Weil conjectures

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1 01.10.2019 – Introduction to the Weil conjectures

Let $\mathbb{F}_q$ be the finite field with q elements and $X/\mathbb{F}_q$ be a smooth, projective, geometrically connected variety. We denote by d the dimension of X. The aim is to count the number of rational points of X: more precisely, we define

$$N_m = \#X(\mathbb{F}_{q^m}).$$

In elementary terms, X is defined by equations in projective space, and we are simply counting the number of solutions to such equations over all extensions of $\mathbb{F}_q$. It turns out that the interesting object to look at is the *exponential* generating function of this collection of numbers N_m:

Definition 1.1 (Zeta function)**.** We set

$$Z_X(T) := \exp\left(\sum_{m\geq 1} N_m \frac{T^m}{m}\right) \in \mathbb{Q}[[T]]$$

and call it the **Zeta function of** X.

Why is this called the Zeta function? There is a close relationship with the Riemann Zeta function which we now discuss.

Let x be a closed point of X. The **degree** of x is by definition

$$\deg(x) := [\kappa(x) : \mathbb{F}_q],$$

where $\kappa(x)$ is the residue field at x (a finite extension of $\mathbb{F}_q$). We have the following equality of formal series:

$$\begin{aligned}\log\left(\frac{1}{1-T^{\deg(x)}}\right) &= \sum_{n\geq 1} \frac{T^{n\deg(x)}}{n} \\ &= \sum_{n\geq 1} \deg(x)\frac{T^{n\deg(x)}}{n\deg(x)},\end{aligned}$$

which shows that the coefficient of $\frac{T^m}{m}$ is

$$\begin{cases} 0, \text{ if } \deg(x) \nmid m \\ \deg(x), \text{ if } \deg(x) \mid m \end{cases}$$

Remark 1.2. The quantity $\deg(x)$ is also the number of points in $X(\mathbb{F}_{q^{\deg(x)}})$ lying above x.

Using the previous remark we find

$$\sum_{x \text{ closed point}} \log\left(\frac{1}{1-T^{\deg(x)}}\right) = \sum_m N_m \frac{T^m}{m},$$

because every rational point corresponds to some x, and the number of rational points corresponding to a given x is precisely its degree. Exponentiating both sides of the previous identity we get

$$Z_X(T) = \prod_{x \text{ closed point}} \frac{1}{1-T^{\deg(x)}},$$

where the right hand side now looks exactly like the Euler product for the Riemann Zeta function.

1.1 Statement of the Weil conjectures

The Weil conjectures are (unsurprisingly) contained in a famous paper by Weil, *Number of solutions of equations over finite fields* (Bulletin of the AMS, 1949). All of Weil's conjectures are now theorems, and we will see their proofs during the course.

Conjecture 1.3 (Rationality)**.** *The Zeta function $Z_X(T)$ is a rational function of T, that is, $Z_X(T) \in \mathbb{Q}(T)$. In fact, there exist polynomials $P_0(T), P_1(T), \ldots, P_{2d}(T)$, all of them with rational coefficients, such that*

$$Z_X(T) = \frac{P_1(T) \cdots P_{2d-1}(T)}{P_0(T) \cdots P_{2d}(T)}$$

Remark 1.4. We will see later that if we normalise the $P_i(T)$ so that $P_i(0) = 1$, then each $P_i(T)$ has integral coefficients.

Conjecture 1.5 (Functional equation)**.** *The Zeta function satisfies the functional equation*

$$Z_X\left(\frac{1}{q^d T}\right) = \pm q^{d\chi/2} T^{\chi} Z_X(T),$$

where χ is the "Euler characteristic of X" (to be defined precisely later).

Remark 1.6. The substitution $T \to p^{-s}$ brings the functional equation into a form very close to the functional equation for the usual Riemann Zeta function.

Conjecture 1.7 (Riemann hypothesis)**.** *Using the normalisation $P_i(0) = 1$ and factoring $P_i(T)$ over $\overline{\mathbb{Q}}$ as $P_i(T) = \prod(1 - \alpha_{ij}T)$, we have:*

1. $P_0(T) = 1 - T$
2. $P_{2d}(T) = 1 - qT$
3. $|\alpha_{ij}| = q^{i/2}$.

Definition 1.8. A **q-Weil number of weight** i is an algebraic number α such that $|\sigma(\alpha)| = q^{i/2}$ for every embedding $\sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Example 1.9. This is a fairly special property: for example, $\alpha = 1+\sqrt{2}$ has very different absolute values under the two possible embeddings $\mathbb{Q}(\sqrt{2}) \hookrightarrow \mathbb{C}$

Remark 1.10.

1. We shall see that the α_{ij} are in fact algebraic *integers*, not just algebraic *numbers*.
2. Statement (3) is independent of the choice of embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$: in particular, α_{ij} is q-Weil number of weight i.
3. The α_{ij} are the reciprocal roots of the polynomials $P_i(T)$.
4. Consider the special case $d = 1$. The only interesting polynomial $P_i(T)$ is then $P_1(T)$ (because $P_0(T), P_2(T)$ are independent of the specific choice of curve), and replacing T by q^{-s} we see that the Riemann hypothesis is indeed the statement that all the zeroes of the Zeta function (as a function of s) have abscissa $1/2$.

Conjecture 1.11. *Suppose X arises by reduction modulo $\mathfrak{p}$ of a flat projective generically smooth scheme $Y/\operatorname{Spec}\mathcal{O}_K$, where K is a finite extension of $\mathbb{Q}$ and $\mathfrak{p}$ is a prime of the ring of integers of K. Then the degree of $P_i(T)$ is the i-th Betti number of the complex variety $(Y \times_{\mathcal{O}_K} \mathbb{C})(\mathbb{C})$.*

Remark 1.12. Starting from $X/\mathbb{F}_q$, one can lift the equations defining X to the ring of integers of some number field, and (with some care) get a Y as in the previous conjecture.

However, one can also run the construction in the other direction: starting from a smooth variety Y/K, we can fix a flat model $\mathcal{Y}$ over $\mathcal{O}_K$. This $\mathcal{Y}$ will not be smooth everywhere, but for all primes $\mathfrak{p}$ of $\mathcal{O}_K$ with finitely many exceptions we will have a smooth fibre $X = \mathcal{Y}_\mathfrak{p}$, for which we can consider the corresponding Zeta function.

1.2 A little history

- The elliptic curve case was proven by Hasse (who gave two proofs, in 1934 and 1935)
- The case when X is an algebraic curve was handled by Weil himself in the 40s (2 proofs). He also tackled the case of abelian varieties.
- Weil also treated the case of diagonal hypersurfaces by elementary methods. Here a *diagonal hypersurface* is a hypersurface defined by an equation of the form $\sum_{i=0} a_i x_i^r$
- There is a proof by Katz that reduces the general case to the case of diagonal hypersurfaces.
- Conjecture 1 was first proven by Dwork in 1959, by p-adic analysis methods. Dwork's proof can be found in Serre's Bourbaki talk [Ser60] on the topic.
- Conjectures 1.3, 1.5 and 1.11 were proven by Grothendieck, Artin, and many others in SGA during the 1960s.
- Conjecture 1.7 was proven by Manin for unirational varieties (1966) and by Deligne for K3 surfaces (1969). The methods used by Deligne for this special case are completely different from those he used to prove the general case, which he established in 1974 in a paper [Del74] usually referred to as 'Weil I'.
- Grothendieck had a strategy to prove Conjecture 1.7 that goes through his famous Standard Conjectures. Unfortunately, these are still very much open!
- Deligne also obtained [Del80] vast generalisations of his results that apply to non-smooth varieties. The arguments in Weil II are much harder than those in Weil I.
- A second, simpler proof of Weil II was obtained by Laumon (1984).
- In Weil II, Deligne also states some open problems, many of which are still open. An important input for those that *have* been solved comes from L. Lafforgue's work on the Langlands correspondence for GL_n.

1.3 Grothendieck's proof of Conjectures 1.3 and 1.5

The key point, as observed by Grothendieck (and already implicit in Weil's work), is that there is a cohomological interpretation of the Zeta function. Fix a prime number $\ell \neq p$. Let $\mathbb{Q}_\ell$ be the field of ℓ-adic numbers, which – as it is well known – is a field of characteristic zero. Let $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. Grothendieck defined certain ℓ-adic étale cohomology groups $H^i\left(\overline{X}, \mathbb{Q}_\ell\right)$, which are finite-dimensional vector spaces over $\mathbb{Q}_\ell$. We briefly recall some properties of these cohomology groups:

1. $H^i(\overline{X}, \mathbb{Q}_\ell$ are finite-dimensional vector spaces over $\mathbb{Q}_\ell$, and are trivial for $i > 2d$.
2. $\overline{X} \mapsto H^i(\overline{X}, \mathbb{Q}_\ell)$ is a contravariant functor.
3. Poincaré duality: the cup product induces perfect pairings
$$H^i(\overline{X}, \mathbb{Q}_\ell) \times H^{2d-i}(\overline{X}, \mathbb{Q}_\ell) \to H^{2d}(\overline{X}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell.$$
4. Lefschetz fixed point formula: let $f : \overline{X} \to \overline{X}$ be a self-map with isolated fixed points of multiplicity 1. Then
$$\#\{\text{fixed points of } f\} = \sum_{i=0}^{2d} (-1)^i \operatorname{tr}\left(f^* \mid H^i(\overline{X}, \mathbb{Q}_\ell)\right).$$
Notice that this last formula makes sense since $H^i(\overline{X}, \mathbb{Q}_\ell)$ is a contravariant functor.

The idea is now to apply (4) with f given by $F : \overline{X} \to \overline{X}$, where F is the (geometric) Frobenius (namely, the identity on X and $g \mapsto g^p$ on functions). It is a fact (a consequence of the definition of the étale cohomology groups) that F acts as multiplication by q^{2d} on $H^{2d}(\overline{X}, \mathbb{Q}_\ell)$. Moreover,
$$X(\mathbb{F}_{q^m}) = \{\text{fixed points of } F^m\}.$$

Proof of Conjecture 1.3. Apply the Lefschetz fixed point formula to F^m. It yields
$$N_m = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}\left(F^{*m} \mid H^i\left(\overline{X}, \mathbb{Q}_\ell\right)\right),$$
and therefore
$$\begin{aligned} Z_X(T) &= \exp\left(\sum_{m\geq 1} N_m \frac{T^m}{m}\right) \\ &= \prod_{i=0}^{2d} \exp\left(\sum_{m=1}^{\infty} \operatorname{Tr}\left(F^{*m} \mid H^i(\overline{X}, \mathbb{Q}_\ell)\right)\right)^{(-1)^i} \\ &= \prod_{i=0}^{2d} \det(\mathrm{Id} - F^*T \mid H^i(\overline{X}, \mathbb{Q}_\ell))^{(-1)^{i+1}} \end{aligned}$$
where in the last equality we have used the well-known Lemma 1.14. Hence one can take
$$P_i(T) = \det(\mathrm{Id} - F^*T \mid H^i(\overline{X}, \mathbb{Q}_\ell)),$$

but so far we have only shown that $P_i(T)$ is a polynomial with coefficients in $\mathbb{Q}_\ell$. We now have

$$Z_X(T) \in \mathbb{Z}[[T]] \cap \mathbb{Q}_\ell(T),$$

where the fact that $Z_X(T) \in \mathbb{Z}[[T]]$ follows from the Euler product description. We would like to deduce that $Z_X(T) \in \mathbb{Q}(T)$. This is done via the theory of **Hankel determinants**: let $f \in K[[T]]$, where K is any field. Write $f = \sum_{i=0}^{\infty} a_i T^i$ and define

$$H_k = \det(a_{i+j+k})_{0 \le i,j \le M}$$

for $k > N$, where both M and N are arbitrary parameters. The theory of Hankel determinants shows that $f \in K(T)$ if and only if $H_k = 0$ for all M, N sufficiently large. Clearly this criterion is independent of the field of coefficients, hence the fact that $Z_X(T)$ is a rational function in $\mathbb{Q}_\ell(T)$ implies the same statement in $\mathbb{Q}(T)$. More precisely: considering $Z_X(T)$ as a formal power series in $\mathbb{Q}_\ell[[T]]$ we find that all Hankel determinants for $M, N \gg 0$ vanish, because $Z_X(T)$ is a rational function in $\mathbb{Q}_\ell(T)$. But on the other hand the coefficients of $Z_X(T)$ are integers, so $Z_X(T)$ is in particular a formal power series with coefficients in $\mathbb{Q}$ *for which all Hankel determinants with* $M, N \gg 0$ *vanish*, so by the converse statement we have that $Z_X(T)$ is a rational function in $\mathbb{Q}(T)$.

To finish the proof (namely, show that $P_i(T)$ has rational coefficients), we use the following elementary Lemma due to Deligne:

Lemma 1.13. *Assume we already know that the α_{ij} are algebraic numbers with absolute value $q^{i/2}$. Write $Z_X(T) = \frac{P(T)}{Q(T)}$, where $P(T), Q(T) \in \mathbb{Q}[T]$. By a classical lemma of Fatou, we may assume $P(0) = Q(0) = 1$. Also write $Z_X(T) = \frac{P_1(T)\dots P_{2d-1}(T)}{P_0(T)\dots P_{2d}(T)}$. Since the different $P_i(T)$ have distinct roots (because they have different absolute value), they are pairwise coprime, so $P(T) = \prod_{i \text{ odd}} P_i(T)$ and $Q(T) = \prod_{j \text{ even}} P_j(T)$. Finally, since the condition $|\alpha_{ij}| = q^{i/2}$ is invariant under the action of* $\mathrm{Gal}(\overline{\mathbb{Q}} \mid \mathbb{Q})$*, the polynomials $P_i(T)$ have rational coefficients.*

□

Lemma 1.14. *Let V be a finite-dimensional vector space, $\varphi \in \mathrm{End}(V)$. Then*

$$\exp\left(\sum_{m=1}^{\infty} \mathrm{Tr}\left(\varphi^m \mid V\right)\right) \frac{T^m}{m} = \det(\mathrm{Id} - \varphi T)^{-1}$$

Proof. The statement is obvious for 1-dimensional vector spaces, and both sides of the equality are multiplicative in short exact sequences of the form $0 \to V' \to V \to V/V' \to 0$. Since the statement does not depend on the ground field, one can work over the algebraic closure and use the existence of eigenvectors to proceed by induction on the dimension of V. □

2 08.10.2019 – Diagonal hypersurfaces

2.1 Proof of Conjecture 1.5

Recall our setup: we let $\mathbb{F}_q$ be a finite field, $X/\mathbb{F}_q$ a smooth projective variety of dimension d, $N_m = \#X(\mathbb{F}_{q^m})$, and

$$\zeta_X(T) = \exp\left(\sum_{m\geq 1} N_m \frac{T^m}{m}\right).$$

We have discussed the Weil conjectures:

1. Rationality: $Z_X(T) \in \mathbb{Q}(T)$.
2. Functional equation: $Z_X\left(\frac{1}{q^d T}\right) = \pm q^{d\chi/2} T^\chi Z_X(T)$.
3. Riemann hypothesis: writing $Z_X(T) = \frac{P_1(T)\cdots P_{2d-1}(T)}{P_0(T)\cdots P_{2d}(T)}$ we have $P_i(T) \in \mathbb{Q}[T]$, and normalising $P_i(T)$ so that $P_i(0) = 1$ we can write

$$P_i(T) = \prod(1 - \alpha_{ij}T)$$

with $|\alpha_{ij}| = q^{i/2}$ under any embedding of α_{ij} in $\mathbb{C}$.

Today we begin with the proof of the functional equation, assuming all the good properties of (ℓ-adic) étale cohomology.

Proof. Proof of Conjecture 1.5 The proof is based on Poincaré duality, that is, the fact that for every $i \in \{0, \ldots, 2d\}$ the vector space $H^i\left(\overline{X}, \mathbb{Q}_\ell\right)$ is dual to $H^{2d-i}(\overline{X}, \mathbb{Q}_\ell)$.

Lemma 2.1. *Let $H^* := \bigoplus_{i=0}^{2d} H^i$ be a graded algebra over a field, and suppose that each H^i is finite-dimensional. Assume that for every i there exists a nondegenerate pairing*

$$\langle\, ,\, \rangle : H^i \times H^{2d-i} \to K,$$

induced by the composition of the product map $H^i \times H^{2d-i} \to H^{2d}$ with a trace map $H^{2d} \to K$.

Let $\varphi = \varphi_0 \oplus \cdots \oplus \varphi_{2d}$ be a graded endomorphism of degree 0 (that is, $\varphi(H^i) \subseteq H^i$ for all i). Assume:

(i) $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$

(ii) $\varphi_{2d} = \mathrm{id}$

Then φ is an automorphism of H^*, so that in particular every φ_i is invertible, and $\varphi_i^{-1} = {}^t\varphi_{2d-1}$, where ^{t} denotes transposition with respect to $\langle\, ,\, \rangle$.

Proof. Let $a \in H^i$ be any nonzero element. Since we have assumed the product to be nondegenerate, there exists $b \in H^{2d-i}$ such that $a \cdot b \neq 0$, so

$$H^{2d} \ni a \cdot b \overset{\text{(ii)}}{=} \varphi(a \cdot b) \overset{\text{(i)}}{=} \varphi(a)\varphi(b),$$

so $\varphi(a)$ is nonzero and φ is injective. Since $\dim H^i$ is finite, this implies that every φ_i (hence also φ) is an automorphism. Finally, for every $a \in H^i$ and $b \in H^{2d-i}$ we have

$$\begin{aligned}\langle \varphi_i^{-1}(a), b\rangle &= \operatorname{Tr}\left(\varphi_i^{-1}(a)\cdot b\right)\\ &= \operatorname{Tr}\left(\varphi_{2d}\left(\varphi_i^{-1}(a)\cdot b\right)\right)\\ = \operatorname{Tr}\left(\varphi_i(\varphi_i^{-1}(a))\cdot \varphi_{2d-i}(b)\right) &= \langle a, \varphi_{2d-i}(b)\rangle,\end{aligned}$$

so – using the fact that $\langle\, ,\, \rangle$ is nondegenerate – we obtain $\varphi_i^{-1} = {}^t\varphi_{2d-i}$ as desired. □

To prove Conjecture 1.5, we apply the Lemma with $H^i = H^i\left(\overline{X}, \mathbb{Q}_\ell\right)$ and $\varphi_i = \frac{F}{\sqrt{q^i}}$ and use that

$$\{\text{eigenvalues of } {}^t\varphi_{2d-i}\} = \{\lambda^{-1} : \lambda \text{ eigenvalue of } \varphi_i\}.$$

It follows that

$$\left\{\frac{q^{i/2}}{\alpha_{ij}} : 1 \le j \le \deg P_i(T)\right\} = \left\{\frac{\alpha_{2d-i}}{q^{\frac{2d-i}{2}}} \;\middle|\; 1 \le j \le \deg P_{2d-i} = \deg P_i\right\} :$$

here we have used $\deg P_i = \deg P_{2d-i}$, which again follows from Poincaré duality since these degrees are the dimensions of $H^i(\overline{X}, \mathbb{Q}_\ell)$, $H^{2d-i}(\overline{X}, \mathbb{Q}_\ell)$ respectively. Equivalently, if α_{ij} is an eigenvalue of F on $H^i(\overline{X}, \mathbb{Q}_\ell)$, then $\frac{q^d}{\alpha_{ij}}$ is an eigenvalue of F on $H^{2d-i}(\overline{X}, \mathbb{Q}_\ell)$. Using this, the statement is reduced to a direct computation. □

Exercise 2.2. *Fill in the details of the previous proof.*

2.2 Diagonal hypersurfaces

In [Wei49], Weil verified his conjectures for the special case of hypersurfaces given by equations of the form

$$\sum_{i=0}^{r} a_i x_i^d = 0,$$

where $(d, q) = 1$. Today we'll study the special case of the Fermat curve $x^d + y^d = z^d$.

Lemma 2.3 (Lemma Z)**.** *$Z_X(T)$ is rational if and only if for some $\alpha_1, \ldots, \alpha_s$ and $\beta_1, \ldots, \beta_r \in \mathbb{C}$ the equality*

$$N_m(X) = \sum_j \beta_j^m - \sum_i \alpha_i^m$$

holds for every m.

