Mathematics Department
The University of Georgia
Math 8150 Homework Assignment 1

1. Describe geometrically the sets of points z in the complex plane defined by the following relations:

   (a) $|z-1|=1$. (b) $|z-1|=2|z-2|$. (c) $1/z=\bar{z}$.

   (d) $\mathrm{Re}(z)=3$ (e) $\mathrm{Im}(z)=a$ with $a\in\mathbb{R}$. (f) $\mathrm{Re}(z)>a$ with $a\in\mathbb{R}$.

   (g) $|z-1|<2|z-2|$.

2. Prove that $|z_1+z_2|\geq ||z_1|-|z_2||$ and explain when equality holds.

3. Prove that the equation $z^3+2z+4=0$ has its roots outside the unit circle. [Hint: what is the maximum value of the modulus of the first two terms if $|z|\leq 1$?]

4. (a) Prove that if $|w_1|=c|w_2|$ where $c>0$, then $|w_1-c^2w_2|=c|w_1-w_2|$.

   (b) Prove that if $c>0$, $c\neq 1$ and $z_1\neq z_2$, then $|\frac{z-z_1}{z-z_2}|=c$ represents a circle. Find its center and radius. [Hint: an easy way is to use part (a)]

5. (a) Let z,w be complex numbers, such that $\bar{z}w\neq 1$. Prove that

$$|\frac{w-z}{1-\bar{w}z}|<1 \quad \text{if } |z|<1 \text{ and } |w|<1,$$

   and also that

$$|\frac{w-z}{1-\bar{w}z}|=1 \quad \text{if } |z|=1 \text{ or } |w|=1.$$

   (b) Prove that for fixed w in the unit disk $\mathbb{D}$, the mapping

$$F: z\mapsto \frac{w-z}{1-\bar{w}z}$$

   satisfies the following conditions:

   (i) F maps $\mathbb{D}$ to itself and is holomorphic.

   (ii) F interchanges 0 and w, namely, $F(0)=w$ and $F(w)=0$.

   (iii) $|F(z)|=1$ if $|z|=1$.

   (iv) $F:\mathbb{D}\mapsto\mathbb{D}$ is bijective. [Hint: Calculate $F\circ F$.]

6. Use n-th roots of unity (i.e. solutions of $z^n-1=0$) to show that

$$2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n}=n\,.$$

   [Hint: $1-\cos 2\theta=2\sin^2\theta,\ \sin 2\theta=2\sin\theta\cos\theta$.]

7. Prove that $f(z) = |z|^2$ has a derivative only at $z = 0$, but nowhere else.

8. Let $f(z)$ be analytic in a domain. Prove that $f(z)$ is a constant if it satisfies any of the following conditions:

   (a) $|f(z)|$ is constant;

   (b) $\mathrm{Re}(f(z))$ is constant;

   (c) $\arg(f(z))$ is constant;

   (d) $\overline{f(z)}$ is analytic;

   How do you generalize (a) and (b)?

9. Let $f(z)$ be analytic. Show that $\overline{f(\bar{z})}$ is also analytic.

10. (a) Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

   (b) Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \ \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

   is a holomorphic function in the region $r > 0$, $-\pi < \theta < \pi$. Also show that $\log z$ defined above is not continuous in $r > 0$.

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Math 8150 Homework Assignment 2

1. Suppose $U(z)$ has continuous second order partial derivatives and $z = f(\zeta)$, $\zeta = \xi + i\eta$ is a holomorphic function. Show that

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = |f'(\zeta)|^2 [\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}] \, .$$

2. Show that $U(x^2 - y^2, 2xy)$ is harmonic if and only if $U(x, y)$ is.

3. Let $a_n \neq 0$ and assume that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$. Show that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

4. Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.

5. Prove the following:

   (a) The power series $\sum_{n=1}^{\infty} nz^n$ does not converge at any point of the unit circle.

   (b) The power series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges at every point of the unit circle.

   (c) The power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges at every point of the unit circle except at $z = 1$.

6. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^z}$$

defines a continuous function for $\text{Re}(z) > 1$. (This is the Riemann ζ function and you will see later that it is analytic in the above region-not just being continuous there- and it can be extended to the complex plane with only 1 removed. You do not need to prove the latter assertions.) [Hint: Show that the series converges uniformly for $\text{Re}(z) \geq 1 + \delta$, where $\delta > 0$ is any positive number.]

7. Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

has radius of convergence 1. Examine convergence at $z = 1$, -1 and i.

8. It was defined that $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Show that $e^z = \lim_{n\to\infty} (1 + \frac{z}{n})^n$.

9. (a) Express $\sin z$ in the form $u + iv$. Do the same for $\cos z$.

