Let L be an oriented link $\hookrightarrow S^3$

Slice genus $g_4(L) = \min\left\{ g(\Sigma) \;\middle|\; \Sigma \hookrightarrow B^4 \text{ smoothly } \& \; \partial\Sigma = L \right\}$

$g_{Top}(L) = \min\left\{ \text{—} \;\middle|\; \text{Same + }\underline{\text{locally}}\text{ }\underline{\text{flat}} \right\}$

Can we compute these for torus knots/links?

$g_4 \sim$ Milnor conj. $\quad g_4(T_{r,s}) = g(T_{r,s}) = \frac{1}{2}(r-1)(s-1)$.

$g_{Top} \sim$ Not fully understood

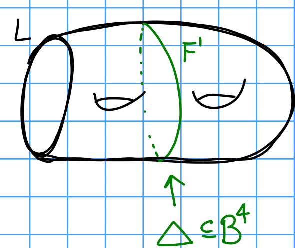
Rudolph('80s): $g_{Top}(T_{r,s}) < g_4(T_{r,s})$

Freedman: $\cdot \underbrace{\Delta_K}_{\text{Alexander poly}} = 1 \implies g_{Top}(K) = 0$

$\cdot \Delta_K = 1$ iff $K = \partial\Delta$, $\Delta \hookrightarrow B_4$, $\pi_1(B_4 \setminus \Delta) \cong \mathbb{Z}$.

Rudolph's Obsv: If L has a Seifert surface F with a sub-surface $F' \subseteq F$ s.t. $\Delta_{\partial F'} = 1$, <u>then</u>

$g_{Top}(L) \leq g(F) - g(F')$



Define $g_{alg} = \min\left\{ g(F) - g(F)' \right\}$

$\geq$

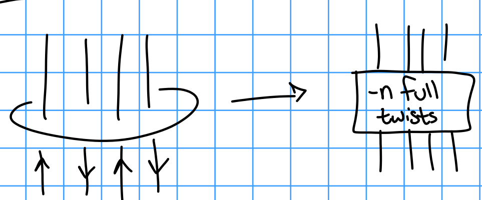
$g_{\mathbb{Z}} = \min\left\{ g(\Sigma) \;\middle|\; \pi_1(B^4 \setminus \Sigma) = \mathbb{Z} \right\}$

Feller: Find subsurfaces using Seifert matrices

$g_{Alg}(L) = \min\left\{ \frac{(m-r+1)-n}{2} \;\middle|\; \begin{array}{l} L \text{ has } r \text{ components} \\ \& \; m\times n \text{ seifert matrix} \\ (+\text{ conditions}) \end{array} \right\}$

$\implies g_{Top}(T_{r,s}) < g_4(T_{r,s}) \quad \& \quad \frac{1}{2} \leq \limsup_{\min\{a,b\}\to\infty} \frac{g_{Top}(T_{a,b})}{g_4(T_{a,b})} < \frac{3}{4}$.

<u>Def</u>: n-twist on L: Take $C \simeq$ Unknot, $lk(C,L) = 0$ do $1/n$-surgery



Thm: Take L, obtain L' from an n-twist then an m-twist; then $|g_{Alg}(L) - g_{Alg}(L')| \leq 1$ if $\sqrt{-mn} \in \mathbb{Z}$.

Thm: $g_{Alg}(T_{a,b}) \leq \frac{1}{3}ab + O(\log a + \log b)$

$$\limsup_{(a,b)\to\infty} \frac{g_{Top}(T_{a,b})}{g_4(T_{a,b})} \leq 14/27 \approx 0.52 \quad (\text{Almost } \tfrac{1}{2})$$

Thm: For $P(K)$ a satellite knot, $g_{Alg} P(K) < g_{Alg} P(U) + g_{Alg}(K)$.

↳ Remarkable feature: there's no winding number