Proof. We have $Z_X(0) = 1$ by definition, so $Z_X(T)$ is rational if and only if we can write $Z_X(T) = \frac{P(T)}{Q(T)}$ with $P(T), Q(T) \in \mathbb{Q}[T]$ with $P(0) = Q(0) = 1$. Assuming rationality, we have

$$Z_X(T) = \frac{\prod(1-\alpha_i T)}{\prod(1-\beta_j T)},$$

where the $\{\alpha_i\}, \{\beta_j\}$ are reciprocal roots of $P(T)$ and $Q(T)$ respectively. Taking the logarithmic derivative of both sides we get

$$\frac{Z_X'(T)}{Z_X(T)} = -\sum_i \frac{\alpha_i}{1-\alpha_i T} + \sum_j \frac{\beta_j}{1-\beta_j T},$$

and multiplying both sides by T and expanding the right hand side as a formal power series in T we have

$$\frac{TZ_X'(T)}{Z_X(T)} = \sum_{m=1}^{\infty} \left(\sum_j \beta_j^m - \sum_i \alpha_i^m \right) T^m.$$

Recalling that $Z_X(T) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m T^m}{m}\right)$, one gets

$$T\frac{Z_X'(T)}{Z_X(T)} = \sum_{m=1}^{\infty} N_m T^m.$$

Comparing coefficients of T^m we get the desired statement. The converse implication is now immediate. □

2.3 Gauss & Jacobi sums

Let $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a multiplicative character.

Definition 2.4 (Gauss sum)**.** We set

$$g(\chi) = \sum_{a \in \mathbb{F}_q} \chi(a) \zeta_p^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)},$$

where – as is customary in number theory – we have set $\chi(0) = 0$.

Remark 2.5. The function

$$\psi(a) := \zeta_p^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)}$$

is an additive character of $\mathbb{F}_q$, that is, a homomorphism $(\mathbb{F}_q, +) \to \mathbb{C}^\times$.

Lemma 2.6 (Lemma G)**.** *(a)* $g(\overline{\chi}) = \chi(-1)\overline{g(\chi)}$*;*

(b) $g(\chi)\overline{g(\chi)} = q$ *if* $\chi \neq 1$*;*

(c) $g(\chi)g(\overline{\chi}) = \chi(-1)q$ *if* $\chi \neq 1$*.*

Proof. Clearly (a) and (b) together imply (c).

(a)

$$\begin{aligned}
\overline{g(\chi)} &= \sum_a \overline{\chi}(a)\overline{\psi(a)} \\
&= \sum_a \overline{\chi}(a)\psi(-a) \\
&= \sum_a \overline{\chi}(-1)\overline{\chi}(-a)\psi(-a) \\
&= \overline{\chi}(-1)g(\overline{\chi}) \\
&= \chi(-1)g(\overline{\chi}),
\end{aligned}$$

where in the last equality we have used the fact that $\chi(-1) \in \{\pm 1\}$ is a real number.

(b)

$$\begin{aligned}
g(\chi)\overline{g(\chi)} &= \sum_{a,b\neq 0} \chi(ab^{-1})\psi(a-b) \\
&\overset{c=ab^{-1}}{=} \sum_{b,c\neq 0} \chi(c)\psi(bc-b) \\
&= \sum_{b,c\neq 0} \chi(c)\psi(b(c-1)) \\
&= \sum_{b\neq 0} \chi(1)\psi(0) + \sum_{c\neq 0,1} \chi(c) \sum_{b\neq 0} \psi(b(c-1)) \\
&= (q-1) + \sum_{c\neq 0,1} \chi(c)(-1) \\
&= q,
\end{aligned}$$

where we have repeatedly used the fact that $\sum_{c\in\mathbb{F}_q} \psi(c) = 0$ if ψ is a nontrivial additive character, and similarly for multiplicative characters.

□

Definition 2.7 (Jacobi sums)**.** If χ_1, χ_2 are multiplicative characters $\mathbb{F}_q^\times \to \mathbb{C}$, we set

$$J(\chi_1, \chi_2) = \sum_{a\in\mathbb{F}_q} \chi_1(a)\chi_2(1-a).$$

Lemma 2.8 (Lemma J)**.** *(a)* $J(1,1) = q-2$

(b) $J(1,\chi) = J(\chi,1) = -1$

(c) $J(\chi,\overline{\chi}) = -\chi(-1)$

(d) If $\chi_1\chi_2 \neq 1$*, then*

$$J(\chi_1,\chi_2) = \frac{g(\chi_1)g(\chi_2)}{g(\chi_1\chi_2)}$$

Remark 2.9. Lemma 2.6 (b) implies in particular $|J(\chi_1,\chi_2)| = \sqrt{q}$.

Proof. (a) and (b) are easy. We prove (c) and (d) simultaneously:

$$\begin{aligned}
g(\chi_1)g(\chi_2) &= \sum_{a,b} \chi_1(a)\chi_2(b)\psi(a+b) \\
&= \sum_{a,c} \chi_1(a)\chi_2(c-a)\psi(c) && (c = a+b) \\
&= \sum_{a\in\mathbb{F}_q, c\in\mathbb{F}_q^\times} \chi_1(a)\chi_2(c-a)\psi(c) + \sum_{a\in\mathbb{F}_q} \chi_1(a)\chi_2(-a).
\end{aligned}$$

Let's compute the two sums separately. The latter one can be evaluated exactly:

$$\sum_{a\in\mathbb{F}_q} \chi_1(a)\chi_2(-a) = \chi_2(-1)\sum_a (\chi_1\chi_2)(a) = \begin{cases} \chi_1(-1)(q-1), & \text{if } \chi_1\chi_2 = 1 \\ 0, & \text{otherwise,} \end{cases}$$

while the former can we rewritten as

$$\begin{aligned}\sum_{\substack{a\in\mathbb{F}_q\\ c\in\mathbb{F}_q^\times}} \chi_1(a)\chi_2(c-a)\psi(c) &= \sum_{\substack{d\in\mathbb{F}_q\\ c\in\mathbb{F}_q^\times}} \chi_1(cd)\chi_2(c(1-d))\psi(c) \qquad (a=c\cdot d)\\ &= \sum_{c\in\mathbb{F}_q^\times, d\in\mathbb{F}_q} \chi_1\chi_2(c)\psi(c)\chi_1(d)\chi_2(1-d)\\ &= g(\chi_1\chi_2)J(\chi_1,\chi_2).\end{aligned}$$

This finishes the proof of (d). For (c), observe that $g(\chi\overline{\chi}) = g(1) = -1$, so that using the previous computations together with Lemma 2.6 we get

$$-J(\chi,\overline{\chi}) + \chi(-1)(q-1) = g(\chi)g(\overline{\chi}) = \chi(-1)q,$$

which proves (c). □

Proposition 2.10 (Main proposition)**.** *Let $X_d = \{x^d + y^d = z^d\} \subset \mathbb{P}^2_{\mathbb{F}_q}$, where $d \geq 2$ and $(d,q) = 1$. Let $e = (q-1,d)$ and $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a character of order e. Then*

$$N_1(X_d) = q + 1 + \sum_{\substack{a,b=1\\ a+b\neq e}}^{e-1} J(\chi^a,\chi^b)$$

To establish the Riemann hypothesis we also need to investigate what happens over extensions. Let $\mathbb{F}_{q^m}/\mathbb{F}_q$ be a finite extension, and let

$$\chi_m = \chi \circ N_{\mathbb{F}_{q^m}/\mathbb{F}_q} : \mathbb{F}_{q^m}^\times \to \mathbb{C}^\times.$$

By definition, $g(\chi_m)$ is a Gauss sum for $\mathbb{F}_{q^m}$.

Proposition 2.11 (Hasse-Davenport relation)**.**

$$-g(\chi_m) = (-g(\chi))^m$$

Corollary 2.12.

$$N_m(X_d) = \#X_d(\mathbb{F}_{q^m}) = q^m + 1 - \sum_{\substack{a,b=1\\ a+b\neq e}}^{e-1} \Big(-J(\chi^a,\chi^b)\Big)^m,$$

so in the spirit of Lemma 2.3 we can write

$$N_m = \underbrace{\sum \alpha_i^m}_{q^m+1} - \underbrace{\sum \Big(-J(\chi^a,\chi^b)\Big)^m}_{\sum \beta_j^m}.$$

The fact that $|\beta_j| = \sqrt{q}$ by Lemma 2.8 then implies the Riemann hypothesis.

Remark 2.13. Replacing $\mathbb{F}_q$ with $\mathbb{F}_{q^m}$ can change the value of e (e.g. $q = 5, m = 2, d = 3$). One checks that this does not affect the proof, and that one can in fact assume $e = d$.

3 15.10.2019 – A primer on étale cohomology

3.1 How to define a good cohomology theory

The basic example to have in mind is the following. Let X be a topological space, and let $\mathcal{C}_X$ be the category whose objects are open subsets of X and in which $\mathrm{Mor}(V, U) = \{\iota : V \hookrightarrow U \text{ open inclusion}\}$. In particular, $\mathrm{Mor}(V, U)$ is nonempty if and only if V is a subset of U, and in that case it consists of precisely one element. A **presheaf** (of abelian groups) on X is a contravariant functor $\mathcal{C}_X \to \mathbf{Ab}$, that is,

- for each open subset U of X we have an abelian group $\mathcal{F}(U)$;
- for each inclusion $V \subseteq U$ we have a restriction morphism $\mathcal{U} \to \mathcal{V}$.

The presheaf $\mathcal{F}$ is a sheaf if and only if for every open cover $\{U_i \to U\}_{i \in I}$ the following sequence is exact:

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j),$$

where the two arrows $\prod \mathcal{F}(U_i) \to \prod_{(i,j)} \mathcal{F}(U_i \cap U_j)$ are induced by the inclusion of $U_i \cap U_j$ into U_i and U_j respectively. Classically, the category of sheaves has enough injectives, and we can define $H^q(X, -)$ as the qth derived functor of the global sections functor $X \mapsto \mathcal{F}(X)$.

Definition 3.1. Let A be an abelian group. The constant sheaf $\mathcal{A}$ associated with A is given by

$$\mathcal{A}(U) := A^{\pi_0(U)},$$

with the obvious restriction maps induced by

Problem 3.2. *The above construction of cohomology is uninteresting if $\mathcal{F}$ is a constant sheaf and X is the underlying topological space of a (reasonable) scheme. The reason is that $\mathcal{A}$ is **flabby**, hence it has trivial cohomology in degree > 0.*

Definition 3.3. A **Grothendieck topology** consists of:

- a category $\mathcal{C}$ with fibre products;
- a set[1] $\mathrm{Cov}\,\mathcal{C}$, whose elements are families of morphisms $\{U_i \xrightarrow{\varphi_i} U\}$ (U is fixed, and we consider several U_i). These elements are called **coverings**, and are supposed to satisfy the following axioms:
  1. the identity $\{U \xrightarrow{\mathrm{id}} U\}$ is in $\mathrm{Cov}\,\mathcal{C}$ for every object U in $\mathcal{C}$;
  2. given $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ and $\{V_{ij} \xrightarrow{\psi_{ij}} U_i\}$ are in $\mathrm{Cov}\,\mathcal{C}$, then so is the family $\{V_{ij} \xrightarrow{\varphi_i \circ \psi_{ij}} U\}$
  3. given $\{U_i \to U\}$ in $\mathrm{Cov}\,\mathcal{C}$ and $V \to U$ a morphism in $\mathcal{C}$, the family $\{V \times_U U_i \to V\}$ is again in $\mathrm{Cov}\,\mathcal{C}$.

[1] there are nontrivial set-theoretical problems here, but we will not discuss them

Definition 3.4. In this situation, a **presheaf** is a contravariant functor $\mathcal{F} : \mathcal{C} \to \mathbf{Ab}$. A presheaf is a sheaf if

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathcal{F}(U_i \times_U U_j),$$

is exact for all coverings $\{U_i \to U\}_{i \in I}$ in the topology.

Theorem 3.5 (Grothendieck)**.** *The category of sheaves has enough injectives, and we can define*

$$H^q(U, \mathcal{F}) := R^q \Gamma(U, \mathcal{F}),$$

where

$$\Gamma(U, -) : \mathcal{F} \mapsto \mathcal{F}(U).$$

3.1.1 Examples

1. the above topological example

2. Let X be a scheme and $\mathcal{C}$ be the category of X-schemes. The family of coverings is given by the collections $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ such that $\bigcup_{i \in I} \varphi_i(U_i) = U$ and every φ_i is étale. We will always use this (Grothendieck) topology, and we will denote by $H^q(X, -)$ the corresponding cohomology theory, usually called **étale cohomology**.

3.2 Basic properties of étale cohomology

1. Let $X = \mathrm{Spec}(k)$, where k is a field with separable closure k_s. Then $H^q(X, \mathcal{F}) := H^q(k, \mathcal{F}(k_s))$, where the latter is (continuous) Galois cohomology.

2. Let A be a finite abelian group and denote by A also the constant sheaf associated with A (same definition as in Definition 3.1). Let X be a smooth variety over $\mathbb{C}$. There exists a canonical isomorphism

$$H^q(X, A) = H^q_{\mathrm{sing}}(X(\mathbb{C}), A).$$

Smoothness is not very important here, but finiteness of A is essential: the statement is not true, for example, with $\mathbb{Z}$ coefficients. In fact, already $H^1(X, \mathbb{Z})$ is different in the two theories: on the one hand, $H^1_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z}) \cong \mathrm{Hom}(\pi_1(X(\mathbb{C})), \mathbb{Z})$, while

$$H^1(X, \mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{cont}}\left(\pi_1^{\mathrm{alg}(X)}, \mathbb{Z}\right) = (0),$$

because $\pi_1(X)^{\mathrm{alg}}$ is (by construction) a profinite group.

3.3 Stalks of an étale sheaf

In the topological case, the stalk of a sheaf $\mathcal{F}$ at a point x is

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U);$$

this is modelled over the set of germs of functions at a point. For étale sheaves we have the following analogue. Given a point $\operatorname{Spec} k(x) \to X$, an **étale neighbourhood of** x is a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k(x)_s \arrow[r] \arrow[d] & U \arrow[d, "\text{étale}"] \\ \operatorname{Spec} k(x) \arrow[r] & X \end{tikzcd}$$

Given a geometric point $\overline{x} : \operatorname{Spec} k(x)_s \to X$ lying above x, we define

$$\mathcal{F}_{\overline{x}} := \varinjlim_{\overline{x} \in U} \mathcal{F}(U),$$

where the indexing category is given by étale neighbourhoods of (the topological image of) $\overline{x}$.

3.4 How to get cohomology groups with coefficients in a ring or field of characteristic 0

Let ℓ be a prime number. The **ring of ℓ-adic integers** is

$$\mathbb{Z}_\ell := \varprojlim_n \mathbb{Z}/\ell^n\mathbb{Z},$$

and the field of ℓ-adic numbers is $\mathbb{Q}_\ell := \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathbb{Q}$. In the same way, when X is a scheme we define

$$H^q(X, \mathbb{Z}_\ell) := \varprojlim_n H^q(X, \mathbb{Z}/\ell^n\mathbb{Z})$$

and

$$H^q(X, \mathbb{Q}_\ell) := H^q(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 3.6. This is not the same as the q-th étale cohomology group of the constant sheaf of group $\mathbb{Z}_\ell$! In particular, $H^1(X, \mathbb{Z}_\ell)$ (cohomology of the constant sheaf) is often zero, while $\varprojlim_n H^q(X, \mathbb{Z}/\ell^n\mathbb{Z})$ usually captures the 'interesting' information.

In general, we can define "ℓ-adic sheaves" and take their cohomology. This is a bit technical, so we only give a sketch. Let $(\mathcal{F}_r)_{r \geq 1}$ be a sequence of sheaves such that $\mathcal{F}_r$ is locally constant[2] with finite stalks. Suppose that for every $r \geq 1$ the sheaf $\mathcal{F}_r$ is killed by ℓ^r, and that $\mathcal{F}_{r+1}/\ell^r\mathcal{F}_{r+1} \cong \mathcal{F}_r$. Then we can define

$$H^q(X, \mathcal{F}) := \varprojlim_n H^q(X, \mathcal{F}_r);$$

one also gets $\mathbb{Q}_\ell$-sheaves by tensoring with $\mathbb{Q}_\ell$, or equivalently, by working in the category of ℓ-adic sheaves up to isogeny.

Remark 3.7. A good reference for ℓ-adic sheaves is [FK88].

[2] the pullback to an étale cover is constant. In this case, since stalks are finite the étale cover can be chosen to be finite

3.4.1 Examples

If X is a variety over a field k with $(n, \operatorname{char} k) = 1$, then we have an étale sheaf of n-th roots of unity,

$$\mu_n(U) := \{f \in \mathcal{O}_U(U) : f^n = 1\}.$$

Notice that $\operatorname{Gal}(k_s/k)$ acts naturally on μ_n. The system $(\mu_{\ell^r})_{r\geq 1}$ defines an ℓ-adic sheaf which is usually denoted by $\mathbb{Z}_\ell(1)$ (and called a **Tate twist**). One also defines $\mathbb{Q}_\ell(1) := \mathbb{Z}_\ell(1) \otimes \mathbb{Q}$ and

$$\mathbb{Z}_\ell(i) := \mathbb{Z}_\ell(1)^{\otimes i}, \quad \mathbb{Q}_\ell(i) := \mathbb{Q}_\ell(1)^{\otimes i}.$$

3.5 Operations on sheaves

Let $\varphi : X \to Y$ be a morphism, $\mathcal{F}$ be an étale sheaf on X. The **pushforward** $\varphi_*\mathcal{F}$ is the sheaf on Y given by

$$\varphi_*\mathcal{F}(U) := \mathcal{F}(X \times_Y U).$$

The functor φ_* has a left adjoint, denoted by φ^*, which to a sheaf $\mathcal{G}$ on Y associates a pullback sheaf $\varphi^*\mathcal{G}$ on X in such a way that

$$\operatorname{Hom}(\varphi^*\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}(\mathcal{F}, \varphi_*\mathcal{G}).$$

The pullback is exact, but the pushforward is not exact on the right, and one may consider the right derived functors $R^q\varphi_*$ of φ_*. If f is locally constant with finite stalks, and φ is proper, then

$$(R^q\varphi_*)_{\overline{y}} \cong H^q\left(X_{\overline{y}}, \mathcal{F}_{X_{\overline{y}}}\right)$$

Theorem 3.8 (Proper smooth base change)**.** *If $\varphi : X \to Y$ is proper and smooth, $\mathcal{F}$ is a locally constant sheaf on X with finite stalks, then the derived pushforwards $R^q\varphi_*\mathcal{F}$ are locally constant with finite stalks on Y.*

3.5.1 The 4th Weil conjecture

Suppose $X/\mathbb{F}_q$ lifts to a smooth proper scheme over an open subscheme of $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a number field K, then $\deg P_i(T) = i$-th Betti number of the corresponding complex variety. Recall that $\deg P_i(T)$ is simply the dimension of the cohomology space $H^i(\overline{X}, \mathbb{Q}_\ell)$, so the fourth Weil conjecture simply states that the dimension of the étale and Betti cohomology groups agree. For the proof, let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ such that $X \to \mathbb{F}_q$ extends to a smooth proper scheme $\mathcal{X}$ over $\operatorname{Spec} \widehat{\mathcal{O}_{K,\mathfrak{p}}}$, where $\widehat{\cdot}$ denotes completion. We have a morphism $\varphi : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$, and we can consider the sheaf $\mathbb{Q}_\ell$ on X and $R^p\varphi_*\mathbb{Q}_\ell$ on $\operatorname{Spec} \mathcal{O}_K$.

Now $\mathcal{O}_{K,\mathfrak{p}}$ is a DVR, so its spectrum consists of two points: a generic point η, with residue field equal to a finite extension of $\mathbb{Q}_p$, and a special point s with residue field $\mathbb{F}_q$. Now smooth proper base change gives

$$\left(R^i\varphi_*\mathbb{Q}_\ell\right)_{\overline{\eta}} \cong \left(R^i\varphi_*\mathbb{Q}_\ell\right)_{\overline{s}},$$

and the two sides of this equality are $H^i(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$ and $H^i(\overline{X}, \mathbb{Q}_\ell)$. The right hand side has dimension $\deg P_i(T)$. We claim that for the left hand side we have

$$H^i(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell) \cong H^i(X_{\mathbb{C}}, \mathbb{Q}_\ell).$$

To see this, one can either use the axiom of choice to prove that the algebraic closure of $\mathbb{Q}_p$ is isomorphic to $\mathbb{C}$, or use the Lefschetz principle to go down from $\overline{\mathbb{Q}_p}$ to a finitely generated extension of $\mathbb{Q}$, then back up to its algebraic closure, and then from there to $\mathbb{C}$. The claim then follows from the fact that étale cohomology is invariant under extension of the field of definition between algebraically closed fields.