   (b) Show that there exists a sequence z_n such that $\sin z_n \to \infty$. Do the same for $\cos z$. (This shows that unlike their real counter parts, $\sin z$ and $\cos z$ are unbounded.)

10. Prove that

$$\text{(a)} \quad \lim_{z \to (2k+1)\pi/2} \tan z = \infty \ , \quad \text{(b)} \quad \lim_{z \to (2k+1)\pi/2} [z - (2k+1)\pi/2] \tan z = -1 \ .$$

   [Hint: The proof of (b) is *very short* if done the "right" way.]

11. Show that if $|\alpha| < r < |\beta|$, then

$$\int_\gamma \frac{1}{(z-\alpha)(z-\beta)} = \frac{2\pi i}{\alpha - \beta}$$

   where γ denotes the circle centered at the origin, of radius r, with positive orientation.

12. Assume f is continuous in the region: $x \geq x_0, \ 0 \leq y \leq b$ and the limit

$$\lim_{x \to +\infty} f(x + iy) = A$$

   exists uniformly with respect to y (independent of y). Show that

$$\lim_{x \to +\infty} \int_{\gamma_x} f(z) dz = iAb \ ,$$

   where $\gamma_x := \{z \mid z = x + it, \ 0 \leq t \leq b\}$.

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Complex Analysis, by Elias M. Stein and Rami Shakarchi,
2.6: 1, 2, 5, 6, 8

1. Assume real value functions u, v of two variables have continuous partial derivatives at (x_0, y_0). Show that $f = u + iv$ has derivative $f'(z_0)$ at $z_0 = x_0 + iy_0$ if and only if

$$\lim_{r\to 0} \frac{1}{\pi r^2} \int_{|z-z_0|=r} f(z)\, dz = 0\ .$$

2. (Cauchy's formula for "exterior" region) Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z\to\infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

3. Let $f(z)$ be bounded and analytic in $\mathbb{C}$. Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists

$$\lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

   Use this to show that $f(z)$ must be a constant (Liouville's theorem).

4. Suppose that $f(z)$ is entire and $\lim_{z\to\infty} f(z)/z = 0$. Show that $f(z)$ is a constant.

5. Let f be analytic on a domain D and let γ be a closed curve in D. For any z_0 in D not on γ, show that

$$\int_\gamma \frac{f'(z)}{(z-z_0)} dz = \int_\gamma \frac{f(z)}{(z-z_0)^2} dz$$

   Give a generalization of this result.

6. Compute $\displaystyle\int_{|z|=1} (z + \frac{1}{z})^{2n} \frac{dz}{z}$ and use it to show that

$$\int_0^{2\pi} \cos^{2n} \theta d\theta = 2\pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

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Math 8150 Homework Assignment 4

Complex Analysis, by Elias M. Stein and Rami Shakarchi,
2.6: 10, 13, 15

1. Prove by *justifying all steps* that for all $\xi \in \mathbb{C}$ we have $e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi} dx$.

   [Hint: You may use that fact in Example 1 on p. 42 of the textbook without proof, i.e., you may assume the above is true for real values of ξ.]

2. Suppose that F is analytic on a region Ω. Define f by

$$f(z) = \begin{cases} \dfrac{F(z) - F(z_0)}{z - z_0} & \text{if } z \neq z_0 \\ F'(z_0) & \text{if } z = z_0 \end{cases}$$

   where z_0 is a point in Ω. Show that f is also analytic on Ω.

3. (1) Show that the series

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

   defines an analytic function in $\mathrm{Re}(z) > 1$. (This statement is stronger than the one on continuity you worked earlier.)

   (2) Find series representation of $\zeta^{(k)}(z)$ $(k \geq 1)$ in $\mathrm{Re}(z) > 1$ and justify you answer.

4. Give an example of a real valued non-zero function $f(x)$ for which $f(x) = 0$ has infinitely many solutions in finite interval.

5. (1) Is there a function $f(z)$ that is analytic at $z = 0$ and $f(1/n) = f(-1/n) = 1/n^3$?

   (2) Is there a function $f(z)$ that is analytic at $z = 0$ and $f(1/(2k+1)) = 0, f(1/2k) = 1/2k$?

   Explain why in each case.

6. Let $f(z), g(z)$ be analytic in a connected open set Ω. Suppose $f(z)g(z) = 0$ for all $z \in \Omega$. Show that either $f(z)$ or $g(z)$ is constantly 0. (This implies $H(\Omega)$ is an integral domain in the language of commutative ring theory.)

7. If the radius R of convergence of the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is finite, show that there is at least one point z_0 on $|z| = R$ at which $f(z)$ is not analytic.

8. Let $f(z)$ be analytic in $D : |z| < 1$ and $f(0) = 0$. Show that $\sum_{n=1}^{\infty} f(z^n)$ converges uniformly to an analytic function on compact subsets of D.

