Enriques surface: X smooth proj, $h^1\mathcal{O}_X = 0$, $K_X \neq 0$, $2K_X = 0$

$$\begin{array}{l} X = \operatorname{Spec}_{\mathcal{O}_X} \mathcal{O}_X \oplus \mathcal{O}_X(K_X) \\ \downarrow \pi \quad 2:1 \text{ unramified} \\ Y \end{array} \qquad X = K3 \text{ surface} \qquad Y = X/\iota$$

ι = base pt free inv, anti-sympl. Enriques inv $\quad i^*\Omega_Y = -\Omega_Y$.

For K3 surfaces:

unpolarized X $\quad$ dim (period domain) $= 20$

polarized (X, L) $\quad$ dim (period dom. or moduli) $= 19$

For Enriques:

unpol'd Y $\quad$ dim $= 10$

pol'd (Y, L) $\quad$ dim $= 10$ $\quad L^2 = 2d$

$\exists$ finitely many moduli of Enriq. surfaces: unpol'd, pol'd (Y, L), $L^2 = 2d$.

Horikawa construction:

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\rho", "2:1"'] & \\ Y & & \mathbb{P}^1 \times \mathbb{P}^1 \supset D \in |-2K_{\mathbb{P}^1\times\mathbb{P}^1}| \end{tikzcd}$$

X K3 with S_2^2 2 invos (π, ρ)

D bideg $(4,4)$

$B = \rho^{-1} D = \operatorname{Fix}(\rho)$

$L' = \rho^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$

$L'^2 = 2\,\mathcal{O}(1,1)^2 = 4$

(X, L') = a "hyperelliptic" deg 4 K3

A generic deg 4 K3 : $\varphi_{|L|} X_4 \hookrightarrow \mathbb{P}^3$ dim = 19

A hyperell deg 4 K3 : $\varphi_{|L|} X \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^1$ dim = 18

A unigonal deg 4 K3 : $|L|$ = sections + elliptic pencil dim = 18

The moduli: [Sterc] described the BB cusps

0-cusps: 5 1-cusps: 9

The KSBA compactification:

for KSBA-stable pair $(X, \epsilon B)$ $0 < \epsilon \ll 1$

eff. ample div

$\overline{M}^{BB}_{Sterc}$ $\overline{M}^{KSBA}_{Sterc}$ = like a "fiber product" of two stratums

$\cup$

M_{Sterc} dim 10 ?

$\overline{M}^{BB}_{Sterc} \to \overline{M}^{BB}_{En,unpol.}$

finite

$M_{Sterc} \to M_{En,unpold}$ dim 10

finite-to-1 to the image

$M_{Sterc} \to M_{deg4,h.e.}$ dim 18

$\overline{M}^{BB}_{deg4,h.e}$

$M_{En,unpold} \to \overline{M}^{BB}_{En,unpol.}$

well-known

Enriq. surf. with a deg 2 numerical pol'n.

$M_{deg4,h.e.} \subset \overline{M}^{KSBA}_{deg4,h.e}$

$\overline{M}^{KSBA}_{deg4,h.e} \to \overline{M}^{BB}_{deg4,h.e}$

described in [AE, invos]

[AE, invos] describes Kulikov degens.