4 22.10.2019 – Deligne's integrality theorem

4.1 More generalities on étale cohomology

4.1.1 Cohomology with compact support

Given an open immersion $j : U \hookrightarrow X$, one can define an **extension by zero** functor

$$\begin{array}{ccc} \mathbf{Sh}(U) & \to & \mathbf{Sh}(X) \\ \mathcal{F} & \mapsto & j_!\mathcal{F}. \end{array}$$

This has the property that for every geometric point $\overline{x}$ of X we have

$$(j_!\mathcal{F})_{\overline{x}} = \begin{cases} \mathcal{F}_{\overline{x}}, & \text{if the topological image of } \overline{x} \text{ is in } |U| \\ 0, & \text{otherwise} \end{cases}$$

Given U and an étale sheaf $\mathcal{F}$ on U, we can then define

$$H^i_c(U, \mathcal{F}) := H^i(X, j_!\mathcal{F})$$

where X is proper and contains U as a dens open subset. For U of finite type over a field, such an X always exist by Nagata's compactification theorem. One shows that the definition does not depend on X.

Let now $f : X \to Y$ be a compatifiable morphism, i.e., such that there exists a diagram

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr, "f"'] & & X^c \arrow[dl, "f^c"] \\ & Y & \end{tikzcd}$$

where X^c is proper over Y and X is a dense open subset of X^c. For a sheaf $\mathcal{F}$ on X we can then define

$$R^i f_!\mathcal{F} := R^i(f^c_* \circ j_!)\mathcal{F}.$$

One (that is, Deligne) proves that this construction does not depend on the compactification X^c, and that if $\mathcal{F}$ is torsion there is a canonical identification

$$(R^i f_!\mathcal{F})_{\overline{y}} = H^i_c\left(X_{\overline{y}}, \mathcal{F}_{\overline{y}}\right)$$

for every geometric point $\overline{y}$ of Y. This extends in the usual way to $\mathbb{Z}_\ell$- and $\mathbb{Q}_\ell$-sheaves.

4.1.2 Three basic theorems

Let $\mathcal{F}$ be a locally constant sheaf of finite $\mathbb{Z}/n\mathbb{Z}$-modules, or of finite-dimensional $\mathbb{Q}_\ell$-vector spaces.

Theorem 4.1 (Cohomological dimension, M. Artin)**.** *Let X be of finite type over a separably closed field k. Then*

$$H^i(X, \mathcal{F}) = H^i_c(X, \mathcal{F}) = (0)$$

for $i > 2d$. In addition, $H^i(X, \mathcal{F}) = (0)$ if $i > \dim X$ and X is affine over a separably closed field k.

Theorem 4.2 (Localisation sequence in compact support cohomology)**.** *Let $Z \hookrightarrow X$ be a closed immersion with complement $U := X \setminus Z$. There is a long exact sequence*

$$\cdots \to H^i_c(U, \mathcal{F}) \to H^i_c(X, \mathcal{F}) \to H^i_c(Z, \mathcal{F} \to H^{i+1}_c(U, \mathcal{F}) \to \cdots$$

Theorem 4.3 (Poincaré duality)**.** *Let X be smooth, connected, and of finite type over a separably closed field and write d for the dimension of X. Then the following hold:*

1. *There is a "trace map" isomorphism $H^{2d}_c(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-d)$;*
2. *Let $\mathcal{F}$ be a locally constant sheaf of finite-dimensional $\mathbb{Q}_\ell$-vector spaces, and define $\mathcal{F}^\vee := \underline{\mathrm{Hom}}(\mathcal{F}, \mathbb{Q}_\ell)$. There exists a perfect pairing*

$$H^i(X, \mathcal{F}) \times H^{2d-i}_c(X, \mathcal{F}^\vee) \xrightarrow{\cup} H^{2d}_c(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-d).$$

Corollary 4.4 (Weak Lefschetz)**.** *Let X be proper and smooth over a separably closed field k and let $d = \dim X$. Let $Z \hookrightarrow X$ be a closed immersion such that $U := X \setminus Z$ is affine. Then:*

1. *$H^i(X, \mathcal{F}) \cong H^i(Z, \mathcal{F})$ for $i < d - 1$;*
2. *$H^{d-1}(X, \mathcal{F}) \hookrightarrow H^{d-1}(Z, \mathcal{F})$ for every locally constant sheaf of $\mathbb{Q}_\ell$-vector spaces.*

Sketch of proof. Write the localisation sequence for $X \setminus Z$ and use the cohomological vanishing from Theorem 4.1 after identifying $H^i_c(U, \mathcal{F})$ with $H^{2d-i}(U, \mathcal{F})$. □

4.2 Generalised zeta functions

Let $X/\mathbb{F}_q$ be a separated scheme of finite type, and let $\mathcal{F}$ be an étale sheaf on X. Denote by $\overline{X}$ the base change $X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ and let $\overline{\mathcal{F}}$ be the pull-back of $\mathcal{F}$ to $\overline{X}$. There is a Frobenius $F : X \to X$, which induces the geometric Frobenius $F : \overline{X} \to \overline{X}$. If $\overline{x}$ is a geometric point of X, pulling back along F induces a morphism

$$F^*_x; \mathcal{F}_{F(\overline{x})} \to \mathcal{F}_{\overline{x}},$$

and more generally

$$F^{*n}_{\overline{x}} : \mathcal{F}_{F^n(\overline{x})} \to \mathcal{F}_{\overline{x}}.$$

In particular, if the topological image of $\overline{x}$ is a closed point defined over the degree n extension of $\mathbb{F}_q$, then $F^n(\overline{x}) = \overline{x}$ and $F^{*n}_{\overline{x}}$ is an endomorphism of $\mathcal{F}_{\overline{x}}$.

Definition 4.5. In this situation, define

$$Z(X, \mathcal{F}, T) := \prod_{\substack{x \in X \\ \text{closed point}}} \det\left(1 - F_{\overline{x}}^{\deg(x)} T \mid \mathcal{F}_{\overline{x}}\right)^{-1}.$$

Remark 4.6. For the constant (ℓ-adic) sheaf $\mathbb{Q}_\ell$ we have $Z(X, \mathbb{Q}_\ell, T) = Z_X(T)$.

There is a generalised Lefschetz trace formula for these zeta functions, which implies in particular

$$Z(X, \mathcal{F}, T) = \prod_{i=0}^{2d} \det\left(1 - F^* T \mid H_c^i(\overline{X}, \mathcal{F})\right)^{(-1)^{i+1}}$$

Exercise 4.7. *Reverse-engineer the general Lefschetz trace formula from the above identity.*

4.3 The integrality theorem

We start with a definition:

Definition 4.8 (Deligne)**.** Let X be a separated scheme of finite type over $\mathbb{F}_q$ and let $\mathcal{F}$ be a locally constant sheaf of finite-dimensional $\mathbb{Q}_\ell$-vector spaces. We say that $\mathcal{F}$ is **integral** if for all geometric points $\overline{x}$ the eigenvalues of $F_{\overline{x}}^{* \deg(x)}$ acting on $\mathcal{F}_{\overline{x}}$ are algebraic integers.

Remark 4.9. As with many of our constructions and definitions, this also makes sense for constructible sheaves. There is also a version of this definition where one considers S-algebraic integers (that is, algebraic numbers that are integral over $\mathbb{Z}[1/S]$).

Theorem 4.10 (Deligne)**.** *In the above situation, if $\mathcal{F}$ is integral, then $H_c^i(\overline{X}, \mathcal{F})$ is also integral for every $i \geq 0$ (that is, the eigenvalues of Frobenius acting on it are algebraic integers).*

Corollary 4.11. *The constant sheaf $\mathcal{F} = \mathbb{Q}_\ell$ is integral, so $H_c^i(\overline{X}, \mathbb{Q}_\ell)$ is too. In particular, the zeroes and poles of $Z_X(T)$ are algebraic integers.*

Proof. Notice first that we can assume that X is reduced by [Sta18, Tag 03SI]. In the case of curves, this guarantees that the singular locus is 0-dimensional.

By induction on $d = \dim X$. For $d = 0$, the theorem is (trivially) true for $H_c^0 = H^0$ (0-dimensional schemes are proper!).

Next we consider the case of curves, namely $d = 1$. The first observation is that there exists a closed subscheme $Z \hookrightarrow X$ of dimension zero such that $H^0(\overline{X}, \mathcal{F}) \hookrightarrow H^0(\overline{Z}, \mathcal{F})$. To see this, recall that we have assumed that each $\mathcal{F}_{\overline{x}}$ is a finite-dimensional $\mathbb{Q}_\ell$-vector space, so it suffices to choose sufficiently many points to get the desired injection. The result for $H_c^0(\overline{X}, \mathcal{F})$ then follows from the 0-dimensional case, because we have injections

$$H_c^0(\overline{X}, \mathcal{F}) \hookrightarrow H^0(\overline{X}, \mathcal{F}) \hookrightarrow H^0(\overline{Z}, \mathcal{F})$$

and the eigenvalues on a subspace are a subset of the eigenvalues on the whole space.

We now consider the case $d = 1$ and $i = 2$. Choose a closed subscheme $Z \hookrightarrow X$ of dimension 0 containing the singular locus of X. The localisation sequence gives

$$(0) = H_c^1(\overline{Z}, \mathcal{F}) \to H_c^2(\overline{U}, \mathcal{F}) \to H_c^2(\overline{X}, \mathcal{F}) \to H_c^2(\overline{Z}, \mathcal{F}) = (0),$$

that is, $H^2_c(\overline{U},\mathcal{F}) \cong H^2_c(\overline{X},\mathcal{F})$. So we may assume that X is smooth, which by Poincaré duality (Theorem 4.3) implies that $H^2_c(\overline{X},\mathcal{F})$ is dual to $H^0(\overline{X},\mathcal{F}^\vee)$, and we win because we already know that the statement holds for $H^0(\overline{X},\mathcal{F}^\vee)$. Notice that we are taking the reciprocal of the eigenvalues *twice*, one because of the presence of the dual sheaf $\mathcal{F}^\vee$ and one because of the duality between $H^2_c(\overline{X},\mathcal{F})$ and $H^0(\overline{X},\mathcal{F}^\vee)$.

To finish the case of curves we still need to handle the case $d = 1, i = 1$. We have an equality of formal power series

$$\det(1 - F^*T \mid H^1_c(\overline{X},\mathcal{F})) = \det(1 - F^*T \mid H^0_c(\overline{X},\mathcal{F}))\det(1 - F^*T \mid H^2_c(\overline{X},\mathcal{F}))Z(X,\mathcal{F},T).$$

As $\mathcal{F}$ is integral, the coefficients of $Z(X,\mathcal{F},T)$ are algebraic integers, so the same is true for $\det(1 - F^*T \mid H^1_c(\overline{X},\mathcal{F}))$. As $\det(1 - F^*T \mid H^1_c(\overline{X},\mathcal{F}))$ is a polynomial with constant coefficient 1 and whose other coefficients are (algebraic) integers, the inverse roots of $\det(1 - F^*T \mid H^1_c(\overline{X},\mathcal{F}))$ are also algebraic integers. But these inverse roots are precisely the eigenvalues of Frobenius, and we are done.

Finally, for the induction step, assume that $d > 1$. Choose an open subscheme $U \hookrightarrow X$ such that $\dim(X \setminus U) < d$ and there exists a *compactifiable* morphism $f : U \to Y$, with Y a curve, such that the fibres of f have dimension $< d$. In other words, we want a diagram

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & U^c \arrow[dl] \\ & Y; & \end{tikzcd}$$

to show that this exists, choose U to be a dense open affine subscheme of X, embed it in projective space, take its closure, and use a suitable pencil of hyperplanes. Using Theorem 4.2 we get

$$H^i_c(\overline{U},\mathcal{F}) \to H^i_c(\overline{X},\mathcal{F}) \to H^i_c(\overline{Z},\mathcal{F})$$

By the induction hypothesis, it suffices to prove the statement for $H^i_c(\overline{U},\mathcal{F})$, that is, we can assume $X = U$. In particular, we can assume that there is a compactifiable morphism $f : X \to Y$ with fibres of dimension $< d$, and we may consider $R^i f_!\mathcal{F}$. Shrinking U further if necessary, we can also assume that f is smooth. There is a potential small problem here, because $R^q f_!\mathcal{F}$ is not necessarily locally constant. In any case, one can work with constructible sheaves, and the problem disappears. In this situation, there exists a Leray spectral sequence

$$E_2^{p,q} = H^p\left(\overline{Y}, R^q f_!\mathcal{F}\right) \Rightarrow H^{p+q}_c(\overline{X},\mathcal{F}).$$

Notice that this is slightly less easy than the usual Leray spectral sequence, because we are working with cohomology with compact support. The spectral sequence is compatible with the action of Frobenius, and $H^{p+q}_c(\overline{X},\mathcal{F})$ is filtred with graded pieces that are subquotients of $H^p\left(\overline{Y}, R^q f_!\mathcal{F}\right)$. Hence it suffices to prove integrality for $H^p\left(\overline{Y}, R^q f_!\mathcal{F}\right)$. Now observe that

$$(R^q f_!\mathcal{F})_{\overline{y}} \cong H^q_c\left(X_{\overline{y}}, \mathcal{F}_{\overline{y}}\right),$$

□

Remark 4.12. We will see later that in the proof of Conjecture 1.7 we shall need a much more refined version of the coarse geometric lemma used in this proof. The advantage in the proof of Deligne's integrality theorem is that the fibres of $f : U \to Y$ can have arbitrarily bad singularities.

5 29.10.2019 – Katz's proof of the Riemann Hypothesis for hypersurfaces (2015)

Theorem 5.1. *Let $X \subset \mathbb{P}^{d+1}_{\mathbb{F}_q}$ be a smooth projective hypersurface. The eigenvalues of Frobenius acting on $H^i\left(\overline{X}, \mathbb{Q}_\ell\right)$ all have absolute value $q^{i/2}$, for $i = 0, \ldots, 2d$.*

Convention. For this lecture, *absolute value* means *absolute value with respect to all embeddings of $\mathbb{Q}_\ell$ into $\mathbb{C}$.*

Remark 5.2. Scholl [Sch11] shows that the case of hypersurfaces implies the case of general smooth projective varieties.

Remark 5.3. The weak Lefschetz theorem implies that $H^i\left(\overline{X}, \mathbb{Q}_\ell\right) \xrightarrow{\sim} H^i\left(\mathbb{P}^d, \mathbb{Q}_\ell\right)$ for $i < d$. Moreover, if k is a separably closed field,

$$H^i\left(\mathbb{P}^d_k, \mathbb{Q}_\ell\right) = \begin{cases} \mathbb{Q}_\ell\left(-\frac{i}{2}\right), & 0 \leq i \leq 2d, i \text{ even} \\ 0, i \text{ odd} \end{cases}$$

To see this, notice that for $k = \mathbb{C}$ this is known from topology (up to the Galois action). Since cohomology does not change by extension of algebraically closed field, the result also holds for every k of characteristic 0. By smooth proper base change, as in Section 3.5.1, this implies the result for arbitrary separably closed k. This proves that the group is $\mathbb{Q}_\ell$. As for the Galois action, we have a Galois-equivariant map

$$\mathbb{Z}/n\mathbb{Z} \cong \operatorname{Pic}(\mathbb{P}^n)/n\operatorname{Pic}(\mathbb{P}^n) \cong H^1\left(\mathbb{P}^1, \mathbb{G}_m\right)/nH^1\left(\mathbb{P}^1, \mathbb{G}_m\right) \to H^2\left(\mathbb{P}^n, \mu_n\right)$$

which is an isomorphism of abelian groups by comparison with the case $k = \mathbb{C}$. Hence it is an isomorphism of Galois modules, which gives the action of Galois on $H^2\left(\mathbb{P}^n, \mu_n\right)$. For $i > 2$, one similarly considers the Galois-equivariant map

$$H^2\left(\mathbb{P}^d, \mu_n\right) \cup \cdots \cup H^2\left(\mathbb{P}^d, \mu_n\right) \to H^{2d}\left(\mathbb{P}^d, \mu_n^{\otimes d}\right),$$

which is again an isomorphism by comparison with topology.

Putting together the previous arguments, we get that for $i < d$

$$H^i\left(\overline{X}, \mathbb{Q}_\ell\right) \cong \begin{cases} \mathbb{Q}_\ell\left(-\frac{i}{2}\right), i \text{ even} \\ 0, i \text{ odd} \end{cases}$$

By Poincaré duality, the same holds for $2d \geq i > d$, hence in particular the Riemann hypothesis holds for all H^i except at most for $i = d$.
Goal. The eigenvalues of Frobenius on $H^d\left(\overline{X}, \mathbb{Q}_\ell\right)$ are of absolute value $q^{d/2}$.

Lemma 5.4 (Key Lemma)**.** *Let $U \subseteq \mathbb{P}^1_{\mathbb{F}_q}$ be a nonempty open subscheme of the projective line, and let $\mathcal{F}$ be a locally constant sheaf of finite $\mathbb{Q}_\ell$-vector spaces on U. Assume that, for every geometric point $\overline{x} \in U$, the inverse characteristic polynomial $P_{\overline{x}} := \det(1 - TF^*_{\overline{x}} \mid \mathcal{F}_{\overline{x}})$ has real coefficients. Assume furthermore that there exists a closed point $x_0 \in U$ such that $\forall \overline{x_0}$ above x_0 the polynomial $P_{\overline{x}_0}$ has roots of absolute value 1. Then the same holds for all closed points.*

Remark 5.5. Here *real* means *real in all embeddings.*

We now show that the Key Lemma implies the Riemann Hypothesis:

Proof. Let F be an equation for X, let n be its degree, and let G be the equation of a hypersurface (of the same degree) for which the result is known. For example, if $n = \deg F$ is prime to q, then we can take $G = \sum_{i=0}^{d} a_i x_i^n$ (see Section 2). If $(n, q) \neq 1$, then one can take

$$G = x_0^n + \sum_{i=0}^{d} x_i x_{i+1}^{n-1},$$

and similar methods involving Gauss sums allow one to prove the Riemann hypothesis directly. Consider now the deformation $tF + (1-t)G = 0$. We consider this as a fibration over the t-line. This fibration may have singular fibres, but the fibres at $t = 0$ and at $t = 1$ are smooth by construction and by assumption respectively. Let $U \subseteq \mathbb{P}^1_{\mathbb{F}_q}$ be the non-empty open locus where the fibres are smooth. Apply Lemma 5.4 to $R^d f_* \mathbb{Q}_\ell \left(-\frac{d}{2}\right)$ and $x_0 = 1$.

Notice that to apply the Key Lemma we need to know that the coefficients of the relevant characteristic polynomials are real. We do in fact know that they are rational, because we know this in all degrees except $i = d$, and we also know that the Zeta function has rational coefficients. □

Remark 5.6. Provided that one knows that the coefficients of the characteristic polynomials of Frobenius are real, the argument also works for $R^j f_* \mathbb{Q}_\ell \left(-\frac{j}{2}\right)$. In particular, if one could find a deformation from an arbitrary variety X to one for which the Riemann hypothesis is known to hold, the Key Lemma would prove the Riemann Hypothesis for X.

5.1 Reminder on the arithmetic fundamental group

Let X be a connected space and let $\overline{x} \to X$ be a geometric point. One can define a profinite group $\pi_1(X, \overline{x})$. There is a correspondence

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{finite étale} \\ \text{covers } Y \to X \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{finite sets endowed} \\ \text{with a continuous action of } \pi_1(X, \overline{x}) \end{array} \right\} \\ Y & \mapsto & Y_{\overline{x}} \end{array}$$

and a similar one

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{locally constant sheaves of} \\ \text{finite-dimensional } \mathbb{Q}_\ell\text{-vector space} \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{c} \text{finite-dimensional continuous} \\ \text{representations of } \pi_1(X, \overline{x}) \text{ over } \mathbb{Q}_\ell \end{array} \right\} \\ \mathcal{F} & \mapsto & \mathcal{F}_{\overline{x}}. \end{array}$$

These objects are usually called *lisse* or *smooth* sheaves, and (essentially) correspond to continuous homomorphisms $\pi_1(X, \overline{x}) \to \mathrm{GL}_n(\mathbb{Z}_\ell)$.

