n-geometric Stack:

$\exists\, \mathcal{U} = \coprod \operatorname{Spec} S_i$ (atlas)

$\downarrow$

$X \to \operatorname{Spec} k$ is $(n-1)$-geometric

$B^n GL_1$ = Classifying stack of étale-loc. trivial $B^{n-1}GL_1$ torsors

$\underline{Pic} = BGL_1$

$\underline{Br} = B^2 GL_1$ ↙ deloop

$\mathbb{G}_m$-gerbes = Pic torsors.

R discrete $\Rightarrow Br(R) \in \mathcal{S}_{\leq 2}$

Can't be rep. by an Artin Stack

$(\in \text{Grpds} \subseteq \mathcal{S}_{\leq 1})$

$F \in Sh^{ét}(dAff_k) \Rightarrow \exists\, \pi_k^{ét} F$

$\mathcal{U} \mapsto \pi_k(F(\mathcal{U}))$, étale sheafify

$\pi_*^{ét} \underline{Br} = \mathbb{G}_m t^2 \oplus \bigoplus_{i \geq 3} \pi_{i-2} \mathcal{O} t^i$.

$\exists\, E^{s,t} = H^s_{ét}(\mathcal{X}, \pi_t^{ét} \underline{Br}) \Rightarrow \pi_{t-s} \underline{Br}(\mathcal{X})$

$\mathcal{X} = (\mathbb{P}^3, \underbrace{\mathcal{O}_{\mathcal{X}} := \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-4)t}_{= \pi_* \mathcal{O}_{\mathcal{X}}})$

$\pi_* \underline{Br}(\mathcal{X})$:

$k \hookrightarrow \pi_0 \underline{Br}(\mathcal{X}) \twoheadrightarrow Br(k)$

$\pi_1 = Pic(\mathbb{P}^3) \cong \mathbb{Z}$.

$\pi_2 = k^\times$

elts. repped by derived Azumaya algebras

Moduli of objs in dg Cats.

$k \in \text{CRing}$, $A \in \text{dgAlg}$

A is hfp $\Leftrightarrow A \in \text{dgAlg}_k^{\omega}$

X/k sm. proper $\Rightarrow D_{qc}(X) \simeq D(A)$ for same A.

& A is hfp.

$R \in sCAlg_k$

$\mathcal{M}_A(R) = \{\text{reps of } A \text{ in } Perf(R)\}$

$= \operatorname{Fun}^{\text{exact}}(Perf(A), Perf(R))$

small stable ∞-cat, repble by a derived stack.

$\mathcal{M}_X := \mathcal{M}_A$ for any A st. $D(X) = D(R)$

$\mathcal{M}_X(R) = Perf(X \times_k R)$

$i\mathcal{C} \subseteq \mathcal{C}$ maximal subgrpd

$M_A := i\mathcal{M}_A$

$\pi_0 M_A = \{\text{objs in } \overset{}{ho\, M_A(R)}\}/\text{iso}$

$\uparrow$ triang. cat

P = perfect R-Mod w/ $A \curvearrowright P$

$\Rightarrow \pi_i M_A(R) = \begin{cases} Hom(P,P)^\times & i = 1 \\ Ext^{1-i}(P,P) & i \geq 2 \end{cases}$

in $ho M_A(R)$.

Thm Let A/k be hfp.

a) M_A is locally geom & lfp

↖ union of n_i-geom. cpts

b) $\operatorname{Spec} R \xrightarrow{p} M_A$

$\Rightarrow p^* L_{M_A} \simeq \mathbb{R}Hom(P,P)^\vee[-1]$

NB $M_k(R) = i\, Perf(R)$.

$\underline{dPic}_X(R) \subseteq \overbrace{M_X(R)}^{H\text{-space}}$ invertible objs in $Perf(X \times_k R)$.

open inclusion

(invertible = open condn)

Let $\mathcal{L}[n] \in \underline{dPic}_X$, then

$L_{\underline{dPic}_X, \mathcal{L}[n]} = R\Gamma(\mathcal{O}_X)^\vee[-1]$

$\in [-1, n-1]$

$\Rightarrow$ rep by derived Artin stack.

Higher $\mathbb{R}\Gamma(\mathcal{O}_X)$ nontrivial $\Rightarrow$ typically not sm.

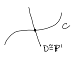
Coarse space: take $\pi_0^{ét}$

MMP

1) Fano $(K_X < 0)$
2) CY $(K_X \sim 0)$
3) Can. pol. $(K_X > 0)$

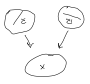
K_X nef $\Rightarrow$ minimal.

Mori bend & break: K_X not nef $\Rightarrow \exists C$ s.t. $K_X \cdot C < 0$, $\forall x \in C$ $\exists D \cong \mathbb{P}^1 \ni x$ (rat'l) w/ $K_X \cdot D < 0$ as well.



- See Mori fibre space.
- $\mathrm{Exc}(f)$ = locus of contracted curves.
- MMP singularities: terminal
- mK_Z being Cartier measures severity of sings.
- Flips:

$$\begin{tikzcd}
X \arrow[rr, dashed, "\exists"] \arrow[dr, dashed, "\psi"'] & & X' \arrow[dl, dashed, "\psi'"] \\
& Z &
\end{tikzcd}$$



$K_X \cdot \bar{C} < 0 \rightarrow K_{X'} \cdot \bar{C}' > 0$, makes K_X more positive. X' has term. sings.

X uniruled (covered by rational curves) or gen type $\Rightarrow$ termination of flips.

$K_{Bl X} \cdot E < 0$.

$\underset{?}{\beta_2}$ $\underset{?}{\text{decreases}}$ under blowups?.

Hironaka: every variety is birational to a smooth variety

Curves w/positive curvature:

$K_X \cdot C < 0$

General type: high deg polynomials.
(e.g. deg $\geq n+2$ hypsfc in $\mathbb{P}^n$)

Min models exist $\forall$ gen type

Constant positive curvature: Kähler-Einstein metric
$\hookrightarrow$ K-stability

See abundance conjecture, follows from alg. foliation thy.

$g = h^0(\omega_X)$, for curves bir $\Rightarrow$ iso.

Goal: min model or Mori fibre space.

Kawamata-Viehweg vanishing

Affine cone of $L \in \mathrm{Cart}(X)_{\mathbb{Q}}$:

$$\mathcal{C}(L) = \mathrm{Spec}\left(\bigoplus_{k=1}^{\infty} R\Gamma(\mathcal{O}_X(kL))\right)$$

$\mathrm{sgn}\ \Lambda_{K3} = (3,19) \Rightarrow \mathrm{sgn}\ L^{\perp} = (2,18)$

$\mathrm{sgn}\ e^{\perp}/e = (1,18) \rightsquigarrow \mathbb{H}^{18}$

$\mathcal{F}_e = \mathrm{Stab}_{\Gamma}(e)$-invt tiling

Quotient fan/tiling by $\rightsquigarrow$ orbifold.

Moduli of slc stable pairs is proper.

$\hookrightarrow K_X + D$ $\mathbb{Q}$-Cart, ample
(X, D) slc

On curves: same as $\overline{\mathcal{M}}_{g,n}$

See Hasset moduli spaces,
discrepancies.
log gen type
Yau-Zaslow formula $n_d = [q^d] \cdot \frac{1}{\Delta(q)}$

Thm (KPP '17) $\Leftarrow$ False for Enriques.

1) $\mathcal{X}$ smooth
2) $K_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}$
3) $\mathcal{X}_0$ SNC

Mumford's semistable reduction:
(1) & (3)

MMP: (2)

KPP: All 3.