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Math 8150 Homework Assignment 5

Complex Analysis, by Elias M. Stein and Rami Shakarchi,
3.8: 1, 2, 4, 5, 7, 8

1. Prove that if
$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n \quad \text{and} \quad \sum_{n=-\infty}^{\infty} c'_n(z-a)^n$$
are Laurent series expansions of $f(z)$, then $c_n = c'_n$ for all n.

2. Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Let $f(z) = \sum_{n=1}^{\infty} c_n z^n$ denote the the power series in the open disc. Show that (1) $c_n \neq 0$ for all large enough n's, and (2) $\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = z_0$.

3. Expand $\frac{1}{1-z^2} + \frac{1}{3-z}$ in a series of the form $\sum_{-\infty}^{\infty} a_n z^n$. How many such expansions are there? In which domain is each of them valid?

4. Let $P(z)$ and $Q(z)$ be polynomials with no common zeros. Assume $Q(a) = 0$. Find the principal part of $P(z)/Q(z)$ at $z = a$ if the zero a is (i) simple; (ii) double. Express your answers explicitly using P and Q.

5. Let $f(z)$ be a non-constant analytic function in $|z| > 0$ such that $f(z_n) = 0$ for infinite many points z_n with $\lim_{n\to\infty} z_n = 0$. Show that $z = 0$ is an essential singularity for $f(z)$. (An example of such a function is $f(z) = \sin(1/z)$.)

6. Let f be entire and suppose that $\lim_{z\to\infty} f(z) = \infty$. Show that f is a polynomial.

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Math 8150 Homework Assignment 6

Complex Analysis, by Elias M. Stein and Rami Shakarchi,

2.7: 4

3.8: 9, 10, 14, 15(b), 17, 19(a)

Work out 3.8.9 as hinted there except modify the contour there more precisely as follows. The contour is the rectangle with vertices $(0,0),(1,0),(1,R),(0,R)$ indented at each of $(0,0),(1,0)$ by a quarter disc of small radius ε. Use an *appropriate* branch of the function $f(z)=\log(1-e^{2i\pi z})=\log(-2ie^{i\pi z}\sin(\pi z))$.

1. (1) Show without using 3.8.9 in the textbook by Stein and Shakarchi that

$$\int_0^{2\pi}\log|1-e^{i\theta}|d\theta=0.$$

   (2) Show the above identity is equivalent to the one in 3.8.9 of the textbook.

2. Evaluate $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^3}dx$, $0<a<4$.

3. (1) Prove the fundamental theorem of algebra using Rouché's theorem.

   (2) Prove the fundamental theorem of algebra using the maximum modulus principle.

1. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^2\theta}$, $a>0$.

2. Assume $f(z)$ is analytic in region D and *gamma* is a rectifiable curve in D with interior in D. Prove that if $f(z)$ is real for all $z\in\Gamma$, then $f(z)$ is a constant.

3. Find the number of roots of $z^4-6z+3=0$ in $|z|<1$ and $1<|z|<2$ respectively.

4. Prove that $z^4+2z^3-2z+10=0$ has exactly one root in each open quadrant.

5. Prove that the equation

$$z\tan z=a,\ \ a>0,$$

   has only real roots in $\mathbb{C}$.

6. Let f be analytic on a bounded region Ω and continuous on the closure $\overline{\Omega}$. Assume $f(z)\neq 0$. Show that $f(z)=e^{i\theta}M$ (where θ is a real constant) if $|f(z)|=M$ (a constant) for $z\in\partial\Omega$.

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Math 8150 Homework Assignment 7

Complex Analysis, by Elias M. Stein and Rami Shakarchi
8.5: 1, 2, 9, 10, 11, 13, 15, 16, 17,
Note: Explain that 8.6.9 does not violate the uniqueness of solution of the Dirichlet problem in a disc.

1. Assume a real valued continuous function $u(x, y)$ has mean value property (MVP) in a region Ω. That is, for each a in Ω there exists $r_0 > 0$ such that

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it})\, dt \;\text{ for }\; 0 < r \leq r_0\,.$$

This exercise guides to a proof of the fact that MVP implies harmonicity using Poisson formula for the solution of the Dirichlet problem in any disc given in class.

(1) Assume that $\overline{\Omega}_1$ is a compact region in Ω. Prove that u cannot attain maximum (resp. minimum) value in in the interior Ω_1 unless it is a constant on Ω_1. (Avoid circular reasoning: you cannot use the equivalence of harmonicity and MVP and the maximum/minmum modulus principle for harmonic functions.)