Let X be a scheme of finite type over $\mathbb{F}_q$. There exists an exact sequence

$$1 \to \pi_1(\overline{X}, \overline{x}) \to \pi_1(X, \overline{x}) \to \mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) \to 1,$$

split by a rational point (if there is one). The group $\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right)$ is isomorphic to $\hat{\mathbb{Z}}$, with generator denoted by F_q. Let x be a closed point of X of degree m,

$$x : \mathrm{Spec}\left(\mathbb{F}_{q^m}\right) \to X.$$

As π_1 is a covariant functor, x induces a map $\mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_{q^m}\right) \to \pi_1(X,x)$ which sends the canonical generator Frob^m of $\mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_{q^m}\right)$ to an element of $\pi_1(X,\overline{x})$ which we denote by F_x.

Remark 5.7. If $\mathcal{F}$ is a lisse sheaf on X, its pullback to $\overline{X}$ corresponds to the restriction of the representation of $\pi_1(X,\overline{x})$ on $\mathcal{F}_{\overline{x}}$ to the geometric fundamental group $\pi_1(\overline{X},\overline{x})$. Moreover,

$$H^0\left(\overline{X},\overline{\mathcal{F}}\right) = \mathcal{F}^{\pi_1(\overline{X})}.$$

If $U/\mathbb{F}_q$ is an affine curve and $\mathcal{F}$ is lisse on U, then $H^0_c(\overline{U},\overline{\mathcal{F}}) = 0$ (either by definition, or because by Poincaré duality this is the same as $H^2(\overline{U},\overline{\mathcal{F}})$, which vanishes for an *affine* curve). Finally,

$$H^2_c(\overline{U},\overline{\mathcal{F}}) = \mathcal{F}_{\pi_1(\overline{X})}(-1),$$

where $\mathcal{F}_{\pi_1(\overline{X})}$ denotes the co-invariants of the action of $\pi_1(\overline{X})$.

Remark 5.8. The crucial observation is the following: for every closed point $x \in U$, the Frobenius F_x acts on $\mathcal{F}_{\pi_1(\overline{X})}$ via $F_q^{\deg(x)}$: indeed, if we work in a quotient where $\pi_1(\overline{X},\overline{x})$ acts trivially, then the action of an element in $\pi_1(X,\overline{x})$ (coming from a point x) depends only on its image in $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$, which depends only on the degree of x.

5.2 Katz's proof

Proposition 5.9 (Deligne)**.** *Let $U/\mathbb{F}_q$ be an affine curve and let $\mathcal{F}$ be a $\mathbb{Q}_\ell$-lisse sheaf over U such that $P_{\overline{x}}$ has real coefficients for all closed $x \in U$. Suppose there exists a closed point x_0 such that all the eigenvalues of F_{x_0} on $\mathcal{F}$ are of absolute value ≤ 1. Then the same holds for all closed points.*

Lemma 5.10 (Rankin's trick)**.** *Under the above assumptions, for all $k \geq 1$ the eigenvalues of F_q on $\left(\mathcal{F}^{\otimes 2k}\right)_{\pi_1(\overline{X})}$ are of absolute value ≤ 1.*

Proof. If β is an eigenvalue of F_q on $\left(\mathcal{F}^{\otimes 2k}\right)_{\pi_1(\overline{X})}$, then for $m = \deg(x_0)$ the number β^m is an eigenvalue of F_{x_0}. But then β^m is a product of $2k$ eigenvalues of F_{x_0} on $\mathcal{F}$, which means $|\beta^m| \leq 1 \Rightarrow |\beta| \leq 1$. □

Proof of Proposition 5.9. We have already seen that

$$H^2_c\left(\overline{U},\overline{\mathcal{F}}^{\otimes 2k}\right) \cong \left(\mathcal{F}^{\otimes 2k}_{\pi_1(\overline{U})}\right)(-1),$$

so the Lemma implies that F_q acts on it with eigenvalues of absolute value $\leq q$. Consider now the Zeta function associated with $\mathcal{F}^{\otimes 2k}$:

$$Z(U,\mathcal{F}^{\otimes 2k},T) := \prod_{x \in U_{(0)}} \det\left(1 - T^{\deg x} F_x T \mid \mathcal{F}^{\otimes 2k}\right)^{-1},$$

which by the Grothendieck-Lefschetz trace formula we know to be equal to

$$\frac{\det(1 - TF \mid H^1_c(\overline{U},\overline{\mathcal{F}}^{\otimes 2k})}{\det(1 - TF \mid H^2_c(\overline{U},\overline{\mathcal{F}}^{\otimes 2k})},$$

where the term corresponding to H_c^0 is trivial by Remark 5.7. The fact that the absolute value of the eigenvalues of Frobenius is bounded by q implies that the Zeta function converges for $|T| < \frac{1}{q}$, because the denominator has no poles there. By the assumption, every factor $\det\left(1 - T^{\deg x} F_x T \mid \mathcal{F}\right)$ has real coefficients, and therefore $\det\left(1 - T^{\deg x} F_x T \mid \mathcal{F}^{\otimes 2k}\right)$ has *positive* real coefficients (this is why the 2 is important!). In particular, if we fix a factor in the Euler product, every coefficient of it is bounded above by the corresponding coefficient in $Z(X, \mathcal{F}^{\otimes 2k}, T)$. This implies that every fixed Euler factor for $\mathcal{F}^{\otimes 2k}$ converges for $|T| < \frac{1}{q}$. Hence for every x closed point of U, and every eigenvalue α of F_x on $\mathcal{F}$, then α^{2k} is an eigenvalue of F_x on $\mathcal{F}^{\otimes 2k}$, and by the previous estimate this implies $|\alpha^{2k}| \leq q^{\deg x}$, which by passing to the limit $k \to \infty$ gives $|\alpha| \leq 1$. □

Lemma 5.11. *Let $U \subseteq \mathbb{P}^1$ be a nonempty affine open subscheme, and let $\mathcal{F}$ be a $\mathbb{Q}_\ell$-lisse sheaf* ***of rank 1*** *on U. Then there exists $m > 0$ such that $\pi_1(\overline{U})$ acts trivially on $\mathcal{F}^{\otimes m}$.*

We briefly postpone the proof of this lemma, and show that Lemma 5.11 and Proposition 5.9 together imply Lemma 5.4.

Proof. By Proposition 5.9, all eigenvalues are of absolute value ≤ 1. To show that they are equal to 1, it suffices to show that their product is 1. Now this product is an eigenvalue of Frobenius acting on $\Lambda^r \mathcal{F}$, where $r = \operatorname{rk} \mathcal{F}$. By Lemma 5.11, there exists $m > 0$ such that $(\Lambda^r \mathcal{F})^{\otimes m}$ has trivial $\pi_1(\overline{U})$-action. It follows that all F_x act on $(\Lambda^r \mathcal{F})^{\otimes m}$ via $F_q^{\deg x}$. In particular, if one of them acts with eigenvalues of absolute value 1, then all do. □

Proof of Lemma 5.11. The sheaf $\mathcal{F}$ corresponds to a representation $\rho : \pi_1(U) \to \mathbb{Z}_\ell^\times = \mathrm{GL}_1(\mathbb{Z}_\ell) \cong \mathbb{F}_\ell^\times \times (1 + \ell\mathbb{Z}_\ell)$. Assume for simplicity that $\ell > 2$: then $1 + \ell\mathbb{Z}_\ell \cong \ell\mathbb{Z}_\ell$ via the logarithm map, hence we can consider $\rho^{\ell-1}$ as a map from $\pi_1(U)$ to $\ell\mathbb{Z}_\ell \hookrightarrow \mathbb{Q}_\ell$. By the étale version of Hurewicz's theorem, this map corresponds to an element of $H^1(U, \mathbb{Q}_\ell)$. In particular, $\rho^{\ell-1}\big|_{\pi_1(U)}$ corresponds to an element of $H^1(\overline{U}, \mathbb{Q}_\ell)$ fixed by Frobenius (because it comes by pullback from something defined over U). Now observe that $H^1(\overline{U}, \mathbb{Q}_\ell)$ is dual to $H_c^1(\overline{U}, \mathbb{Q}_\ell)(-1)$, and – setting $Z := \mathbb{P}^1 \setminus U$ – by the long exact sequence for compact support cohomology we get

$$H^0\left(\overline{Z}, \mathbb{Q}_\ell\right) \to H_c^1(\overline{U}, \mathbb{Q}_\ell) \to H^1(\mathbb{P}^1_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell) = (0).$$

Since Frobenius acts trivially on $H^0\left(\overline{Z}, \mathbb{Q}_\ell\right)$, and we have just shown that $H_c^1(\overline{U}, \mathbb{Q}_\ell)$ is a quotient of $H^0\left(\overline{Z}, \mathbb{Q}_\ell\right)$, we get that Frobenius acts trivially on $H_c^1(\overline{U}, \mathbb{Q}_\ell)$, hence it has no invariants when acting on $H_c^1(\overline{U}, \mathbb{Q}_\ell)(-1)$, because of the twist. Hence $\rho^{\ell-1}|_{\pi_1(\overline{U})}$ must be trivial, which is what we wanted to show. □

6 05.11.2019 – Deligne's original proof of the Riemann Hypothesis

6.1 Reductions in the proof of the Weil Conjectures

We now only have one remaining conjecture, namely the Riemann Hypothesis 1.7.

Theorem 6.1. *The eigenvalues of Frobenius on $H^i\left(\overline{X}, \mathbb{Q}_\ell\right)$ have absolute value $q^{i/2}$.*

Remark 6.2. The proof will be by induction on the dimension of X, starting from $\dim X = 0$.

Remark 6.3. We continue with our running convention that *absolute value* means *absolute value under every embedding.* Also recall that $X/\mathbb{F}_q$ is a smooth projective variety.

Lemma 6.4. *It is enough to prove the theorem after finite extension of the base field.*

Proof. Let $\mathbb{F}_{q'}/\mathbb{F}_q$ be a finite extension. The hypothesis is that $F^{[\mathbb{F}_{q'}:\mathbb{F}_q]}$ has eigenvalues of absolute value $(q')^{i/2}$, which implies that the eigenvalues of F have absolute value $q^{i/2}$. □

Lemma 6.5. *It is enough to prove the theorem for $i = d = \dim X$.*

Proof. By Poincaré duality it is enough to prove the statement for $i \leq d$. Let $\overline{Y} \hookrightarrow \overline{X}$ be a smooth connected hyperplane section (which exists by Bertini's theorem). By Lemma 6.4, we may assume that such a smooth section is defined over $\mathbb{F}_q$. By weak Lefschetz, it is enough to prove the statement for Y: indeed, weak Lefschetz yields

$$H^i(\overline{X}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i(\overline{Y}, \mathbb{Q}_\ell) \quad \text{for } i < d-1$$

and

$$H^{d-1}(\overline{X}, \mathbb{Q}_\ell) \hookrightarrow H^{d-1}(\overline{Y}, \mathbb{Q}_\ell),$$

so the induction hypothesis on Y gives the statement for $H^i(X, \mathbb{Q}_\ell)$ with $i < d$, so only the case $i = d$ remains as claimed. □

Lemma 6.6. *It is enough to prove the following for d* ***even****: for every smooth projective X, the eigenvalues α of F on $H^d(\overline{X}, \mathbb{Q}_\ell)$ satisfy*

$$q^{\frac{d}{2}-\frac{1}{2}} \leq |\alpha| \leq q^{\frac{d}{2}+\frac{1}{2}}. \tag{1}$$

Proof. If k is even, α^k is an eigenvalue of Frobenius on $H^{kd}(X^k, \mathbb{Q}_\ell)$ by the Künneth formula. The hypothesis implies

$$q^{kd/2-1/2} \leq |\alpha|^k \leq q^{kd/2+1/2},$$

which by taking k-th roots and passing to the limit $k \to \infty$ yields $|\alpha| = q^{d/2}$. □

6.2 Geometric and topological ingredients: Lefschetz pencils

Let $\mathbb{P}$ be a projective space of dimension > 1 over an algebraically closed field k. Let $\check{P}$ be the dual projective space: points of $\check{P}$ correspond to hyperplanes in $\mathbb{P}$. We denote the bijection (on closed points) by $t \in \check{P} \longleftrightarrow H_t \subseteq \mathbb{P}$. If $A \subseteq \mathbb{P}$ is a subspace of codimension 2, the hyperplanes $H \supset A$ are parametrised by the points of a line $D \subseteq \check{P}$. These hyperplanes form a **pencil** with axis A.

Let $\overline{X} \subseteq \mathbb{P}$ be a smooth projective variety of dimension $d = n+1$. Define the incidence variety

$$\tilde{X} := \{(x, t) \in \overline{X} \times D : x \in H_t\},$$

which as usual comes equipped with maps

$$\begin{tikzcd} \overline{X} & \tilde{X} \arrow[l, "\pi"'] \arrow[d, "f"'] \\ & D. \end{tikzcd}$$

Assume that $\tilde{X} \cap A$ is smooth and has codimension 2 in $\overline{X}$. The fibre of f over $t \in D$ is $\overline{X} \cap H_t$. The fibre of π over $x \in \overline{X}$ is:

- $(x, D) \cong D$ if $x \in \overline{X} \cap A$;
- a single point (x, H_x) if $x \notin A$.

This remark suggests (and the definition of $\tilde{X}$ shows) that $\tilde{X}$ is the blowup of $\overline{X}$ in $\tilde{X} \cap A$. The map $f : \tilde{X} \to D$ is a proper surjective map (a fibration, in the sense of algebraic geometry).

Theorem 6.7. *By choosing A appropriately (in fact, sufficiently generically), and after performing some Veronese embedding in characteristic $p > 0$, we may arrange for the following to hold:*

1. *there exists a finite set of points $S \subseteq D$ such that the fibre $\tilde{X}_t$ is smooth for all $t \notin S$*
2. *the fibres $\tilde{X}_s$ for $s \in S$ have only one singularity, and it is quadratic (see Definition 6.8).*

Definition 6.8. A singularity is quadratic if the completion of the corresponding local ring is isomorphic to $k[[t_1, \ldots, t_n]]/(Q(t_1, \ldots, t_n))$ with Q a nondegenerate quadratic form.

Before moving on, we give a general overview of Deligne's argument to prove the estimate in Equation (1).

6.3 Strategy of proof of estimate (1)

Let $k = \overline{\mathbb{F}_q}$ and let $X/\mathbb{F}_q$ be a smooth projective variety. Take a Lefschetz pencil on $\overline{X}$. Up to extending $\mathbb{F}_q$, we may assume the following objects to all be defined over $\mathbb{F}_q$:

- the base D and axis A of the pencil, hence the pencil itself;
- the points of S;
- some other point $u_0 \in D \setminus S$;
- a smooth hyperplane section $\overline{Y}$ of the fibre X_{u_0}.

Notice that $\dim \overline{Y} = d - 2 = n - 1$: we wish our induction to only involve varieties of *even* dimension.

Theorem 6.9 (Blow-up formula for étale cohomology)**.** *There is a Frobenius-equivariant isomorphism*

$$H^i\left(\tilde{X}, \mathbb{Q}_\ell\right) = H^i\left(\overline{X}, \mathbb{Q}_\ell\right) \oplus H^{i-2}\left(A \cap \overline{X}, \mathbb{Q}_\ell(-1)\right).$$

Remark 6.10. The proof of the blow-up formula goes through the projective bundle formula and some localisation sequences, so it exists for almost any cohomological theory (in particular, for étale cohomology).

The blow-up formula (together with its compatibility with the Frobenius action) shows that it is enough to prove (1) for $\tilde{X}$ instead of $\overline{X}$, hence we may assume that there is a Lefschetz pencil $\overline{X} \to D$.

We have a Frobenius-equivariant Leray spectral sequence (for normal étale cohomology: everything is proper, so we don't need compact support cohomology)

$$H_2^{p,q} = H^p\left(D, R^q f_* \mathbb{Q}_\ell\right) \Rightarrow H^{p+q}\left(\overline{X}, \mathbb{Q}_\ell\right)$$

We know that the stalks of $R^q f_* \mathbb{Q}_\ell$ at points $t \in D$ are the cohomology groups $H^q(\overline{X}_t, \mathbb{Q}_\ell)$. Assume for now that all fibres of f are smooth. Notice that this almost never happens, and avoiding this assumption is a substantial difficulty in the actual proof; for now, however, we just discuss this toy case.

By proper smooth base change, the sheaves $R^q f_* \mathbb{Q}_\ell$ are locally constant on $D \cong \mathbb{P}^1$, hence they are constant (because $\mathbb{P}^1$ has no finite connected étale covers). It follows that $R^q f_* \mathbb{Q}_\ell$ is the constant sheaf associated with the abelian group $H^q(\overline{X}_{u_0}, \mathbb{Q}_\ell)$ (because this is the fibre at the point u_0, hence at every point by constancy). The groups $E_2^{p,q}$ contributing to $H^{n+1}(\overline{X}, \mathbb{Q}_\ell)$ are

$$H^0(D, R^{n+1} f_* \mathbb{Q}_\ell), \quad H^1(D, R^n f_* \mathbb{Q}_\ell), \quad H^2(D, R^{n-1} f_* \mathbb{Q}_\ell),$$

hence it is enough to prove that (1) holds for the eigenvalues of Frobenius acting on these groups.

1. $H^1(D, \mathcal{F})$ vanishes for any constant sheaf of finite-dimensional $\mathbb{Q}_\ell$-vector spaces: one reduces first to $\mathbb{Z}_\ell$-sheaves, and then to the constant sheaf $\mathbb{Z}/n\mathbb{Z}$, for which it is well-known that H^1 vanishes (because $\mathbb{P}^1$ has no nontrivial covers, for example).

2. Using Poincaré duality we find

$$H^2(D, R^{n-1} f_* \mathbb{Q}_\ell) = H^2(D, H^{n-1}(\overline{X}_{u_0}, \mathbb{Q}_\ell)) = H^0(D, H^{n-1}(\overline{X}_{u_0}, \mathbb{Q}_\ell)^\vee)^\vee(-1),$$

   and since $\dim \overline{X}_u = n$ we obtain $H^{n-1}(\overline{X}_u, \mathbb{Q}_\ell) \hookrightarrow H^{n-1}(\overline{Y}, \mathbb{Q}_\ell)$, so the claim follows from the case $d-2$ by induction.

3. A similar argument works for $H^0\left(D, R^{n+1} f_* \mathbb{Q}_\ell\right) \cong H^{n+1}\left(\overline{X}_{u_0}, \mathbb{Q}_\ell\right)$ (one again applies Poincaré duality and Weak Lefschetz).

Of course, the difficulty is that when f has singular fibres the sheaves $R^{n+1} f_* \mathbb{Q}_\ell$ are in general not locally constant. In particular, $H^1(D, R^n f_* \mathbb{Q}_\ell)$ will prove to be hard to understand.

6.4 Lefschetz theory over $\mathbb{C}$

Lefschetz is the study of the cohomology of Lefschetz pencils. We shall start with the local picture – what happens around a single bad fibre – and then move on the global aspects.

6.4.1 Local theory

Consider a small complex disc D around the singular fibre.

$$\begin{tikzcd} X_t \arrow[r, hook] \arrow[d, "\text{smooth}"'] & X \arrow[d] & X_0 \arrow[l, hook'] \arrow[d, "\text{one quadratic singularity}"] \\ \{t \neq 0\} \arrow[r, hook] & D & \{0\} \arrow[l, hook'] \end{tikzcd}$$

As X_0 is a deformation retract of X, we have an isomorphism $H^i(X_0, \mathbb{Z}) \xleftarrow{\sim} H^i(X, \mathbb{Z})$, and we also have a restriction map $H^i(X_t, \mathbb{Z})$. By composing (the inverse of) the former with the latter we obtain a (co)specialisation map

$$\mathrm{cosp} : H^i(X_0, \mathbb{Z}) \to H^i(X_t, \mathbb{Z}).$$

Monodromy induces an action of the fundamental group of the punctured disc,

$$\pi_1(D \setminus \{0\}, t) \cong \mathbb{Z},$$

on $H^i(X_t, \mathbb{Z})$.

