example 5.7.5.

pole at zero. The result is $2\pi i/2i$, so

$$\int_{-\infty}^{\infty} \frac{\sin ax}{x}\, dx = \pi \tag{5.7.14}$$

The last subtlety concerns the fact that while (5.7.14) is independent of a, for $a = 0$, the integral is zero! This lack of continuity was discovered by Cauchy and concerned him. It shows the difficulty of interchanging limits and conditionally convergent integrals (or sums). □

Example 5.7.6. For $0 < a < 1$, compute

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx \tag{5.7.15}$$

Exponentials and trigonometric functions are brothers. Since for $x > 0$, this function vanishes as $e^{-(1-a)x}$, and for $x < 0$ as $e^{-a|x|}$, the integral is absolutely convergent. Illustrating that contour integrals are an art, we close this contour by letting

$$F(z) = \frac{e^{az}}{1+e^z} \tag{5.7.16}$$

and note that

$$F(z + 2\pi i) = e^{2\pi i a} F(z) \tag{5.7.17}$$

Thus, we take the rectangle in Figure 5.7.4 with corners at $\pm R$ and $\pm R + 2\pi i$. The contributions of the vertical edges go to zero as $R \to \infty$. Thus, the contour integral as $R \to \infty$ goes to

$$(1 - e^{2\pi i a}) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx \tag{5.7.18}$$

On the other hand, there is one pole within the contour at $z = i\pi$ where the residue is

$$e^{i\pi a} \left[\frac{d}{dz}(1+e^z)\bigg|_{z=i\pi} \right]^{-1} = -e^{i\pi a}$$

Thus,

$$(5.7.18) = -2\pi i\, e^{i\pi a} \tag{5.7.19}$$

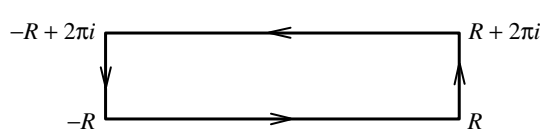

Figure 5.7.4. Contour for Example 5.7.6.

Since $-2\pi i e^{i\pi a}/(1-e^{2\pi i a}) = \pi/\sin(\pi a)$, we see that (see also (5.7.30) and Problem 10)

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin(\pi a)} \tag{5.7.20}$$

□

5.7.4. Examples with Branch Cuts.

Example 5.7.7. Compute for $a > 0$,

$$\int_0^{\infty} \frac{\log(x)}{x^2+a^2}\,dx \tag{5.7.21}$$

For $-\pi/2 < \arg(z) < 3\pi/2$, define

$$F(z) = \frac{\log(z)}{z^2+a^2} \tag{5.7.22}$$

and integrate $F(z)$ over the contour from $-R$ to R along $\mathbb{R}$ and then along the semicircle in the upper half-plane. The contribution of the semicircle is $O\big((R\log(R))/R^2\big)$ as $R \to \infty$, and so goes to zero.

Since $\log(-x) = \log(x) + i\pi$ for this branch of $\log(z)$, the integral over the real axis piece as $R \to \infty$ goes to

$$2\int_0^{\infty} \frac{\log(x)}{x^2+a^2} + i\pi \int_0^{\infty} \frac{dx}{x^2+a^2} \tag{5.7.23}$$

On the other hand, for large R, the contour has one pole at $z = ia$ with residue $\log(ia)/2ia = (\log(a) + i\pi/2)/2ia$. Thus, if I is the integral in (5.7.21), we get

$$2I + i\pi \int_0^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a}\left[\log(a) + \frac{i\pi}{2}\right] \tag{5.7.24}$$

We could just take real parts (but, in fact, the imaginary parts are equal by (5.7.10)). Thus,

$$\int_0^{\infty} \frac{\log(x)\,dx}{x^2+a^2} = \frac{\pi}{2a}\log(a) \tag{5.7.25}$$

In fact, this can be computed by conformal invariance alone (see Problem 6).

□

Example 5.7.8. This integral is also evaluated in Problem 10. For $0 < a < 1$, compute

$$\int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx \tag{5.7.26}$$

We define on $\mathbb{C} \setminus [0,\infty)$,

$$f(z) = \frac{(-z)^{a-1}}{1+z} \tag{5.7.27}$$

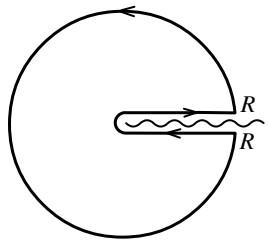

Figure 5.7.5. Contour for Example 5.7.8.