(2) Let $\overline{D}_r(a) \in \Omega$ be a disk in Ω. Let $g(z) = u(x, y)$ for $z \in \partial D_r(a)$ and let $U(z)$ be the solution of the Dirichlet problem in $\overline{D}_r(a)$ with boundary values $g(z)$. Show that $u(z) - U(z) = 0$ in $\overline{D}_r(a)$ and conclude that u is harmonic in Ω.

[Note: The same method idea in (1) also yields the uniqueness of the solution of the Dirichlet problem in a disc.]

2. (1) Let $f(z) \in H(\mathbb{D})$, $\mathrm{Re}(f(z)) > 0$, $f(0) = a > 0$. Show that

$$\left|\frac{f(z) - a}{f(z) + a}\right| \leq |z|, \quad |f'(0)| \leq 2a.$$

(2) Show that the above is still true if $\mathrm{Re}(f(z)) > 0$ is replaced with $\mathrm{Re}(f(z)) \geq 0$.

3. Assume $f(z)$ is analytic in $\mathbb{D}$ and $f(0) = 0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta} z$). Show that $\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of $\mathbb{D}$, where $f^{n+1}(z) = f(f^n(z))$.

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Math 8150 Homework Assignment 8

Complex Analysis, by Elias M. Stein and Rami Shakarchi
8.5: 5, 8, 12, 14

1. Let z_1, z_2 be distinct points and $k > 0$. Let C be the circle/line: $\dfrac{|z-z_1|}{|z-z_2|} = k$. Show that z_1 and z_2 are reflections of each other along C.

   Hint: See problem 4 in Homework Assignment 1.

2. (1) Let f be a fractional linear transformation. Show that f preserve the cross ratio, i.e., $(f(z_1), f(z_2), f(z_3), f(z_4)) = (z_1, z_2, z_3, z_4)$.

   (2) Let C be a line or circle. Let z_1, z_2, z_3 be any three distinct points on C. Let z^* be the reflection of z along C. Show that $(z_1, z_2, z_3, z^*) = \overline{(z_1, z_2, z_3, z)}$.

3. Find the following fractional linear transformations:

   (1) it maps $0, i, -i$ to $1, -1, 0$.

   (2) it maps $0, 1, \infty$ to $1, \infty, 0$.

   (3) it maps $0, 1, 2$ to $1, \infty, 0$.

   Be cautious of what it means when ∞ is involved.

4. Let $\Omega = \{z : \ |z-1| < \sqrt{2},\ |z+1| < \sqrt{2}\}$. Find a bijective conformal map from Ω to the upper half plane $\mathbb{H}$.

5. Find the fractional linear transformation that maps the circle $|z| = 2$ into $|z+1| = 1$, the point -2 into the origin, and the origin into i.

6. Let $\Omega = \mathbb{D} \backslash (-1, -1/2]$. Find a bijective conformal map from Ω to the unit disk $\mathbb{D}$. How do you find the most general form of all such maps (you don't have to explicitly describe the general form, just explain the strategy for obtaining it)?

7. Let $\Omega = \mathbb{C} \backslash [0, \infty)$. Is there an analytic isomorphism from Ω to $\mathbb{C}$? If yes, exhibit one such isomorphism. If no, explain why.

8. (1) Show that if f is analytic in an open set containing the disc $|z-a| \le R$, then

$$|f(a)|^2 \le \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R |f(a + re^{i\theta})|^2 r\, dr\, d\theta$$

   (2) Let Ω be a region and $M > 0$ a fixed positive constant. Let $\mathcal{F}$ be the family of all analytic functions f on Ω such that $\displaystyle\iint_\Omega |f(z)|^2 dx dy \le M$. Show that $\mathcal{F}$ is a normal family.

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Math 8150 Homework Assignment 9

Complex Analysis, by Elias M. Stein and Rami Shakarchi
5.6: 1, 2, 10, 11, 12 13, 14, 15
5.7: 1, 2
Note that I have distributed in class:
12.30: 2, 4, 9, 12, 14.
12.40: 2, 4, 8, 11

1. The purpose of this exercise is to exhibit the remarkable example in (2) using (1). Try to understand its significance and implications/ramifications.

   (1) Assume that $c_n \geq 0$ and the radius of convergence of $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is 1. Show that $z = 1$ is a singular point. [For any power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ with radius of convergence $R < \infty$, a point $Re^{i\phi}$ on the boundary is called *singular* if the radius of convergence of the Taylor series of f at $re^{i\phi}$ is $R - r$ for any $r \in (0, R)$. If each point on the boundary is singular, the boundary is called the *natural boundary*.]

   (2) Let $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^{2^n}}{2^n}$. Show that f is analytic in the open disc $\mathbb{D}$, continuous on the closed disc $\overline{\mathbb{D}}$, and *each point* on $\partial\mathbb{D}$ is singular.