Theorem 6.11 (Lefschetz)**.** *The following hold:*

1. *for $i \neq n, n+1$ the cospecialisation map* cosp *is an isomorphism, and the monodromy action is trivial;*
2. *there exists a canonical element $\delta \in H^n(X_t, \mathbb{Z})$, called the* ***vanishing cycle*** *and well-defined up to sign, such that we have an exact sequence*

$$0 \to H^n(X_0, \mathbb{Z}) \xrightarrow{\mathrm{cosp}} H^n(X_t, \mathbb{Z}) \xrightarrow{\alpha} \mathbb{Z} \to H^{n+1}(X_0, \mathbb{Z}) \xrightarrow{\mathrm{cosp}} H^{n+1}(X_t, \mathbb{Z}) \to 0,$$

where the map α sends $\xi \in H^n(X_t, \mathbb{Z})$ to $\xi \cup \delta \in H^{2n}(X, \mathbb{Z}) = \mathbb{Z}$ by Poincaré duality.

The action of $\pi_1(D \setminus \{0\}) = \mathbb{Z} \cdot \gamma$ on $H^{n+1}(X_t, \mathbb{Z})$ is trivial, and on $H^n(X_t, \mathbb{Z})$ is given by

$$\gamma \cdot x = x \pm (x, \delta)\delta,$$

where $(\cdot, \cdot)$ is the Poincaré duality pairing, and the sign is determined by n mod 4 (once γ is fixed): if $n \equiv 0, 1 \pmod 4$ then the sign is $+$, and if $n \equiv 2, 3 \pmod 4$ the sign is $-$. This result is called the Picard-Lefschetz formula. Further interesting properties of this situation:

- The monodromy action on $H^n(X_t, \mathbb{Z})$ is compatible with the Poincaré duality pairing $(\cdot, \cdot)$.
- The orthogonal of $\langle \delta \rangle$ with respect to the Poincaré pairing is $H^n(X_t, \mathbb{Z})^{\pi_1(D \setminus \{0\}, t)}$: this is an immediate consequence of the Poincaré-Lefschetz formula).

7 12.11.2019 – Deligne's original proof of the Riemann Hypothesis II

7.1 Lefschetz theory over $\mathbb{C}$

Aim: describe the cohomology of the fibres of a Lefschetz pencil.

We have seen that (up to blowing-up a subvariety of codimension 2) we may find a pencil in which every fibre has at worst one quadratic singularity.

7.1.1 Local theory

Let D be a small complex disc around a point 0 where the fibre is singular. Recall our convention that $\dim X = d = n+1$.

$$\begin{tikzcd} X_t \arrow[r, hook] \arrow[d] & X \arrow[d] & X_0 \arrow[l, hook'] \arrow[d] \\ \{t \neq 0\} \arrow[r, hook] & D & \{0\} \arrow[l, hook'] \end{tikzcd}$$

We have seen that the cospecialisation map $H^i(X_0, \mathbb{Z}) \to H^i(X_t, \mathbb{Z})$ is an isomorphism for $i \neq n, n+1$. Furthermore, there exist a canonical element $\delta \in H^n(X_t, \mathbb{Z})$ and an exact sequence

$$\begin{array}{ccccccccc} 0 \to H^n(X_0, \mathbb{Z}) & \xrightarrow{\text{cosp}} & H^n(X_t, \mathbb{Z}) & \xrightarrow{\alpha} & \mathbb{Z} & \to H^{n+1}(X_0, \mathbb{Z}) & \xrightarrow{\text{cosp}} & H^{n+1}(X_t, \mathbb{Z}) \to 0 \\ & & \xi & \mapsto & \xi \cup \delta & & & \end{array}$$

We also have the Picard-Lefschetz formula: given $\gamma \in \pi_1(D \setminus \{0\})$ and $x \in H^n(X_t, \mathbb{Z})$ we have

$$\gamma \cdot x = x \pm (x, \delta)\delta.$$

7.1.2 Global theory

We will now consider a Lefschetz pencil $X \to D \cong \mathbb{P}^1$ and let $U = D \setminus S$, where $S \subseteq D$ is the set of points over which the fibre is singular. Fix $u \in U$ and consider the fundamental group $\pi_1(U, u)$: it is generated by loops γ_s around the points $s \in S$.

It would be nice to add a picture

The group $\pi_1(U, u)$ acts by monodromy on $H^i(X_u, \mathbb{Z})$ for all i, and the local theory describes the action of every γ_s: for every $s \in S$ we have a vanishing cycle $\delta_s \in H^n(X_u, \mathbb{Z})$, and $\gamma_s \in \pi_1(U, u)$ acts by

$$\gamma_s \cdot x = x \pm (x, \delta_s)\delta_s. \tag{2}$$

Theorem 7.1 (Lefschetz)**.** *The following hold:*

- *$\pi_1(U, u)$ acts trivially on $H^i(X_u, \mathbb{Z})$ for $i \neq n$;*
- *the elements δ_s are conjugate under the action of $\pi_1(U, u)$.*

Proposition 7.2. *Let E be the subspace of $H^n(X_u, \mathbb{Q})$ generated by the δ_s. Then E is stable by the action of $\pi_1(U, u)$, and its orthogonal $E^\perp$ (with respect to Poincaré duality) is $H^n(X_u, \mathbb{Q})^{\pi_1(U,u)}$.*

Proof. This is an immediate consequence of the Picard-Lefschetz formula (see for example Equation (2)): if $x \in E$ then $\gamma_s \cdot x \in E$, and x is orthogonal to all the δ_s if and only if it is stable under the action of every γ_s. □

Proposition 7.3. *The action of $\pi_1(U, u)$ on $E/E \cap E^\perp$ is absolutely irreducible.*

Remark 7.4. Notice that $E \cap E^\perp$ is in fact 0, but this is a consequence of the Hard Lefschetz theorem. Over $\mathbb{C}$, one can prove Hard Lefschetz independently of what we are doing, but the proof of Hard Lefschetz in étale cohomology relies on the Weil conjectures, so we will not be able to assume that $E \cap E^\perp = (0)$ in our appications.

Proof. If $F \subseteq E \otimes \mathbb{C}$ is stable by $\pi_1(U, u)$, and F is not contained in $(E \cap E^\perp) \otimes \mathbb{C}$, then there exists $x \in F$ and $s \in S$ such that $(x, \delta_s) \neq 0$. By Picard-Lefschetz we have $\gamma_s x - x = \pm(x, \delta_s)\delta_s$, and this is an element of F. Furthermore, the vanishing cycles are all conjugate, so all δ_s are in F, which proves $F = E \otimes \mathbb{C}$. $\square$

Let n be odd (this will be the case in our applications, where the total space is even-dimensional and the fibres are odd-dimensional). The duality pairing on $\left(E/E \cap E^\perp\right) \otimes \mathbb{C}$ is non-degenerate. Hence we get a representation

$$\rho : \pi_1(U, u) \to \mathrm{Sp}\left((E/E \cap E^\perp) \otimes \mathbb{C}\right)$$

Remark 7.5. When n is even, it is interesting to study the self-product (δ_s, δ_s), which turns out to be equal to 2.

7.2 Lefschetz theory in étale cohomology

Consider a Lefschetz pencil

$$\begin{array}{c} \overline{X} \\ \downarrow \\ D \cong \mathbb{P}^1 \end{array}$$

defined over an algebraically closed field k. Write $d = \dim \overline{X} = n + 1 = 2m + 2$ and let $s \in D$ be a point where $\overline{X}_s$ is singular. The completion of the local ring $\mathcal{O}_{D,s}$ is $k[[t]]$. Let $B = \operatorname{Spec} k[[t]]$ and consider the Cartesian diagram.

$$\begin{tikzcd} \overline{X}_B \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\ B \arrow[r] & D \cong \mathbb{P}^1 \end{tikzcd}$$

Let $\overline{\eta}$ be a geometric generic point of B and let s be the closed point of B. For any $\ell \neq \mathrm{char}(k)$ we have a cospecialisation map

$$H^i(\overline{X}_s, \mathbb{Q}_\ell) \xleftarrow{\sim} H^i(\overline{X}, \mathbb{Q}_\ell) \to H^i(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell),$$

where the first arrow is an isomorphism by the proper base change theorem (which replaces the argument with deformation retracts).

Let $I = \mathrm{Gal}\left(\overline{k((t))}/k((t))\right) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$ be the inertia subgroup. The action of I on $H^i(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$ is the analogue of the local monodromy action.

7.2.1 Local theory

There exists a vanishing cycle $\delta \in H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)(m)$, well-defined up to sign.

Theorem 7.6. *Assume that $\delta \neq 0$ (if $\delta = 0$ life is actually easier, so this is harmless). The cospecialisation map*

$$\cos : H^i\left(\overline{X}_s, \mathbb{Q}_\ell\right) \to H^i\left(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell\right)$$

is an isomorphism for $i \neq n$. Furthermore, there is an exact sequence

$$\begin{array}{ccccc} 0 \to H^n(\overline{X}_s, \mathbb{Q}_\ell) \to & H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell) & \to & \mathbb{Q}_\ell(m-n) & \to 0 \\ & x & \mapsto & x \cup \delta \in H^{2n}(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)(m) & \end{array} \tag{3}$$

and a Picard-Lefschetz formula: for all $x \in H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$ and $\sigma \in I$ we have

$$\sigma(x) = x \pm t_\ell(\sigma)(x, \delta)\delta,$$

where $t_\ell : I \to \mathbb{Z}_\ell(1)$ is the cyclotomic character, defined by

$$\sigma(t^{1/\ell^n}) = t_\ell(\sigma) \cdot t^{1/\ell^n}.$$

Notice that (x, δ) is an element in $\mathbb{Q}_\ell(m-n)$ and δ is in $H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$, so the product carries a Galois action of weight $2m - n = -1$, which is compensated by the cyclotomic character.

Corollary 7.7. *Suppose the vanishing cycle is nonzero. There is a canonical isomorphism $H^n(\overline{X}_s, \mathbb{Q}_\ell) \cong H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)^I$, and this space has codimension 1 in $H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$.*

7.2.2 Global theory

Consider again

$$\begin{array}{c} \overline{X} \\ \downarrow \\ D \end{array}$$

and let $S = \{s \in D : \overline{X}_s \text{ is singular}\}$. Also define $\overline{U} := D \setminus S$. There is a problem in translating the classical theory over to the étale setting:

Problem 7.8. *$\pi_1(\overline{U}, \overline{u})$ is huge in characteristic p. For example, $\pi_1(\mathbb{A}^1)$ has any finite p-group as a quotient!*

Instead of the full $\pi_1(\overline{U}, \overline{u})$, one considers its tame quotient $\pi_1^t(\overline{U}, \overline{u})$, which classifies étale covers $\overline{V} \to \overline{U}$ extending to $\overline{W} \to D$ tamely ramified at the points of S.

Remark 7.9. There is a general definition of the tame fundamental group for the complement of any divisor in any proper variety.

Good news. As a consequence of Picard-Lefschetz, the action of $\pi_1(\overline{U}, \overline{u})$ on $H^n(\overline{X}_{\overline{u}}, \mathbb{Q}_\ell)$ factors via $\pi_1^t(\overline{U}, \overline{u})$. Furthermore, $\pi_1^t(\overline{U}, \overline{u})$ is generated by the inertia groups I_s for $s \in S$.

Results. The sheaves $R^i\pi_*\mathbb{Q}_\ell$ are locally constant for $i \neq n$: this is an immediate consequence of the local theory, since it can be checked on stalks. Furthermore, the action of the tame fundamental group $\pi_1^t(\overline{U}, \overline{u})$ on these groups is trivial. Let $\overline{\eta}$ be a geometric generic point of $\overline{U}$. For every $s \in S$, there is a vanishing cycle $\delta_s \in H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell))$ (after twisting by $-m$). If E is the subspace generated by the vanishing cycles, then E is stable by the action of $\pi_1^t(\overline{U}, \overline{u})$, and

$$E^\perp = H^n(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell)^{\pi_1^t(\overline{U}, \overline{u})}.$$

This is proven in the same way as the classical case, starting from Picard-Lefschetz.

Remark 7.10. Picard-Lefschetz is proven in SGA7 by reduction to the complex case. Illusie has later given an independent, purely algebraic proof, which however is not easier than the original one.

Furthermore, the δ_s are conjugated under the action of $\pi_1(\overline{U}, \overline{u})$, and as over $\mathbb{C}$ this implies that $E/E \cap E^\perp$ is an absolutely irreducible representation, and we get (for odd n) a representation

$$\rho : \pi_1^t(\overline{U}, \overline{u}) \to \mathrm{Sp}\left(E/E \cap E^\perp\right).$$

Theorem 7.11 (Kazhdan-Margulis)**.** *The image of ρ is Zariski-dense and ℓ-adically open in* $\mathrm{Sp}\left(E/E \cap E^\perp\right)$.

Remark 7.12. This result is not too hard, and is proven by comparing Lie algebras. We shall only use the Zariski part of this theorem.

Corollary 7.13. *Consider the inclusion $j : \overline{U} \hookrightarrow D$ and let $\pi : \overline{X} \to D$ be the Lefschetz pencil.*

1. *The natural adjunction map*

$$R^n\pi_*\mathbb{Q}_\ell \to j_*j^*R^n\pi_*\mathbb{Q}_\ell$$

   is an isomorphism.

2. *$j^*R^n\pi_*\mathbb{Q}_\ell$ is locally constant on $\overline{U}$, and has a filtration by locally constant subsheaves*

$$0 \subseteq \mathcal{E} \cap \mathcal{E}^\perp \subseteq \mathcal{E} \subseteq j^*R^n\pi_*\mathbb{Q}_\ell$$

   where $\mathcal{E}$ is the sheaf with stalks $\mathcal{E}_{\overline{u}} = E_{\overline{u}}$. Moreover, $\pi_1(\overline{U}, \overline{\eta})$ acts trivially on the subsheaf $\mathcal{E} \cap \mathcal{E}^\perp$ (because $E^\perp = H^n\left(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell\right)^{\pi_1^t(\overline{U}, \overline{u})}$), hence in particular $\mathcal{E} \cap \mathcal{E}^\perp$ is a constant sheaf. Similarly, by Picard-Lefschetz the monodromy action on $j^*R^n\pi_*\mathbb{Q}_\ell/\mathcal{E}$ is trivial, and so this is also a constant sheaf.

Proof. We only give details for part 1. This can be checked on stalks: for points in $\overline{U}$ they are the same, and for $s \in S$ we have $(j_*j^*R^n\pi_*\mathbb{Q}_\ell)_{\overline{s}} = H^n\left(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell\right)^{I_s}$, and by Picard-Lefschetz (more precisely, by Corollary 7.7) we have $H^n\left(\overline{X}_{\overline{\eta}}, \mathbb{Q}_\ell\right)^{I_s} = H^n\left(\overline{X}_s, \mathbb{Q}_\ell\right)$. □

7.3 The big picture: how is all this used in Deligne's proof?

Goal: prove that the eigenvalues of the Frobenius F on $H^{n+1}(\overline{X}, \mathbb{Q}_\ell)$ satisfy the estimate of Equation (1).

We have already seen that it is enough to prove this for the eigenvalues of F on

1. $H^0\left(D, R^{n+1}\pi_*\mathbb{Q}_\ell\right)$
2. $H^1\left(D, R^n\pi_*\mathbb{Q}_\ell\right)$
3. $H^2\left(D, R^{n-1}\pi_*\mathbb{Q}_\ell\right)$

The first and the last of these spaces behave as in the case of a smooth fibration. Using the filtration above,

$$0 \subseteq j_*\left(\mathcal{E} \cap \mathcal{E}^\perp\right) \subseteq j_*\mathcal{E} \subseteq j_*\left(R^n\pi_*\mathbb{Q}_\ell\right),$$

and observing that two of the successive quotients are constant, one is left with handling the cohomology $j_*\mathcal{E}/j_*(\mathcal{E} \cap \mathcal{E}^\perp)$. This will be done by a (clever) combination of the techniques introduced today and of the manipulations of L-functions we have seen last time.

8 19.11.2019 – Conclusion of the proof (modulo two technical lemmas)

8.1 Where do we stand?

After many reductions, we are in the following situation: we have a (smooth projective) variety $X/\mathbb{F}_q$ and a Lefschetz pencil

$$\begin{array}{c} X \\ \downarrow \\ D_0 \cong \mathbb{P}^1 \end{array}$$

which by base-change induces $\overline{X} \to D := D_0 \times \overline{\mathbb{F}_q}$. We denote by d and n the dimensions of X and of the fibres of π respectively (so $d = n+1$).

We are reduced to considering the cohomology groups

$$H^0\left(D, R^{n+1}\pi_*\mathbb{Q}_\ell\right), \quad H^1\left(D, R^n\pi_*\mathbb{Q}_\ell\right), \quad H^2\left(D, R^{n-1}\pi_*\mathbb{Q}_\ell\right),$$

and we need to prove that the eigenvalues of Frobenius acting on these spaces satisfy (1), that is,

$$q^{\frac{d}{2}-\frac{1}{2}} \leq |\alpha| \leq q^{\frac{d}{2}+\frac{1}{2}}.$$

Let S be the locus of points over which the fibre is singular, and assume that the corresponding vanishing cycles δ_s are nonzero for all $s \in S$.

Remark 8.1. It is possible for one or more of the vanishing cycles to be zero. This situation can be handled by similar methods, and is not significantly harder.

In this case we obtain that $R^{n+1}\pi_*\mathbb{Q}_\ell$ and $R^{n-1}\pi_*\mathbb{Q}_\ell$ are locally constant (see Corollary 7.13), hence constant because $\mathbb{P}^1$ is simply-connected. These sheaves can then be handled by the same methods we used in the case when all the fibres of the Lefschetz pencil are smooth, see section 6.3.

We have also seen (Corollary 7.13 again) that the adjunction map

$$j_*j^*R^n\pi_*\mathbb{Q}_\ell \xleftarrow{\sim} R^n\pi_*\mathbb{Q}_\ell$$

is an isomorphism, and the sheaf $j^*R^n\pi_*\mathbb{Q}_\ell$ has a filtration

$$0 \subseteq j_*\left(\mathcal{E} \cap \mathcal{E}^\perp\right) \subseteq j_*\mathcal{E} \subseteq j_*\left(R^n\pi_*\mathbb{Q}_\ell\right),$$

where $\mathcal{E}$ is the locally constant sheaf with fibre $E_{\overline{u}}$ at $\overline{u} \in U$. We have remarked that $\mathcal{E} \cap \mathcal{E}^\perp$ is now known to be 0, but this is proven as a *consequence* of the Weil conjectures (including the Riemann hypothesis), so we cannot assume that this is the case.

Finally, we have discussed the Picard-Lefschetz formula: given $s \in S$, we have a vanishing cycle $\delta_s \in H^n(X_{\overline{u}}, \mathbb{Q}_\ell(n))$ and an element $\gamma_s \in \pi_1^t(\overline{U}, \overline{u})$ (a generator for the inertia group at s), and for the natural action of the tame fundamental group on $H^n(X_{\overline{u}}, \mathbb{Q}_\ell(n))$ we have

$$\gamma_s \cdot x = x \pm t_\ell(\gamma_s)(x, \delta_s)\delta_s.$$

This formula shows that the action of $\pi_1^t(\overline{U}, \overline{u})$ is trivial on $\mathcal{E} \cap \mathcal{E}^\perp$, as well as on the quotient $j^*R^n\pi_*\mathbb{Q}_\ell/\mathcal{E}$ (indeed, the action of the generators γ_s can only change any given cohomology class by a vanishing cycle, which is in $\mathcal{E}$).