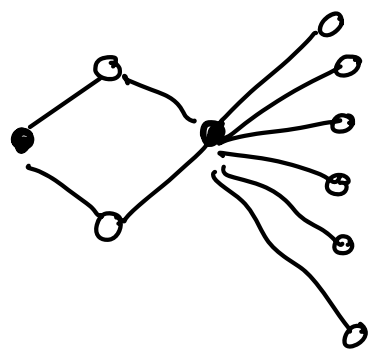
The answer for the BB cusps

$\overline{M}^{BB}_{En,2}$ [Sterc] 5 0-cusps

9 (?) 1-cusps

$\overline{M}^{BB}_{En,unpol'd}$

$\overline{M}^{BB}_{deg\,4,\,h.e}$

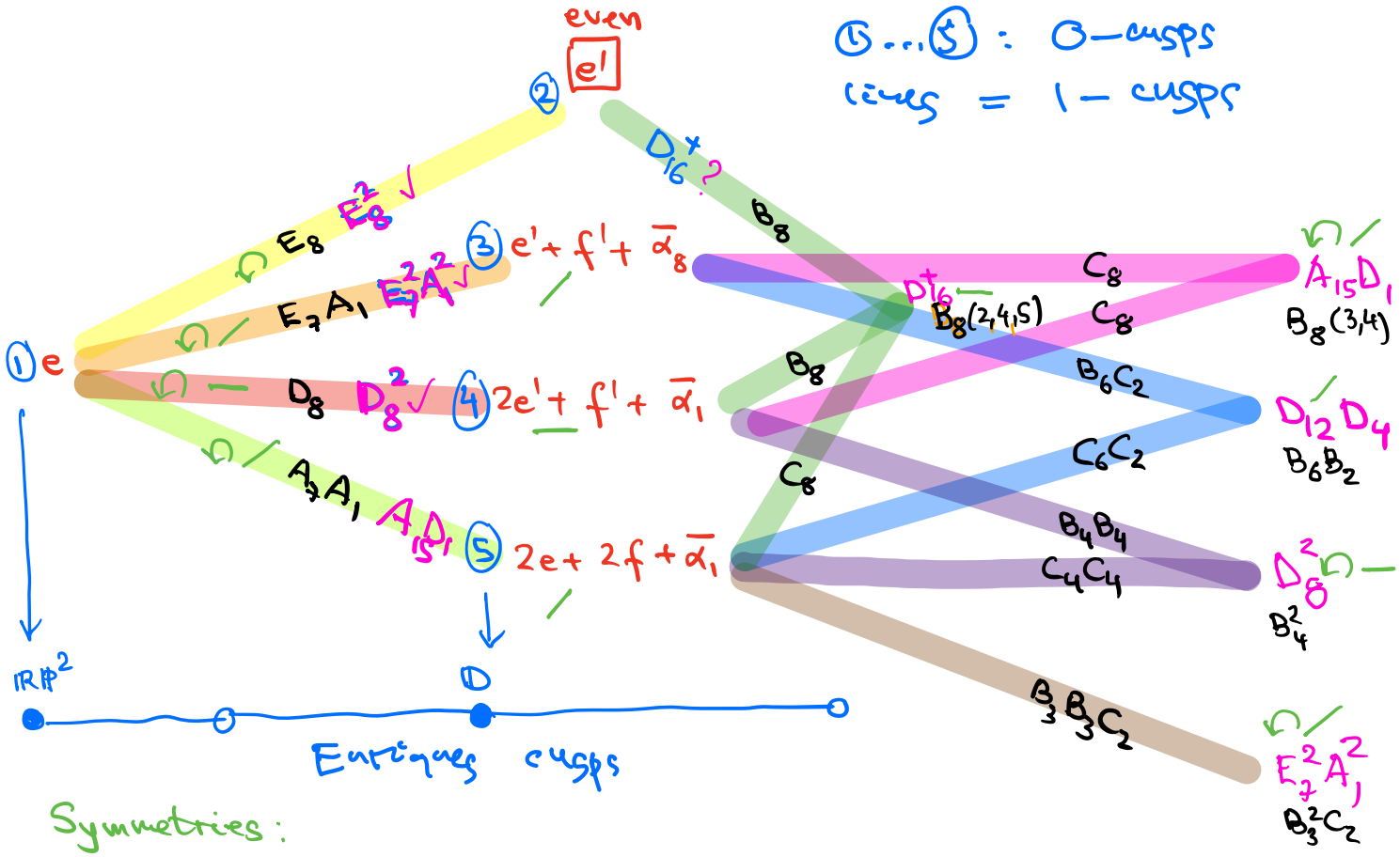


● : 0-cusp : pt of $\overline{M}^{BB}$

○ : 1-cusp : curve on $\overline{M}^{BB}$

Q: How do Sterc's BB-cusps map to BB-cusps of $\overline{M}^{BB}_{En,un}$ and $\overline{M}^{BB}_{deg\,4,\,h.e.}$?

A (myself):



Symmetries:

E_8^2 ↺ $\quad$ $E_7^2 A_1^2$ ↺/ $\quad$ D_8^2 — | ↺

D_{16}^+ — $\quad$ $A_{15} D_1$ ↺/ $\quad$ $D_{12} D_4$ /

1-cusps of $\overline{M}^{BB}_{\deg 1, 4}$: degenerations $A_{15} D_1$ etc

The main task: to describe IAS^2 with 2 invos π, ρ, of Enriques type and of deg 4, n.e. – type.

Each of these individually is done in [AE, invos]. Need to find pictures where $\exists$ both, for each of the 5 0-cusps of $\overline{M}^{BB}_{\text{Sterns}}$.

1-cusps = degens of $\mathbb{A}S^2$ when a sphere collapses to an interval!