where we take the branch of $(-z)^{a-1}$ which is positive for $z < 0$, that is, for $z = re^{i\theta}$, $0 < \theta < 2\pi$,

$$(1+z)f(z) = r^{a-1}e^{i(\theta-\pi)(a-1)} \tag{5.7.28}$$

Thus, for $x > 0$,

$$f(x \pm i0) = \frac{e^{\pm i\pi(1-a)}}{1+x}\, x^{a-1} \tag{5.7.29}$$

Consider the contour that goes above the cut from 0 to R, circles to around just below R, and back to 0 (see Figure 5.7.5). The contribution of the circle to $\oint f(z)\,dz$ is bounded by $R^{a-1}(2\pi R)/(R-1)$ goes to zero as $R \to \infty$. Thus, as $R \to \infty$, the integral is equal to $[e^{i\pi(1-a)} - e^{-i\pi(1-a)}]$ (integral in (5.7.26)).

f has one pole at $z = -1$ and the residue is 1. Thus, since $e^{i\pi} = -1$, the integral is $(2\pi i)/(e^{i\pi a} - e^{-i\pi a}) = \pi/\sin(\pi a)$ and

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin(\pi a)} \tag{5.7.30}$$

□

The reader may have noticed that the right sides of (5.7.30) and (5.7.20) agree. This is no coincidence—the change of variable $x = e^y$ in (5.7.30) turns one integral into the other! (See also Problem 10.)

Example 5.7.9. In Section 5.7.2, we saw how to evaluate $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx$ if

$$\deg Q \geq \deg P + 2 \tag{5.7.31}$$

where $Q(x)$ has no zeros on $\mathbb{R}$. One need only close the contour in the upper half-plane. Here, we'll consider $\int_0^\infty \frac{P(x)}{Q(x)}\,dx$ if Q has no zeros on $[0,\infty)$ and (5.7.31) holds. A simple example (which can be computed by other means!) is

$$\int_0^\infty \frac{dx}{(x+1)^2} \tag{5.7.32}$$

Let $f(z) = \log(-z)/(z+1)^2$ and consider the integral of f along the contour shown in Figure 5.7.5. Since $|f(z)| = o(\frac{1}{|z|})$ at infinity, the circle makes a vanishing contribution as $R \to \infty$. Since $\log(-x+i0) - \log(-x-i0) = -2\pi i$, the combined integral along the cut is $-2\pi i I$, where I is the integral in (5.7.32). Near $z=-1$, $\log(-z) = -(z+1) + O((z+1)^2)$, so the residue of f at -1 is -1. Therefore, $-2\pi i I = -2\pi i$, that is, $I = 1$ (which can be obtained from the antiderivatives). Clearly, the same method works for any $\int_0^\infty \frac{P(x)}{Q(x)}\,dx$ of the type discussed (see Problem 11). □

Example 5.7.10. Compute

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \tag{5.7.33}$$

Of course, this can be done via trigonometric substitution $x = \sin\theta$ to get $\int_{-\pi/2}^{\pi/2} d\theta = \pi$. But we can do it and more complicated examples (see Problem 8) using contour integration. Put a branch cut for $(1-z^2)^{-1/2}$ from -1 to 1 and take the branch which is positive for $z = x+i0$, $-1 < x < 1$, so the branch which is positive for $z = iy$, $y > 0$. The integrand at $z = x - i0$ is $-(1-x^2)^{-1/2}$, so a contour around the top returning on the bottom (which is clockwise) is $-2 \times$ (5.7.33). This can be deformed to a contour around a large circle, so large that the Laurent series at infinity converges.

Near $z = \infty$,

$$f(z) = \frac{i}{\sqrt{z^2-1}} = \frac{i}{z}\left(1 - \frac{1}{z^2}\right)^{-1/2} \tag{5.7.34}$$

where we pick $\sqrt{-1} = i$ rather than $-i$ to have $f(iy) > 0$ for $y > 0$. Since $\oint \frac{dz}{z^\ell} = 0$ for $\ell > 1$, we see

$$(5.7.33) = -\frac{1}{2}(2\pi i)(i) = \pi \tag{5.7.35}$$

□

5.7.5. Gaussian and Related Integrals.

Example 5.7.11 (Gaussian Integral)**.** Evaluate the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx \tag{5.7.36}$$

using residue calculus.

Remark. The simplest method of evaluating this integral uses polar coordinates; see Proposition 4.11.10 of Part 1 or Theorem 9.6.6. Part 1 also had a third proof using the Fourier inversion formula; see Problem 11 of Section 6.2. We will also find a method using Euler's gamma function in Sec-

tion 9.6 (see especially the remark following Corollary 9.6.5). Problems 26 and 28 sketch two other proofs.

This is a tricky example since there are no poles and no obvious closed contours to use.