8.2 Study of the filtration on $R^n\mathbb{Q}_\ell$

Pushing forward the filtration we have on $j^*R^n\pi_*\mathbb{Q}_\ell$ by j_* we get a filtration on $j_*j^*R^n\mathbb{Q}_\ell = R^n\mathbb{Q}_\ell$:

$$0 \to j_*(\mathcal{E} \cap \mathcal{E}^\perp) \subset j_*\mathcal{E} \subset R^n\pi_*\mathbb{Q}_\ell.$$

Denote by E the fibre $\mathcal{E}_{\overline{u}}$. We now distinguish two cases:

- Case 1: $\delta_s \notin E^\perp$ for all $s \in S$. We have an exact sequence

$$0 \to E \to H^n(X_{\overline{u}}, \mathbb{Q}_\ell) \to H^n(X_{\overline{u}}, \mathbb{Q}_\ell)/E \to 0. \tag{4}$$

  The local inertia group I_s at s acts trivially on $H^n(X_{\overline{u}}, \mathbb{Q}_\ell)/E$ (again by Picard-Lefschetz), and the inclusion

$$H_n\left(X_{\overline{u}}, \mathbb{Q}_\ell\right)^{I_s} \subset H^n(X_{\overline{u}}, \mathbb{Q}_\ell)$$

  has codimension 1 (this follows from the exact sequence (3)) . The assumption on δ_s implies that restricting to E we obtain an inclusion $E^{I_s} \subset E$ which is still of codimension 1. In particular, we see that taking invariants in (4) gives another exact sequence

$$0 \to E^{I_s} \to H^n(X_{\overline{u}}, \mathbb{Q}_\ell)^{I_s} \to H^n(X_{\overline{u}}, \mathbb{Q}_\ell)/E \to 0.$$

  At the level of sheaves, this gives an exact sequence of sheaves

$$0 \to j_*\mathcal{E} \to R^n\pi_*\mathbb{Q}_\ell \to \begin{array}{c}\text{constant}\\ \text{sheaf}\end{array} \to 0.$$

  The constancy of the quotient follows from the fact that the monodromy action around the singular points is trivial (for each singular point). The long exact sequence in cohomology then yields

$$H^1\left(D, j_*\mathcal{E}\right) \twoheadrightarrow H^1\left(D, R^n\pi_*\mathbb{Q}_\ell\right).$$

  Now consider the other relative quotient in the filtration,

$$0 \to E \cap E^\perp \to E \to E/(E \cap E^\perp) \to 0,$$

  equipped with its natural I_s-action. The action is trivial on $E \cap E^\perp$ by Picard-Lefschetz, and the inclusions $E^{I_s} \subseteq E$ and $\left(E/E \cap E^\perp\right)^{I_s} \subseteq E/(E \cap E^\perp)$ are of codimension 1. As above, we obtain a sequence

$$0 \to \begin{array}{c}\text{constant}\\ \text{sheaf}\end{array} \to j_*\mathcal{E} \to j_*\left(\mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp\right),$$

  hence an injection

$$0 \to H^1\left(D, j_*\mathcal{E}\right) \hookrightarrow H^1\left(D, j_*\left(\mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp\right)\right).$$

  It follows that it suffices to study the eigenvalues of Frobenius on $H^1(D, j_*\mathcal{E}/(\mathcal{E} \cap \mathcal{E}^\perp))$ (this contains $H^1(D, j_*\mathcal{E})$, which in turn surjects onto the relevant cohomology group $H^1\left(D, R^n\pi_*\mathbb{Q}_\ell\right)$).

- Case 2: $\forall s \in S$, the vanishing cycle δ_s is in $E^\perp$. To see that this is indeed the complementary case to case (1) above, simply recall that all the δ_s are conjugated to each other. In particular, $E \subseteq E^\perp$. Fix $s \in S$ and consider the corresponding inertia group I_s. As before, the action of I_s is trivial on $E^\perp$ by Picard-Lefschetz, and we obtain a slightly more complicated exact sequence

$$0 \to E^\perp \to H^n\left(X_{\overline{u}}, \mathbb{Q}_\ell\right)^{I_s} \to H^n\left(X_{\overline{u}}, \mathbb{Q}_\ell\right)/E^\perp \to H^n\left(X_{\overline{u}}, \mathbb{Q}_\ell\right)/\langle \delta_s \rangle^\perp \to 0.$$

The inertia action on $E^\perp$ is trivial, and – as $E \subseteq E^\perp$ – so is the action on $H^n(X_{\overline{u}}, \mathbb{Q}_\ell)/E^\perp$. It follows that the whole sequence consists of I_s-invariant groups. Moreover,

$$H^n(X_{\overline{u}}, \mathbb{Q}_\ell)/\langle \delta_s \rangle^\perp \cong \mathbb{Q}_\ell(m-n)$$

by the exact sequence (3). We split the exact sequence above into two pieces,

$$0 \to \begin{array}{c}\text{constant}\\ \text{sheaf}\end{array} \to R^n\pi_*\mathbb{Q}_\ell \to \mathcal{F} \to 0$$

and

$$0 \to \mathcal{F} \to \begin{array}{c}\text{constant}\\ \text{sheaf}\end{array} \to \bigoplus_{s\in S} \mathbb{Q}_\ell(m-n) \to 0.$$

The two exact sequences give

$$\begin{tikzcd} H^1(D, R^n\pi_*\mathbb{Q}_\ell) \arrow[r, hook] & H^1(D, \mathcal{F}) \\ & H^0(D, \mathbb{Q}_\ell(m-n)) \arrow[u, two heads] \end{tikzcd}$$

and by definition Frobenius acts as multiplication by $q^{n-m} = q^{\frac{n+1}{2}}$, which implies that Frobenius has the desired weight on $H^1(D, R^n\pi_*\mathbb{Q}_\ell)$.

8.3 The sheaf $\mathcal{E}/\left(\mathcal{E} \cap \mathcal{E}^\perp\right)$

From now on, let $\mathcal{F} := \mathcal{E}/\left(\mathcal{E} \cap \mathcal{E}^\perp\right)$. We have to prove that the eigenvalues of Frobenius on $H^1\left(D, j_*\mathcal{F}\right)$ satisfy

$$q^{\frac{d}{2}-\frac{1}{2}} \leq |\alpha| \leq q^{\frac{d}{2}+\frac{1}{2}}.$$

Lemma 8.2. *It suffices to show* $|\alpha| \leq q^{\frac{d}{2}+\frac{1}{2}}$.

Proof. Poincaré duality. □

Lemma 8.3. *It is enough to prove the estimate for the eigenvalues of Frobenius on* $H^1_c(U, \mathcal{F})$.

Proof. There is an exact sequence

$$0 \to j_!\mathcal{F} \to j_*\mathcal{F} \to j_*\mathcal{F}/j_!\mathcal{F} \to 0,$$

and the sheaf $j_*\mathcal{F}/j_!\mathcal{F}$ has only finitely many nonzero stalks. This induces

$$H^1_c(U, \mathcal{F}) = H^1(D, j_*\mathcal{F}) \to H^1(D, j_*\mathcal{F}) \to 0,$$

because the H^1 of an étale sheaf supported at finitely many points is zero (this can be checked for example by using Čech cohomology, which coincides with étale cohomology at the level of H^1). □

Lemma 8.4. *Assume that for every closed point $x \in U$ the local Frobenius $F_{\overline{x}}^{\deg x}$ acts on $\mathcal{F}_{\overline{x}}$ with eigenvalues that are algebraic numbers and satisfy $|\alpha| \leq \left(q^{\deg x}\right)^{n/2}$. Assume moreover that the characteristic polynomial of $F_{\overline{x}}^{\deg x}$ has rational coefficients. Then the eigenvalues of Frobenius on $H_c^1(\overline{U}, \mathcal{F})$ satisfy $|\alpha| \leq q^{\frac{d+1}{2}}$.*

Proof. Postponed until next time. □

We are thus reduced to the following

Lemma 8.5 (Main lemma)**.** *Let $\mathcal{F}$ be a locally constant sheaf of finite-dimensional $\mathbb{Q}_\ell$-vector spaces on U such that*

(a) for all closed points $x \in U$, the local Frobenius $F_{\overline{x}}^{\deg(x)*}$ acts on $\mathcal{F}_{\overline{x}}$ with a characteristic polynomial in $\mathbb{Q}[T]$.

(b) There exists a $\pi_1(U, \overline{\eta})$-invariant and $\mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right)$-equivariant alternating, non-degenerate form

$$\mathcal{F}_{\overline{x}} \times \mathcal{F}_{\overline{x}} \to \mathbb{Q}_\ell(-n)$$

for all geometric points $\overline{x} \to U$.

(c) the image of $\pi_1(\overline{U}, \overline{u})$ in $\mathrm{Sp}(\mathcal{F}_{\overline{x}})$ is Zariski dense.

Then the eigenvalues of $F_{\overline{x}}^{\deg(x)*}$ on $\mathcal{F}_{\overline{x}}$ satisfy $|\alpha| \leq \left(q^{\deg x}\right)^{n/2}$.

We now show that the main lemma implies the theorem:

Proof. We apply the lemma to our $\mathcal{F} := \mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp$. We will prove assumption (a) next time; (b) is simply Poincaré duality on fibres, and (c) is Kazhdan-Margulis (Theorem 7.11). We then get the inequality stated in the main lemma, which implies the theorem by Lemma 8.4. □

8.4 Proof of the Main Lemma

Reduction 1. It is enough to prove that the eigenvalues of Frobenius $F_{\overline{x}}^{\deg(x)*}$ on $\mathcal{F}_{\overline{x}}^{2k}$ satisfy $|\beta| \leq (q^{\deg x})^{kn+1}$ for every $k \geq 0$. Indeed, α^{2k} is among the eigenvalues β on $\mathcal{F}_{\overline{x}}^{\otimes 2k}$, and we get the assumption by passing the limit $k \to \infty$.

Reduction 2. It is enough to prove that the eigenvalues of Frobenius on $H_c^2\left(\overline{U}, \mathcal{F}^{\otimes 2k}\right)$ have absolute value at most q^{kn+1}.

This is very similar to Proposition 5.9, but we repeat the argument. Recall the elementary identity

$$\det\left(1 - F_{\overline{x}}^{\deg(x)*} T^{\deg x} \mid \mathcal{F}_{\overline{x}}^{\otimes 2k}\right) = \exp\left\{\sum_{m=1}^{\infty} \mathrm{Tr}\left(F_{\overline{x}}^{\deg(x)*m} \mid \mathcal{F}_{\overline{x}}^{\otimes 2k}\right) \frac{T^{m \deg(x)}}{m}\right\}$$

By assumption, for $k = \frac{1}{2}$ this is a polynomial (in particular, a power series in $\mathbb{Q}[[T]]$), hence $\mathrm{Tr}\left(F_{\overline{x}}^{\deg(x)*m} \mid \mathcal{F}_{\overline{x}}\right)$ is a rational number. But for any $k \geq 0$ we have

$$\mathrm{Tr}\left(F_{\overline{x}}^{\deg(x)*m} \mid \mathcal{F}_{\overline{x}}^{\otimes 2k}\right) = \mathrm{Tr}\left(F_{\overline{x}}^{\deg(x)*m} \mid \mathcal{F}_{\overline{x}}\right)^{2k},$$

so if we write

$$Z_{\overline{x}}(\mathcal{F}^{\otimes 2k}, T) := \det\left(1 - F_{\overline{x}}^{\deg(x)*} T^{\deg x} \mid \mathcal{F}_{\overline{x}}^{\otimes 2k}\right)$$

we see that for all integer $k \geq 0$ this is a power series with *positive* coefficients.

Notation 8.6. *We denote by ρ the radius of convergence of a complex power series.*

We obtain

$$\rho\left(Z_{\overline{x}}\left(\mathcal{F}^{\otimes 2k}, T\right)\right) \leq \rho\left(\prod_{x \in U} Z_{\overline{x}}(\mathcal{F}^{\otimes 2k}, T)\right) = \rho\left(Z(U, \mathcal{F}^{\otimes 2k}, T)\right);$$

on the other hand,

$$Z(U, \mathcal{F}^{\otimes 2k}, T) = \frac{\det\left(1 - F^* T \mid H_c^1(\overline{U}, \mathcal{F}^{\otimes 2k}\right)}{\det\left(1 - F^* T \mid H_c^2(\overline{U}, \mathcal{F}^{\otimes 2k})\right)},$$

where the H_c^0 vanishes because it's dual to the H^2 of an affine curve, which vanishes.

Now α is a pole of the RHS if and only if it is a pole of the LHS, and $\alpha^{\deg x}$ is a pole of $Z_{\overline{x}}(\mathcal{F}^{\otimes 2k}, T)$ if and only if α is a pole of $Z(U, \mathcal{F}^{\otimes 2k}, T)$.

Main proof. We have to consider

$$H_c^2\left(\overline{U}, \mathcal{F}^{\otimes 2k}\right) \cong H^0\left(\overline{U}, \mathcal{F}^{\otimes 2k}(-1)\right)^\vee = \mathrm{Hom}\left(\mathcal{F}_{\overline{x}}^{\otimes 2k}, \mathbb{Q}_\ell\right)^{\pi_1(\overline{U})}(-1).$$

By assumption (c), we have

$$\mathrm{Hom}\left(\mathcal{F}_{\overline{x}}^{\otimes 2k}, \mathbb{Q}_\ell\right)^{\pi_1(\overline{U})} = \mathrm{Hom}\left(\mathcal{F}_{\overline{x}}^{\otimes 2k}, \mathbb{Q}_\ell\right)^{\mathrm{Sp}(\mathcal{F}_{\overline{x}})}$$

Recall the following general fact:

Theorem 8.7. *Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional symplectic space over a field K. For every partition $\mathcal{P}$ of $\{1, 2, \ldots, 2n\}$ in pairs $\{a_i, b_i\}$, an $\mathrm{Sp}(V)$-invariant function on $V^{\otimes 2k}$ is given by*

$$\begin{array}{rccc} f_{\mathcal{P}}: & V^{\otimes 2k} & \to & K \\ & v_1 \otimes \cdots \otimes v_{2k} & \mapsto & \prod_i \langle v_{a_i}, v_{b_i}\rangle. \end{array}$$

Furthermore, as $\mathcal{P}$ varies, these give a basis for the $\mathrm{Sp}(V)$-invariant functions $V^{\otimes 2k} \to K$.

We obtain

$$H_c^2\left(\overline{U}, \mathcal{F}^{\otimes 2k}\right) \cong \mathrm{Hom}\left(\mathcal{F}_{\overline{x}}^{\otimes 2k}, \mathbb{Q}_\ell\right)^{\mathrm{Sp}(\mathcal{F}_x)}(-1) \cong \bigoplus \mathbb{Q}_\ell(-kn)^{\oplus|\mathcal{P}|}(-1).$$

Notice that the weight $-kn$ comes from the fact that the bilinear form $\langle\cdot,\cdot\rangle$ has weight $-n$, and we are taking the product of k such forms. The eigenvalues of Frobenius on $\mathbb{Q}_\ell(-kn-1)$ are q^{kn+1}, and we are done.

9 26.11.2019 – Leftovers

Our first leftover is Lemma 8.4, which we now restate. Consider a Lefschetz pencil

$$\begin{array}{c} X \\ \downarrow \\ D \cong \mathbb{P}^1 \end{array}$$

where everything lives over the finite field $\mathbb{F}_q$. Let $j : U \hookrightarrow D$ be the open subset of D over which the fibres are smooth. Write $d = n + 1 = \dim X$, and recall that have constructed a sheaf of vanishing cycles $\mathcal{E}$. We are interested in the subsheaf $j_*\mathcal{E}$ of $R^n\pi_*\mathbb{Q}_\ell$, and this led us to consider the sheaf

$$\mathcal{F} := \mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp.$$

Lemma 9.1 (Lemma 8.4)**.** *Assume that for every closed point $x \in U$ the local Frobenius $F_{\overline{x}}^{\deg x}$ acts on $\mathcal{F}_{\overline{x}}$ with eigenvalues that are algebraic numbers and satisfy $|\alpha| \leq \left(q^{\deg x}\right)^{n/2}$. Assume moreover that the characteristic polynomial of $F_{\overline{x}}^{\deg x}$ has rational coefficients. Then the eigenvalues of Frobenius on $H_c^1(\overline{U}, \mathcal{F})$ satisfy $|\alpha| \leq q^{\frac{d+1}{2}}$.*

Remark 9.2. If one uses the axiom of choice (and therefore the fact that the algebraic closure of $\mathbb{Q}_\ell$ embeds in $\mathbb{C}$), the lemma can also be proven without assuming rationality of the coefficients of the characteristic polynomials of Frobenius. However, we will prove rationality later today, so we might as well assume it.

Proof. By Poincaré duality we have $H_c^0(\overline{U}, \mathcal{F}) \cong H^2(\overline{U}, \mathcal{F}^\vee(-1))^\vee$, which vanishes because $\overline{U}$ is affine of dimension 1 over an algebraically closed field (Theorem 4.1). Similarly,

$$H_c^2(\overline{U}, \mathcal{F}) \cong H^0(\overline{U}, \mathcal{F}^\vee(-1))^\vee.$$

As $\mathcal{F}$ is locally constant, taking H^0 is the same as taking π_1-invariants, and after dualising this statement we get

$$H_c^2(\overline{U}, \mathcal{F}) \cong H^0(\overline{U}, \mathcal{F}^\vee(-1))^\vee \cong \left(E/E \cap E^\perp\right)_{\pi_1(\overline{U}, \overline{\eta})}(-1).$$

By Kazhdan-Margulis (Theorem 7.11) we have

$$\left(E/E \cap E^\perp\right)_{\pi_1(\overline{U}, \overline{\eta})} = \left(E/E \cap E^\perp\right)_{\mathrm{Sp}\left(E/E \cap E^\perp\right)}$$

It is a fact from invariant theory that

$$\mathrm{Hom}(V, K)^{\mathrm{Sp}(V)} = (0)$$

for any finite-dimensional K-vector space V. In our setting, this implies that $H_c^2(\overline{U}, \mathcal{F})$ is trivial. As a consequence, the Zeta function

$$Z(U, \mathcal{F}, T) = \prod_{x \in U_{(0)}} \det\left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right)^{-1}$$

is equal (by the Lefschetz trace formula) to

$$\det\left(1 - F^*T \mid H^1_c(\overline{U}, \mathcal{F})\right).$$

Using the assumption that characteristic polynomials of Frobenius have rational coefficients, we have

$$\det\left(1 - F^*T \mid H^1_c(\overline{U}, \mathcal{F})\right) = \prod_{x \in U_{(0)}} \det\left(1 - F_{\overline{x}}^{\deg x}\, T \mid \mathcal{F}_{\overline{x}}\right) \in \mathbb{Q}[[T]] \subset \mathbb{C}[[T]],$$

and we now show that this power series converges for $|T| \leq \frac{1}{q^{\frac{d+1}{2}}} = \frac{1}{q^{\frac{n}{2}+1}}$.

Let $\alpha_{x,i}$, for $1 \leq i \leq r = \operatorname{rk} \mathcal{F}_{\overline{x}}$, be the eigenvalues of $\mathrm{F}_{\overline{x}}^{\deg x}$ on $\mathcal{F}_{\overline{x}}$. Then

$$Z(U, \mathcal{F}, T)^{-1} = \prod_{x,i} (1 - \alpha_{x,i} T^{\deg(x)}),$$

and it is enough to show that

$$\sum_{x,i} |\alpha_{x,i} T^{\deg(x)}\| < \infty$$

for $|T| \leq \frac{1}{q^{\frac{n}{2}+1}}$. We have

$$\sum_{x,i} \|\alpha_{x,i} T^{\deg x}\| \leq \sum_{x,i} \|\alpha_{x,i}\| \cdot \|T\|^{\deg x} \leq r \sum_x (q^{\deg x})^{n/2} |T|^{\deg x} \leq r \sum_m (q^m)^{n/2+1} |T^m|,$$

where we have used the trivial estimate

$$\#\{x \in U \text{ closed point} : \deg x \leq m\} \leq q^m.$$

Finally, the sum $r \sum_m (q^m)^{n/2+1} |T^m|$ converges for $|T| < \frac{1}{q^{n/2+1+\varepsilon}}$, as desired. □

9.1 Rationality theorem

Our second (and final) leftover is the following rationality result:

Theorem 9.3 (Théorème de rationalité, Weil I)**.** *For every closed point $x \in U$ the characteristic polynomial*

$$\det\left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right)$$

is in $\mathbb{Q}[T]$.

We will prove this under the assumption $q > |S| = |X \setminus U|$. Let X_x be the fibre of $\pi : X \to \mathbb{P}^1_{\mathbb{F}_q}$ above x. We already know that $Z(X_x, T)$ is a rational function of T with rational coefficients.

Lemma 9.4. *There exist units*[3] *$\alpha_i, \beta_j \in \overline{\mathbb{Q}_\ell}^\times$, independent of x, such that*

$$Z(X_x, T) = \frac{\prod_i (1 - \alpha_i^{\deg x} T)}{\prod_j (1 - \beta_j^{\deg x} T)} \det\left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right).$$

[3] that is, algebraic elements of valuation 0

Proof. We have

$$Z(X_x,T)=\prod_{i=0}^{2n}\det\left(1-F_{\overline{x}}^{*}T\mid\left(R^i\pi_*\mathbb{Q}_\ell\right)_{\overline{x}}\right)^{(-1)^{i+1}},$$

and we have seen[4] that the sheaves $\left(R^i\pi_*\mathbb{Q}_\ell\right)_{\overline{x}}$ are constant for $i\neq n$. In particular, for $i\neq n$ the operator $F_{\overline{x}}^{\deg x}$ acts on $R^i\pi_*\mathbb{Q}_\ell$ acts as $A_i^{\deg x}$, where $A_i\in \mathrm{G}_{r_i}(\mathbb{Q}_\ell)$ is a constant matrix. The eigenvalues are elements of $\overline{\mathbb{Q}_\ell}$ of valuation 0, because (by definition of an ℓ-adic sheaf) the action of any automorphism preserves an integral structure, so all its eigenvalues are integral. It follows that

$$Z(X_x,T)=\frac{\prod_i\left(1-(\alpha_i')^{\deg x}T\right)}{\prod_j\left(1-(\beta_j')^{\deg x}T\right)}\det\left(1-F_{\overline{x}}^{\deg x}T\mid R^n\pi_*\mathbb{Q}_\ell\right).$$

To deal with $R^n\pi_*\mathbb{Q}_\ell$, recall that we have a filtration

$$0\subseteq j_*(\mathcal{E}\cap\mathcal{E}^\perp)\subseteq j_*\mathcal{E}\subseteq R^n\pi_*\mathbb{Q}_\ell$$

where the first and last relative quotients are constant. Repeating the same argument as above for these constant sheaves, we get some more powers of constant matrices, and all that is left is then the factor $\det\left(1-F_{\overline{x}}^{\deg x}T\mid R^n\pi_*\mathbb{Q}_\ell\right)$ as claimed. □

The idea is now that if $1-F_{\overline{x}}^{\deg x}$ has an eigenvalue which is not an algebraic number, then – using the fact that $Z(X_x,T)$ is a rational function with rational coefficients – this eigenvalue must cancel with some factor $1-\beta_j^{\deg x}T$, and this *for all* x. The next lemma essentially shows that this cannot happen.

Lemma 9.5. *Fix $N>1$. There does not exist any integral $\lambda\in\mathbb{Q}_\ell^\times$ such that $\lambda^{\deg x}$ is an eigenvalue of $F_{\overline{x}}^{\deg x}$ on $\mathcal{F}_{\overline{x}}$ for all $x\in U$ with $(\deg x,N)=1$.*

Proof. Assume such a λ exists. Let $E/\mathbb{Q}_\ell$ be a finite extension such that $\lambda\in\ mathcalO_E^\times$. Notice that the homomorphism

$$\begin{array}{ccc}\mathbb{Z} & \to & \mathcal{O}_E^\times\\ n & \mapsto & \lambda^{-n}\end{array}$$

extends to $\widehat{\mathbb{Z}}\to\mathcal{O}_E^\times$. To see this, let ℓ^k be the order of the finite residue field of $\mathcal{O}_E$, observe that λ^{ℓ^k-1} is in $1+\mathfrak{m}_E$, and recall that one can take arbitrary ℓ-adic powers of elements in $1+\mathfrak{m}_E$.

Consider now the exact sequence

$$1\to\pi(\overline{U},\overline{x})^t\to\pi_1(U,\overline{x})^t\to\mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right)\to 0,$$

where $\mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right)\cong\widehat{\mathbb{Z}}$. Consider the character

$$\chi:\pi_1(U,\overline{x})\to\widehat{\mathbb{Z}}\to\mathcal{O}_E^\times,$$

[4] under the assumption that the vanishing cycles are nonzero. But if they are zero, the sheaf $\mathcal{F}$ is trivial and there is nothing to prove

and observe that $\chi\left(F_{\overline{x}}^{\deg x}\right) = \lambda^n$. It follows that

$$\det(1 - F_{\overline{x}}^{\deg x}\chi(F_{\overline{x}}^{\deg x})) = 0,$$

and by Chebotarev we know that the $F_{\overline{x}}^{\deg x}$, for all x of degree prime to N, are dense in $\pi_1(U, \overline{x})^t$. We get that for all $g \in \pi_1(U, \overline{u})^t$ the determinant $\det(1 - g\chi(g))$ vanishes. Now use the fact that there exists $x_0 \in U$ of degree 1. Let F_0 be the associated Frobenius. For all $h_0 \in \pi_1(\overline{U}, x_0)^t$, the operator $F_0 h_0$ has eigenvalue λ on $\mathcal{F}_{\overline{x}}$. For the Poincaré duality pairing on $\mathcal{F}_{\overline{x}}$ we have

$$\langle F_0 h_0(x), F_0 h_0(y)\rangle = \langle F_0(x), F_0(y)\rangle = q^n \langle x, y\rangle,$$

hence

$$\{q^{-n/2} F_0 h_0 : h_0 \in \pi_1^t(\overline{U}, \overline{x})\} \subseteq \mathrm{Sp}(\mathcal{F}_{\overline{x}}),$$

and by Kazhdan-Margulis again this implies that $q^{-n/2} F_0 h_0$ are dense in $\mathrm{Sp}(\mathcal{F}_{\overline{x}})$. This implies that $\lambda q^{-n/2}$ is a common eigenvalue for all the elements of $\mathrm{Sp}(\mathcal{F}_{\overline{x}})$, which is a contradiction. □

Notation 9.6. *If f is a polynomial of the form $\prod_k (1 - \gamma_k T)$, we write*

$$f^{(m)} := \prod_k (1 - \gamma_k^m T).$$

Lemma 9.7. *Assume that $f \in \overline{\mathbb{Q}_\ell}[T]$, with $f(0) = 1$, is such that*

$$f^{(\deg x)} Z(X_x, T) \in \overline{\mathbb{Q}_\ell}[T] \quad \forall x \in U \text{ closed point}$$

Then $\prod_j (1 - \beta_j T) \mid f$, where the β_j are as in the statement of Lemma 9.4.

Proof. Write $f = \prod_k (1 - \gamma_k T)$. Then

$$\frac{\prod_k (1 - \gamma_k^{\deg x} T)(1 - \alpha_i^{\deg x} T)}{\prod (1 - \beta_j^{\deg x} T)} \det\left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right)$$

is a polynomial for all closed points x. Let N be a common multiple of all r such that $\beta_j^r = \gamma_k^r$ or $\alpha_i^r = \beta_j^r$ for some $\alpha_i, \beta_j, \gamma_k$ and r is minimal with this property. Also if $\beta_j = \alpha_i$ or $\beta_j = \gamma_k$, then simplify. If no β_j remains we are done. If β_j does remain, then $\beta_j^{\deg x}$ is an eigenvalue of $F_{\overline{x}}^{\deg x}$ on $\mathcal{F}_{\overline{x}}$ for all x such that $(\deg x, N) = 1$. But this is impossible by Lemma 9.5. □

Corollary 9.8. *1. $\prod_j (1 - \beta_j T)$ is a polynomial with rational coefficients.*

2. $\prod_i (1 - \alpha_i^{\deg x} T) \det\left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right)$ is a polynomial with rational coefficients.

Proof. The second part is a consequence of the first: this is obvious for $\deg x = 1$, using that $Z(X_x, T)$ has rational coefficients. Moreover, if $\prod (1 - \beta_j T)$ has rational coefficients, so does $\prod (1 - \beta_j^r T)$ by Galois theory, which proves it for $\deg x = r$. For part (1), let

$$\mathcal{S} := \{f \in \overline{\mathbb{Q}_\ell}[T] : f(0) = 1, f^{\deg x} Z(X_x, T) \in \overline{\mathbb{Q}_\ell}[T]\}.$$

Then by Lemma 9.7 we have that the greatest common divisor of all the polynomials in $\mathcal{S}$ is $\prod(1-\beta_j T)$, and on the other hand $\mathcal{S}$ is stable under the action of $\mathrm{Aut}(\overline{\mathbb{Q}_\ell})$. Hence in particular the coefficients of $\prod(1-\beta_j T)$ are in

$$\overline{\mathbb{Q}_\ell}^{\mathrm{Aut}(\overline{\mathbb{Q}_\ell})} = \mathbb{Q}$$

as desired. □

The corollary implies in particular that all the roots of $\det\left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right)$ are algebraic numbers. It remains to show that the set of such roots is stable under the Galois action.

Proof of Theorem 9.3. We already know that $\prod_i (1-\alpha_i^{\deg x} T) \det\left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right)$ is a polynomial with rational coefficients. If all the Galois conjugates of the α_i are among the α_i, then the same is true for the $\alpha_i^{\deg(x)}$ for all x. If there exists an eigenvalue of $F_{\overline{x}}^{\deg x}$ that is a conjugate of some $\alpha_i^{\deg x}$, then already the α_i are not stable by Galois. Hence there exists a conjugate $\tilde{\alpha}_i$ of some α_i such that $\tilde{\alpha}_i^{\deg x}$ is an eigenvalue of $F_{\overline{x}}^{\deg x}$ on $\mathcal{F}_{\overline{x}}$ for all x. But this again contradicts Lemma 9.5. □

Remark 9.9. To completely formalise this proof one needs to take the expression $\prod(1-\alpha_i T)$ of the minimal degree such that $\prod(1-\alpha_i T)\det\left(1 - F_{\overline{x}}^{\deg x} T \mid \mathcal{F}_{\overline{x}}\right)$ is a polynomial with rational coefficients.

10 03.12.2019 – A brief overview of Weil II

10.1 $\overline{\mathbb{Q}_\ell}$-sheaves

Let $E/\mathbb{Q}_\ell$ be a finite extension. The ring $\mathcal{O}_E$ is a finite-dimensional $\mathbb{Z}_\ell$-algebra, and one can generalise the construction of $\mathbb{Z}_\ell$-sheaves to this case to obtain a $\mathcal{O}_E$-sheaf. Tensoring with E (over $\mathcal{O}_E$) we get the notion of an E-sheaf, and the category of $\overline{\mathbb{Q}_\ell}$-sheaves is the direct limit of the categories of E-sheaves for $E \subset \overline{\mathbb{Q}_\ell}$.

Remark 10.1. In particular, a $\mathbb{Q}_\ell$-sheaf is also a $\overline{\mathbb{Q}_\ell}$-sheaf.

10.2 Purity

Definition 10.2. Let X be a scheme of finite type over $\mathbb{F}_q$ and let $\mathcal{F}$ be a $\overline{\mathbb{Q}_\ell}$-sheaf on X. We say that $\mathcal{F}$ is **(punctually) pure** if $\exists w \in \mathbb{Z}$ such that the eigenvalues of $F_{\overline{x}}^{\deg x}$ on $\mathcal{F}_{\overline{x}}$ satisfy

$$|\tau(\alpha)| = \left(q^{\deg x}\right)^{w/2}$$

for all $\tau : \overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \mathbb{C}$ and for all closed points x of X. If we want to specify w, we say that $\mathcal{F}$ is **pure of weight** w**.** We say that $\mathcal{F}$ is **mixed** if it has a finite filtration such that all successive quotients are pure. We say that it is **mixed of weights** $\leq n$ if (for some filtration) the successive quotients are pure, and weights of such quotients are all $\leq n$.

Remark 10.3. There exist more general notions of being τ-pure and τ-mixed for a fixed embedding τ.

Theorem 10.4 (Main theorem of Weil II)**.** *Let $f : X \to Y$ be a morphism of schemes of finite type over $\mathbb{F}_q$. If $\mathcal{F}$ is a constructible $\overline{\mathbb{Q}_\ell}$-sheaf on X that is mixed of weights $\leq n$. Then, for every integer $i \geq 0$, the sheaf $R^i f_! \mathcal{F}$ is also mixed of weights $\leq n + i$.*

Corollary 10.5. *In the case $Y = \operatorname{Spec} \mathbb{F}_q$, we get that $H^i_c(\overline{X}, \mathcal{F})$ is mixed of weights $\leq n + i$.*

Corollary 10.6. *Assume moreover that X is smooth. If $\mathcal{F}$ is mixed of weights $\geq n$, then $H^i(\overline{X}, \mathcal{F})$ is mixed of weights $\geq n + i$.*

Proof. Poincaré duality. □

Corollary 10.7. *If X is smooth and $\mathcal{F}$ is pure of weight n, then*

$$\operatorname{Im}\left(H^i_c(\overline{X}, \mathcal{F}) \to H^i(\overline{X}, \mathcal{F})\right)$$

is pure of weight $n + i$. In particular, if X is proper and smooth, then $H^i(\overline{X}, \mathcal{F})$ is pure of weight $n + i$.

Remark 10.8. This immediately implies Weil I (which is the case $\mathcal{F} = \mathbb{Q}_\ell$, of weight 0). Also notice that here we assume that X is only proper, and not necessarily projective.

Theorem 10.9 (Semisimplicity theorem)**.** *If X is smooth and $\mathcal{F}$ is lisse and pure, then $\mathcal{F}$ is semisimple, that is, a direct sum of irreducible subsheaves. In particular, if $f : X \to Y$ is proper and smooth, the sheaves $R^i f_* \mathbb{Q}_\ell$ are semisimple (because they are pure, by Weil I).*

Remark 10.10. Let $f : X \to Y$ be a proper and smooth morphism of schemes over $\mathbb{C}$. It is again true that the sheaves $R^i f_* \mathbb{Q}_\ell$ are semisimple, and this can be deduced from the finite field case.

Proof. Let $\overline{\mathcal{F}'} \subseteq \overline{\mathcal{F}}$ on $\overline{X}$ be the largest semisimple lisse subsheaf (in other words, the – automatically direct – sum of all the irreducible subsheaves). By construction, $\overline{\mathcal{F}'}$ is stable by Frobenius, and therefore it comes from a subsheaf $\mathcal{F}'$ of $\mathcal{F}$ over X. Let $\mathcal{F}'' := \mathcal{F}/\mathcal{F}'$. We have an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \tag{5}$$

that gives rise to a class in $\operatorname{Ext}^1_X(\mathcal{F}'', \mathcal{F}')$. We can map this class to $\operatorname{Ext}^1_{\overline{X}}\left(\overline{\mathcal{F}''}, \overline{\mathcal{F}'}\right)^F$, where the superscript F denotes the subset of elements fixed by Frobenius. Hence we get

$$\operatorname{Ext}^1_X\left(\mathcal{F}'', \mathcal{F}'\right) \to \operatorname{Ext}^1_{\overline{X}}\left(\mathcal{F}'', \mathcal{F}'\right)^F = H^1\left(\overline{X}, \underline{\operatorname{Hom}}_{\overline{X}}(\mathcal{F}'', \mathcal{F}')\right)^F$$

Since $\mathcal{F}', \mathcal{F}''$ are both pure of weight w, the sheaf $\underline{\operatorname{Hom}}(\mathcal{F}'', \mathcal{F}')$ is of weight 0, hence $H^1(\overline{X}, \underline{\operatorname{Hom}}(\mathcal{F}'', \mathcal{F}'))$ is of weights ≥ 1. In particular it has no weight-0 part, hence it contains no nontrivial Frobenius-invariant element. This proves that the pullback of sequence (5) to the algebraic closure splits (because the class defining the extension vanishes over the algebraic closure). But this is a contradiction, because one can then enlarge $\overline{\mathcal{F}'}$ (simply add to it a simple subsheaf of $\mathcal{F}''$), contradicting its maximality. □

Remark 10.11. Using Theorem 10.4 it is possible to define, on every lisse mixed sheaf $\mathcal{F}$, an increasing **weight filtration** by lisse subsheaves $W_i\mathcal{F}$ such that the graded quotients $\operatorname{gr}^W_i \mathcal{F}$ is pure of weight i and the filtration is functorial in $\mathcal{F}$. Moreover, morphisms $\mathcal{F} \to \mathcal{G}$ are even **strictly** compatible with the filtration W: the pullback of the weight filtration on $\mathcal{G}$ gives the weight filtration on $\mathcal{F}$.

10.3 Some reductions in the proof of Theorem 10.4

The following is the key case: X is a smooth projective curve over $\mathbb{F}_q$, $j : U \hookrightarrow X$ is a dense open, and $\mathcal{F}$ is lisse and pure of weight w on U. Then $H^i\left(\overline{X}, j_*\mathcal{F}\right)$ is pure of weight $w+i$ for $i = 0, 1, 2$.

Remark 10.12. Besides Deligne's original proof, there is an argument by Laumon using the Fourier transform for ℓ-adic sheaves.

We now give a sketch of how this special case implies the general one.

10.4 Dévissages

1. if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact, and the conclusion of the theorem holds for $\mathcal{F}', \mathcal{F}''$, then it also holds for $\mathcal{F}$. One can therefore assume that $\mathcal{F}$ is pure.

2. let $U \xrightarrow{i} X \xleftarrow{j} Z = X \setminus U$ with U open. Then if the theorem holds for $i^*\mathcal{F}, j^*\mathcal{F}$ it also holds for $\mathcal{F}$.

3. It is slightly harder to check that the same holds on Y: if V is an open subset of Y, T is its closed complement, and the statement holds upon restriction to V, T, then it holds on all of Y.

4. if $f = g \circ h$, then the theorem for g and h implies the theorem for f (by the spectral sequence for composite functors)

5. one may assume that Y is reduced, and also replace it with a purely inseparable cover if necessary

6. the case where $\mathcal{F}$ is pure and f is of relative dimension 0 is easy (one reduces to finite morphisms).

10.5 Structure of the main proof

One uses Noetherian induction on the relative dimension. After the above dévissages, one arrives at the following situation:

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & D \arrow[l, hook', "i"'] \arrow[dl] \\ & Y & \end{tikzcd}$$

where $\overline{X}$ is a smooth projective relative curve over Y and D is an étale divisor. We have

$$R^i f_! \mathcal{F} = R^i \overline{f}_* j_! \mathcal{F},$$

and the key case says that $R^i \overline{f}_* j_* \mathcal{F}$ is pure of weight $w+i$ (if $\mathcal{F}$ is pure of weight i). There is an exact sequence

$$0 \to j_!\mathcal{F} \to j_*\mathcal{F} \to i_! i^* j_* \mathcal{F} \to 0,$$

which (by passing to the corresponding long exact sequence) shows that it is enough to prove that $i^* j_* \mathcal{F}$ is mixed of weights at most w (in fact, it would be enough to prove that the weights are at most $w+i$, but they turn out to be at most w).

10.6 Local monodromy

Let X be a smooth curve over $\mathbb{F}_q$, $U \subset X$ an open dense subscheme, $s \in X \setminus U$ a closed point, $\overline{\eta}$ a geometric generic point of U. Let $\mathcal{F}$ be a lisse sheaf over X. The stalk $\mathcal{F}_{\overline{\eta}}$ has an action of the local Galois group G_s. The structure theory for the Galois group of local fields gives

$$1 \to I \to G_s \to \widehat{\mathbb{Z}} \to 0$$

and

$$1 \to P \to I \to \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \to 0,$$

where the map $I \to \mathbb{Z}_\ell(1)$ is the cyclotomic character.

Theorem 10.13 (Grothendieck's local monodromy theorem)**.** *There exists $I' \subseteq I$ open of finite index such that the action of I' on $V = \mathcal{F}_{\overline{\eta}}$ is unipotent. If ρ is the representation of G_s on V, we may write*

$$\rho(\sigma) = \exp(N t_\ell(\sigma)) \quad \forall \sigma \in I'$$

for a suitable nilpotent operator N.

Remark 10.14. The action of $\sigma \in I'$ on V factors through t_ℓ (by the usual argument with passing to a finite quotient of $\mathrm{GL}(V)$). Once we know this, we have a map $\mathbb{Z}_\ell(1) \to \mathrm{End}(V)$, and since the image consists of unipotent operators we can take the logarithm by using the defining power series.