The arithmetic will be easier for e^{-x^2} rather than $e^{-x^2/2}$; we'll then scale to get (5.7.36). A special role will be played by the complex number

$$\beta = \sqrt{\pi}\, e^{i\pi/4} \tag{5.7.37}$$

which obeys

$$\beta^2 = \pi i \tag{5.7.38}$$

Thus,

$$e^{-(z+\beta)^2} = e^{-z^2-\beta^2-2\beta z} = -e^{-z^2} e^{-2\beta z} \tag{5.7.39}$$

The minus sign is promising, but $e^{-2\beta z}$ looks problematic. However, since $e^{-2\beta^2} = e^{-2\pi i} = 1$, $h(z) = e^{-2\beta z}$ obeys $h(z+\beta) = h(z)$.

This leads us to define

$$f(z) = \frac{e^{-z^2}}{1 + e^{-2\beta z}} \tag{5.7.40}$$

As noted, the denominator is invariant under $z \to z + \beta$, so by (5.7.39),

$$f(z) - f(z+\beta) = e^{-z^2} \tag{5.7.41}$$

We therefore consider the contour, Γ_R, which is a parallelogram with corners, $-R, R, R+\beta, -R+\beta$ (Figure 5.7.6). By (5.7.41),

$$\int_{-R}^{R} e^{-z^2}\, dz = \int_{\text{bottom–top of } \Gamma_R} f(z)\, dz \tag{5.7.42}$$

It is an easy exercise (Problem 12) to see the contribution of the sides goes to zero as $R \to \infty$, so by the residue theorem,

$$\int_{-\infty}^{\infty} e^{-z^2}\, dz = 2\pi i \sum_{\substack{\text{poles} \\ z_j \text{ of } f(z) \text{ in } \{z \mid 0 < \operatorname{Im} z < \operatorname{Im} \beta\}}} \operatorname{Res}(f; z_j) \tag{5.7.43}$$

The poles of $f(z)$ are solutions of

$$e^{-2\beta z} = -1 \tag{5.7.44}$$

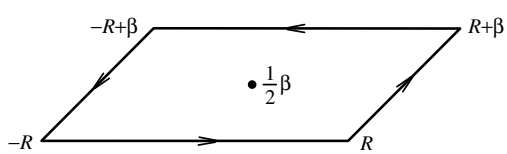

Figure 5.7.6. Contour for Example 5.7.11.

that is,

$$2\beta z = (2n+1)\pi i = (2n+1)\beta^2 \tag{5.7.45}$$

or

$$z_n = (n+\tfrac{1}{2})\beta, \qquad n = 0, \pm 1, \dots \tag{5.7.46}$$

Only z_0 is in the strip $\{z \mid 0 < \operatorname{Im} z < \operatorname{Im} \beta\}$ (see Figure 5.7.6) and the residue there is

$$\frac{e^{-\frac{1}{4}\beta^2}}{\frac{d}{dz}(1+e^{-2z\beta})\big|_{z=\beta/2}} = \frac{e^{-\beta^2/4}}{-2\beta e^{-\beta^2}} = \frac{e^{-i\pi/4}}{2\beta} = \frac{1}{2i\sqrt{\pi}}$$

Thus, $2\pi i \sum_{z \text{ in strip}} \operatorname{Res}(f;z) = \sqrt{\pi}$, and we see that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} \tag{5.7.47}$$

By scaling, for $a > 0$,

$$\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\pi/a} \tag{5.7.48}$$

and, in particular,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi} \tag{5.7.49}$$

From this, one can easily see (Problem 13) that for all $w \in \mathbb{C}$,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{iwx} \frac{dz}{\sqrt{2\pi}} = e^{-\frac{1}{2}w^2} \tag{5.7.50}$$

□

Example 5.7.12 (Fresnel Integrals)**.** Evaluate

$$\int_0^{\infty} \sin(x^2)\, dx \quad \text{and} \quad \int_0^{\infty} \cos(x^2)\, dx \tag{5.7.51}$$

Obviously, these integrals are not absolutely convergent. They are conditionally convergent; indeed, if

$$f(R) = \int_0^R e^{ix^2}\, dx \tag{5.7.52}$$

then, changing variables to $y = x^2$, we have

$$f(R) = \int_0^{R^2} e^{iy} \frac{dy}{2\sqrt{y}} \tag{5.7.53}$$

and Problem 3 implies $\lim_{R\to\infty} f(R)$ exists.