Notice furthermore that we can consider N as a map $V(1) \to V$.

Lemma 10.15. *There exists a unique increasing filtration $M_\bullet$ of V such that N sends $M_i V(1)$ into $M_{i-2}V$ and for all k the operator N^k induces an isomorphism $\mathrm{gr}_k^M V(k) \xrightarrow{\sim} \mathrm{gr}_{-k}^M(V)$.*

Remark 10.16. The construction shows that $M_0 = \ker N$.

Theorem 10.17 (Deligne)**.** *In the above situation, if $\mathcal{F}$ is pure of weight w, then $\mathrm{gr}_k^M(V)$ is pure of weight $w + k$.*

10.7 Conclusion of the proof

Theorem 10.17 is applied as follows. Recall our setting: we have

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & D \arrow[l, hook', "i"'] \arrow[dl] \\ & Y & \end{tikzcd}$$

Let $V = \mathcal{F}_{\overline{\eta}}$, which gives rise to

$$\mathcal{F}_{\overline{s}} = V^{I_s} \subseteq V^{\ker N} = M_0(V).$$

This shows that only the negative parts of the monodromy filtration contribute, so – as $\mathcal{F}$ is pure of weight w – the only nontrivial contributions come from $\mathrm{gr}_k^M(V)$ for $k \leq 0$, which have weights $w + k \leq w$ by Theorem 10.17.

10.8 Aside: the weight-monodromy conjecture

Let K be any local field and X/K be smooth and proper. Let $V = H^i(\overline{X}, \mathbb{Q}_\ell)$ for some ℓ prime to the characteristic. Applying Theorem 10.13 we get a monodromy filtration on V.

Conjecture 10.18 (Weight-monodromy conjecture)**.** $\mathrm{gr}_k^M(V)$ *is pure of weight* $i + k$.

Remark 10.19. If X has good reduction X_0, by smooth proper base change one gets that $H^i(\overline{X}, \mathbb{Q}_\ell)$ is the same as $H^i(\overline{X_0}, \mathbb{Q}_\ell)$, which is pure by Weil I.

Remark 10.20. Scholze has proven the weight-monodromy conjecture for $\operatorname{char} K = 0$ and for X a smooth complete intersection in $\mathbb{P}^n$.

10.9 A conjecture from Weil II

Conjecture 10.21 (Deligne, Weil II)**.** *Let X be a normal connected scheme over $\mathbb{F}_q$ and let $\mathcal{F}$ be a lisse irreducible $\overline{\mathbb{Q}_\ell}$-sheaf such that its determinant (seen as a representation of the fundamental group) is a character of finite order of $\pi_1(X, \overline{x})$.*

1. *$\mathcal{F}$ is pure*
2. *the characteristic polynomials of Frobenius on $\mathcal{F}_{\overline{x}}$, as x varies over the closed points $x \in X$, all have coefficients in a fixed number field $E \subset \overline{\mathbb{Q}_\ell}$*
3. *("companions") given a $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ as above, for all $\ell' \notin \{\ell, p\}$, there exists a $\overline{\mathbb{Q}_{\ell'}}$-sheaf $\mathcal{G}$ with the same Frobenius eigenvalues*
4. *Moreover, 'on espère de petits camarades cristallins', i.e. a similar statement should hold for $\ell = p$.*

Remark 10.22. Deligne shows that, after twisting by some $\overline{\mathbb{Q}_\ell}(i)$, we may always achieve the condition that $\det \mathcal{F}$ is a character of finite order.

Remark 10.23. Parts (1)-(3) are now proven (based on work of Lafforgue).

11 10.12.2019 – Appendix: an application of Deligne's integrality theorem

Conjecture 11.1 (Manin, Lang)**.** *If $X/\mathbb{F}_q$ is a (smooth projective) Fano variety, then $X(\mathbb{F}_q) \neq \emptyset$; more precisely, $X(\mathbb{F}_q) \equiv 1 \pmod{q}$.*

Remark 11.2. A special case of this is the Chevalley-Warning theorem (which is the case of hypersurfaces of low degree).

Theorem 11.3 (Esnault, 2002)**.** *The conjecture is true more generally for smooth projective varieties $X/\mathbb{F}_q$ such that $\mathrm{CH}_0(X_{\overline{k(X)}}) \cong \mathbb{Z}$.*

Remark 11.4. Recall that the Chow group of 0-cycles is

$$\mathrm{CH}_0(X) = \operatorname{coker}\left(\bigoplus_{x \in X_1} k(x)^\times \xrightarrow{\mathrm{div}} \bigoplus_{x \in X_0} \mathbb{Z}\right)$$

The condition on CH_0 appearing in Theorem 11.3 is satisfied for all *rationally chain connected* varieties over an algebraically closed field. Here rationally chain connected means that any two points can be joined by a chain of rational curves.

Theorem 11.5 (Kollár, Miyaoka, Mori)**.** *Fano varieties are rationally chain connected.*

Remark 11.6. In Theorem 11.3 one could (equivalently) consider the base-change of X to any algebraically closed field Ω containing the field of rational functions $k(X)$.

Remark 11.7. There are generalisations of Theorem 11.3 to singular varieties. For example, Esnault herself has proven that if X is the (projective) special fibre of a smooth projective variety Y over $\mathbb{Q}_p$, and Y satisfies

$$\mathrm{CH}_0\left(Y_{\overline{k(Y)}}\right) \cong \mathbb{Z},$$

then the conclusion holds for X.

Remark 11.8. Esnault's original proof relies on p-adic techniques and rigid cohomology. We will see a modification of the argument, due to Faltings, which allows one to work with 'good old' étale cohomology instead.

11.1 Idea of proof

Let $\overline{X} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. The Lefschetz trace formula gives

$$\#X(\mathbb{F}_q) = \sum_{i=0}^{d} (-1)^i \operatorname{Tr}\left(F^* \mid H^i\left(\overline{X}, \mathbb{Q}_\ell\right)\right),$$

and $H^0(\overline{X}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$. It follows that the trace of Frobenius on $H^0(\overline{X}, \mathbb{Q}_\ell)$ is 1, hence to prove the theorem it is therefore enough to show that, for $i > 0$, the eigenvalues of F^* on $H^i(\overline{X}, \mathbb{Q}_\ell)$ are divisible by q. Notice that this last statement makes sense because of Deligne's integrality theorem 4.10: the eigenvalues of Frobenius are algebraic *integers*.

11.2 Generalities

11.2.1 Cohomology with support in a closed subscheme

If Z in X is a closed subscheme, one can define

$$H^0_Z(X, \mathcal{F}) = \ker\left(H^0(X, \mathcal{F}) \to H^0(X \setminus Z, \mathcal{F})\right);$$

this is a left-exact functor, whose right derived functors are by definition $\mathcal{F} \mapsto H^i_Z(X, \mathcal{F})$, cohomology with compact support on Z.

There is a long exact sequence

$$\cdots \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}) \to \cdots$$

where $U := X \setminus Z$, and more generally, if $Y \subseteq Z$ is another closed subset, there is a long exact sequence

$$\cdots \to H^i_Y(X, \mathcal{F}) \to H^i_Z(X, \mathcal{F}) \to H^i_{Z \setminus Y}(X \setminus Y, \mathcal{F}) \to \cdots \tag{6}$$

There is a purity isomorphism: if $\mathcal{F} = \mathbb{Z}/n\mathbb{Z}(j)$, $Z \subseteq X$ is a smooth pair of varieties over a field, and c is the codimension of Z in X, then there is an isomorphism

$$H^i_Z(X, \mathcal{F}) \cong H^{i-2c}(Z, \mathcal{F}(-c))$$

Remark 11.9. By work of Gabber, the purity isomorphism is now known to exist more generally for regular pairs.

11.2.2 Cycle map

Let X be a smooth variety and let $Z^i(X)$ be the group of cycles of codimension i on X. There is a cycle map

$$Z^i(X) \to H^{2i}(X, \mathbb{Z}/n\mathbb{Z}(i)).$$

In particular, we want to associate a class in $H^{2i}(X, \mathbb{Z}/n\mathbb{Z}(i))$ with with any closed subscheme Z of codimension i. If Z happens to be smooth, by purity we have

$$H^0(Z, \mathbb{Z}/n\mathbb{Z}) \cong H^{2i}_Z(X, \mathbb{Z}/n\mathbb{Z}(i)) \to H^{2i}(X, \mathbb{Z}/n\mathbb{Z}(i)),$$

and we define the cycle class of Z to be the image of $1 \in H^0(Z, \mathbb{Z}/n\mathbb{Z})$ into $H^{2i}(X, \mathbb{Z}/n\mathbb{Z}(i))$. If Z is not smooth, let $Y \subset Z$ be the singular locus, and consider the long exact sequence (6) for the pair $Y \subset Z$:

$$H^\alpha_Y(X, \mathbb{Z}/n\mathbb{Z}(\cdot)) \to H^\alpha_Z(X, \mathbb{Z}/n\mathbb{Z}(\cdot)) \to H^\alpha_{Z\setminus Y}(X \setminus Y, \mathbb{Z}/n\mathbb{Z}(\cdot)) \to H^{\alpha+1}_Y(X, \mathbb{Z}/n\mathbb{Z}(\cdot))$$

For $\alpha < 2i = 2\operatorname{codim}_X(Z)$ the cohomology groups $H^\alpha_Y(X, \mathbb{Z}/n\mathbb{Z}(\cdot))$ and $H^{\alpha+1}_Y(X, \mathbb{Z}/n\mathbb{Z}(\cdot))$ vanish: if Y is smooth, this follows by purity for dimension reasons. Even if Y is not smooth, one can proceed by (Noetherian) induction by taking Y_{n+1} to be the singular locus of Y_n, and the conclusion is still the same. Now we take $\alpha = 2i$ and $\cdot = i$ to get

$$0 \to H^{2i}_Z(X, \mathbb{Z}/n\mathbb{Z}(i)) \to H^{2i}_{Z\setminus Y}(X \setminus Y, \mathbb{Z}/n\mathbb{Z}(i)) \to 0,$$

and by purity we have

$$H^{2i}_{Z\setminus Y}(X \setminus Y, \mathbb{Z}/n\mathbb{Z}(i)) \cong H^0(Z \setminus Y, \mathbb{Z}/n\mathbb{Z}(i)).$$

Finally, we have a canonical map $H^0(Z, \mathbb{Z}/n\mathbb{Z}(i)) \to H^0(Z \setminus Y)$: the image of $1 \in H^0(Z, \mathbb{Z}/n\mathbb{Z}(i))$ in $H^{2i}_Z(X, \mathbb{Z}/n\mathbb{Z}(i))$ is then the value of cycle class map in Z.

Definition 11.10. Two cycles α_1, α_2 are rationally equivalent if $\alpha_1 - \alpha_2$ is a $\mathbb{Z}$-linear combination of $T(0) - T(\infty)$, where $T \subseteq X \times \mathbb{P}^1$ is a closed subscheme.

Fact. Let

$$\mathrm{CH}^i(X) := Z^i(X)/\text{rational equivalence}$$

be the i-th Chow group of X. The cycle map factors through $\mathrm{CH}^i(X) \to H^{2i}(X, \mathbb{Z}/n\mathbb{Z}(i))$.

11.2.3 Correspondences

Consider the obvious diagram

$$\begin{tikzcd} & \overline{X} \times \overline{X} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \overline{X} & & \overline{X} \end{tikzcd}$$

We have pullback maps $p_1^*, p_2^* : H^i(\overline{X}, \mathbb{Q}_\ell(j)) \to H^i(\overline{X} \times \overline{X}, \mathbb{Q}_\ell(j))$ and, if X is smooth and projective, by Poincaré duality we get push-forward maps

$$p_{1*}, p_{2*} : H^{4d-i}(\overline{X} \times \overline{X}, \mathbb{Q}_\ell(j)) \to H^{2d-i}(\overline{X}, \mathbb{Q}_\ell(d-j))$$

Now take $\alpha \in \mathrm{CH}^d(\overline{X} \times \overline{X})$, which induces $[\alpha] \in H^{2d}\left(\overline{X} \times \overline{X}, \mathbb{Q}_\ell(d)\right)$. We can then consider the following chain of maps:

$$H^i(\overline{X}, \mathbb{Q}_\ell(j)) \xrightarrow{p_1^*} H^i\left(\overline{X} \times \overline{X}, \mathbb{Q}_\ell(j)\right) \xrightarrow{-\cup[\alpha]} H^{2d+i}(\overline{X} \times \overline{X}, \mathbb{Q}_\ell(j+d)) \xrightarrow{p_{2,*}} H^i(\overline{X}, \mathbb{Q}_\ell(j)).$$

In particular, given α, we obtain $[\alpha]_* : H^i(\overline{X}, \mathbb{Q}_\ell(j)) \to H^i(\overline{X}, \mathbb{Q}_\ell(j))$, called a *correspondence action.*

11.3 Bloch's lemma

Lemma 11.11 (Bloch, decomposition of the diagonal)**.** *Assume X is a smooth projective variety over a field and such that $\mathrm{CH}_0(X_{\overline{k(X)}}) \cong \mathbb{Z}$. Let $\Delta \subset X \times X$ be the diagonal. Then, for some positive integer N, there exists a decomposition*

$$N \cdot \Delta \sim \Gamma_1 + \Gamma_2 \in \mathrm{CH}^d(X \times X) \quad (\textit{rational equivalence})$$

such that Γ_1 is supported on $\star \times X$, where $\star \subseteq X$ is a 0-dimensional closed subscheme, and Γ_2 is supported on $X \times (X \setminus D)$, where D is a divisor.

Proof. Let $\eta : \operatorname{Spec} k(X) \to X$ be the generic point. Then Δ induces, by pullback, an element Δ_η of $Z_0(X_{k(X)})$. The assumption implies that $\mathrm{CH}_0(X_{\overline{k(X)}} \setminus \star_{\overline{k(X)}}) = 0$ for some 0-dimensional $\star$ (which, over the algebraic closure, becomes a bunch of points, one of which is such that one can concentrate the support of any 0-cycle on it, up to rational equivalence).
Fact. If V is a variety over a field F, and if we write $\overline{V}$ for $V \times_F \overline{F}$, we have that

$$\ker\left(\mathrm{CH}_0(V) \to \mathrm{CH}_0(\overline{V})\right)$$

is torsion.

Proof of the fact. If α is an element in the kernel, there exists a finite extension L/F such that α becomes 0 in $\mathrm{CH}_0(X_L)$. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
\bigoplus_{x \in V_1} k(x)^\times \arrow[r] \arrow[d] & \bigoplus_{x \in V_0} \mathbb{Z} \arrow[r] \arrow[d] & \mathrm{CH}_0(V) \arrow[d] \\
\bigoplus_{x \in (V_L)_1} k(x)^\times \arrow[r] \arrow[d, "N_{L/F}"'] & \bigoplus_{x \in (V_L)_0} \mathbb{Z} \arrow[r] \arrow[d] & \mathrm{CH}_0(V_L) \arrow[d] \\
\bigoplus_{x \in V_1} k(x)^\times \arrow[r] & \bigoplus_{x \in V_0} \mathbb{Z} \arrow[r] & \mathrm{CH}_0(V) :
\end{tikzcd}$$

then the composition of the vertical arrows is $[L : F]$, and an easy diagram chasing then shows that if α is in $\ker \mathrm{CH}_0(V) \to \mathrm{CH}_0(V_L)$, then it is killed by $[L : F]$. (Notice that the vertical maps at the level of $\bigoplus \mathbb{Z}$ are multiplication by the local degree of the field extensions). $\square$

By the fact, some multiple $N\Delta_\eta$ of Δ_η vanishes. Let $\star \times X$ be the closure of $\star_{k(X)}$ in $X \times X$. There exists some Γ_1, supported on $\star \times X$, such that $N \cdot \Delta - \Gamma_1$ maps to 0 in $\mathrm{CH}_0(X_{k(X)})$. Now it is an easy fact that

$$\mathrm{CH}_0(X_{k(X)}) = \varinjlim_{U \text{ open in } X} \mathrm{CH}_d(X \times U),$$

and open subsets of the form $X \setminus D$ are cofinal in the system of all open subsets, so

$$\mathrm{CH}_0(X_{k(X)}) = \varinjlim_{D} \mathrm{CH}_d(X \times (X \setminus D)).$$

The fact that $N\Delta - \Gamma_1$ is 0 in the limit now means precisely that, for some D, the cycle $N\Delta - \Gamma_1$ is of the desired form Γ_2. This follows from the exact sequence

$$\mathrm{CH}_d(X \times D) \to \mathrm{CH}_d(X \times X) \to \mathrm{CH}_d(X \times (X \setminus D)).$$

□

11.4 Proof of Theorem 11.3

Lemma 11.11, applied to the diagonal of $X \times X$, implies that

$$N[\Delta]_* = [\Gamma_1]_* + [\Gamma_2]_* \text{ on } H^i(\overline{X}, \mathbb{Q}_\ell);$$

moreover, $[\Delta]_*$ is the identity, while $[\Gamma_1]_* = 0$, because the action factors via the d-dimensional cohomology of the 0-dimensional variety $\star$. For every $\alpha \in H^i(\overline{X}, \mathbb{Q}_\ell)$ we then have

$$N\alpha \in \ker\left(H^i(\overline{X}, \mathbb{Q}_\ell) \to H^i(\overline{X \setminus D}, \mathbb{Q}_\ell)\right)$$

But $N : H^i(\overline{X}, \mathbb{Q}_\ell) \to H^i(\overline{X}, \mathbb{Q}_\ell)$ is an isomorphism, so the same is true for α itself. Recall once more that we have a purity isomorphism: if $Z \subset \overline{X}$ is smooth,

$$H^i_Z(\overline{X}, \mathbb{Q}_\ell(j)) \cong H^{i-2c}(Z, \mathbb{Q}_\ell(j-c)),$$

where c is the codimension of Z in X. If moreover we are over $\mathbb{F}_q$, this implies that the action of Frobenius on $H^i_Z(\overline{X}, \mathbb{Q}_\ell(j))$ is q^c times the action on $H^{i-2c}(Z, \mathbb{Q}_\ell(j))$. Applying this with $Z = \overline{D}$ (we are assuming, for now, that D is smooth) and $c = 1$, we have seen that α comes from $H^i_D(\overline{X}, \mathbb{Q}_\ell) \cong H^{i-2}(D, \mathbb{Q}_\ell(-1))$. The eigenvalues of the action of Frobenius on this latter space are of the form $q^2 \times$ eigenvalues on $H^{i-2}(D, \mathbb{Q}_\ell)$, hence they are divisible by q^2 (by theorem 4.10). Finally, if D is not smooth, we can find a chain of subschemes $\{\mathrm{pt}\} \subset D_0 \subset D_1 \subset \cdots \subset D_r = D$ such that $D_g \setminus D_{g-1}$ is smooth. The long exact sequence

$$\cdots \to H^i_{D_{g-1}}(\overline{X}, \mathbb{Q}_\ell) \to H^i_{D_g}(\overline{X}, \mathbb{Q}_\ell) \to H^i_{D_g \setminus D_{g-1}}(\overline{X} \setminus D_{g-1}, \mathbb{Q}_\ell)$$

then implies the theorem by Noetherian induction.

References

[Del74] Pierre Deligne. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.*, (43):273–307, 1974.

[Del80] Pierre Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.

[FK88] Eberhard Freitag and Reinhardt Kiehl. *Étale cohomology and the Weil conjecture*, volume 13 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988. Translated from the German by Betty S. Waterhouse and William C. Waterhouse, With an historical introduction by J. A. Dieudonné.

[Sch11] Anthony J. Scholl. Hypersurfaces and the Weil conjectures. *Int. Math. Res. Not. IMRN*, (5):1010–1022, 2011.

[Ser60] Jean-Pierre Serre. Rationalité des fonctions ζ des variétés algébriques. In *Séminaire Bourbaki : années 1958/59 - 1959/60, exposés 169-204*, number 5 in Séminaire Bourbaki, pages 415–425. Société mathématique de France, 1960. talk:198.

[Sta18] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2018.

[Wei49] André Weil. Numbers of solutions of equations in finite fields. *Bull. Amer. Math. Soc.*, 55:497–508, 1949.