In fact (see Problem 14), uniformly in $\alpha \geq 0$, for R fixed,

$$\sup_{R'>R\geq 1} \left| \int_R^{R'} e^{(i-\alpha)x^2}\, dx \right| \leq cR^{-1} \tag{5.7.54}$$

That implies $\lim_{R\to\infty}\int_0^R e^{(i-\alpha)x^2}\,dx$ converges uniformly in $\alpha \geq 0$, which justifies interchanging limits to conclude

$$\lim_{R\to\infty}\int_0^R e^{ix^2}\,dx = \lim_{\alpha\downarrow 0}\int_0^\infty e^{(i-\alpha)x^2}\,dx \tag{5.7.55}$$

In the region $\operatorname{Re} a > 0$, both sides of (5.7.48) are analytic (the left by Theorem 3.1.6). Since they are equal for a real, they are equal for all a with $\operatorname{Re} a > 0$ and, in particular, for $a = \alpha - i$ with $\alpha > 0$.

Thus, by (5.7.55),

$$\int_0^\infty e^{ix^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{-i}} = \frac{1}{2\sqrt{2}}\,(1+i)\sqrt{\pi} \tag{5.7.56}$$

Taking real and imaginary parts,

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}} \tag{5.7.57}$$

These are called *Fresnel integrals*; more generally, the *Fresnel functions* are defined by

$$C(x) = \int_0^x \cos(y^2)\,dy, \qquad S(x) = \int_0^x \sin(y^2)\,dy \tag{5.7.58}$$

The curve

$$E(t) = C(t) + iS(t) \tag{5.7.59}$$

(see Figure 5.7.7) is called the *Euler* or *Cornu spiral*. It enters in optics and in civil engineering. There is more about these functions in the Notes and in Problems 15 and 16. □

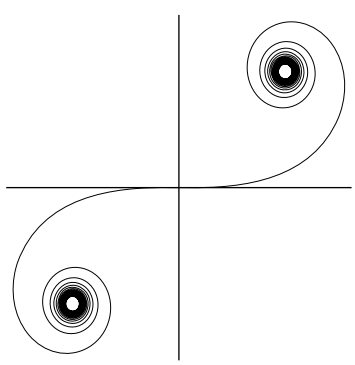

Figure 5.7.7. The Euler spiral, $x = C(t)$, $y = S(t)$.

5.7.6. Infinite Sums Via the Residue Calculus. Remarkably, some infinite sums can be calculated as finite sums of residues. Besides the theorem below, one can use contour integration on sums to prove the Poisson summation formula (Theorem 6.6.10 of Part 1), see Problem 29, and to prove a suitable version of the Nyquist–Shannon sampling theorem (Theorem 6.6.16 of Part 1), see Problem 20.

Theorem 5.7.13. *Let f be a rational function $f(z) = P(z)/Q(z)$, where*

(i) $\deg(Q) \geq 2 + \deg(P)$

(ii) *P and Q have no common zeros and Q is nonvanishing at every $n \in \mathbb{Z}$. Then*

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{\text{zeros } z_k \text{ of } Q} \operatorname{Res}(fg; z_k) \tag{5.7.60}$$

where

$$g(z) = \pi \cot(\pi z) \tag{5.7.61}$$

Remarks. 1. Since $|f(n)| \leq C|n|^{-2}$ for n large, the sum is absolutely convergent.

2. If f has a simple pole at z_k, then

$$\operatorname{Res}(fg; z_k) = g(z_k)\operatorname{Res}(f; z_k) \tag{5.7.62}$$

Proof. Let

$$F(z) = f(z)g(z) \tag{5.7.63}$$

which is meromorphic on all of $\mathbb{C}$. Let Γ_n be the rectangle with corners at $(\pm(n + \frac{1}{2}), \pm n)$ oriented counterclockwise. We claim that

$$\lim_{n\to\infty} \frac{1}{2\pi i} \oint_{\Gamma_n} F(z)\, dz = 0 \tag{5.7.64}$$

We note first that

$$g(z + 1) = g(z) \tag{5.7.65}$$

since $\cos(\pi z)$ and $\sin(\pi z)$ are periodic. When $z = \frac{1}{2} + iy$ $e^{\pm i\pi z} = \pm i e^{\pm \pi y}$, so

$$|g(\tfrac{1}{2} + iy)| = \pi|\tanh(\pi y)| \leq \pi \tag{5.7.66}$$

Therefore, on the vertical sides of Γ_n, $|g(z)|$ is bounded by π.

On the horizontal sides, where $|e^{i\pi z}|$ and $|e^{-i\pi z}|$ are $e^{\pi n}$ and $e^{-\pi n}$ or vice-versa,

$$|g(x \pm in)| \leq \pi\left(\frac{e^{2n\pi} + 1}{e^{2n\pi} - 1}\right) = \pi(1 + O(e^{-2n\pi})) \tag{5.7.67}$$

which, for n large, is certainly bounded by 2π.