Vector Bundles and K-Theory

Skeleton Lecture Notes 2011/2012

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http://www.mathematik.uni-kl.de/∼wirthm/de/top.html

Contents

1 Vector Bundle Basics

Definition Let X be a topological space. A *family of real vector spaces* over X consists of

- a continuous map $E \xrightarrow{\pi} X$ from another topological space to X, and
- for each $x \in X$ a structure of finite dimensional real vector space on the fibre $E_x := \pi^{-1}\{x\}$.

It is required that the topology of E_x as a subspace of E coincides with its standard topology as a real vector space.

The space X is called the *base*, E the *total space*, and π the *projection* of the family. The family is often referred to just by E when X and π are implied by the context.

A *section* of the family E is a map[1] $s\colon X \longrightarrow E$ such that $\pi \circ s = \mathrm{id}_X$.

Definition Let $E \xrightarrow{\pi} X$ and $F \xrightarrow{\chi} X$ be families over X. A *homomorphism* from E to F is a map $h\colon E \to F$ such that

- the diagram

$$\begin{array}{ccc} E & \xrightarrow{\ h\ } & F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\chi} & \\ & X & \end{array}$$

is commutative, and

- for each $x \in X$ the restricted map $h_x\colon E_x \to F_x$ is linear.

Of course, h is an isomorphism if there exists a homomorphism $k\colon F \to E$ with $k \circ h = \mathrm{id}_E$ and $h \circ k = \mathrm{id}_F$. This clearly happens if and only if h is bijective and h^{-1} is continuous. The families E and F are called *isomorphic* if there exists an isomorphism between them.

Examples (1) Any (finite dimensional real) vector space V gives rise to the *product family* $X \times V \xrightarrow{\mathrm{pr}} X$. A family $E \xrightarrow{\pi} X$ is called *trivial* if it is isomorphic to such a product family.

(2) We let $E = X \times V$ as in the previous example but giving the factor X the discrete topology (while X as the base space retains the original one). The result is a family in the sense of the definition, but it is quite far away from the intuitive notion of a continuously parametrised family of vector spaces.

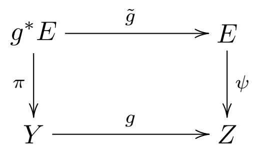
(3) Put $E = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$ and let $\pi\colon E \to \mathbb{R}$ project (x, y) to x. Then $E \xrightarrow{\pi} X$ is a family of vector spaces which certainly is not trivial since the dimension of the fibre E_0 is one while all other fibres are zero vector spaces.

Definition Let $E \xrightarrow{\psi} Z$ be a family of vector spaces over the base Z. A subspace $Y \subset Z$ gives rise to the commutative diagram

$$\begin{array}{ccc} E|Y & \hookrightarrow & E \\ {\scriptstyle\psi'}\downarrow & & \downarrow{\scriptstyle\psi} \\ Y & \hookrightarrow & Z \end{array}$$

[1] As usual in topology we use map as shorthand for a *continuous* map between topological spaces.

with $E|Y := \psi^{-1}Y$ and ψ' obtained from ψ by restriction; it involves the new family $E|Y \xrightarrow{\psi'} Y$ called the *restriction* of E over Y. More generally, let Y be any space and $g: Y \to Z$ a map. We then have a commutative diagram

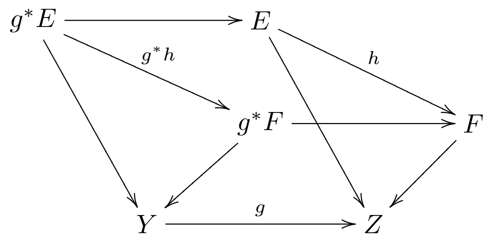


with

$$g^*E := \{(y,v) \in Y \times E \mid g(y) = \psi(v)\} \quad \text{and} \quad \pi(y,v) = y,\ \tilde{g}(y,v) = v,$$

and $g^*E \xrightarrow{\chi} Y$ is a family of vector spaces over Y said to be *induced* by g, or the *pull-back* of E under g.

This construction is functorial in two independent ways. Firstly fix g and let $E \xrightarrow{h} F$ be a homomorphism into another family $F \to Z$ over Z. Then a unique homorphism of families $g^*h: g^*E \longrightarrow g^*F$ is induced that lets the diagram



commute, and we clearly have

$$g^* \operatorname{id}_E = \operatorname{id}_{g^*E} \quad \text{as well as} \quad g^*(k \circ h) = g^*k \circ g^*h$$

for composable homomorphisms h and k. — On the other hand if we now fix the family E and vary g we have though not equalities but canonical isomorphisms

- $\operatorname{id}_Z^* E \simeq E$ and
- $(g \circ f)^* E \simeq f^* g^* E$ if $f: X \to Y$ is another map.

In the special case where $g: Y \to Z$ is the inclusion map of a subspace we recover the restricted family $E|Y \simeq g^*E$ up to canonical isomorphism.

Definition A family E of vector spaces over X is said to be *locally trivial* if every $x \in X$ has a neighbourhood $U \subset X$ such that $E|U$ is trivial: any isomorphism between $E|U$ and a product family over U will be called a *trivialisation* of E over U. In this case the family will be called a *vector bundle* over X. Every family induced from a vector bundle turns out to be locally trivial too, so that we have the notion of induced vector bundle.

The function $\operatorname{rank}_E: X \to \mathbb{N}$ assigns to each point $x \in X$ the vector space dimension of the fibre $\dim E_x$; while rank_E makes sense for every family E over X, in the case of a vector bundle it is a locally constant — or, equivalently, continuous — function. It is called the *rank function*, and if it happens to be constant its value it is simply called the *rank* of the family or vector bundle. We see that the family (3) of our example is not a vector bundle while the product family (1) is one, of course. — A vector bundle of constant rank one is often called a *line bundle*.

Example (4) One way to construct interesting vector bundle is via eigenspaces of families of linear endomorphisms. Consider for each $t \in \mathbb{R}$ the matrix

$$a(t) := \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \in \operatorname{Mat}(2 \times 2, \mathbb{R}).$$

It is orthogonal of determinant -1, so we know it describes a reflection in a uniquely determined line, its 1-eigenspace $E(t) \subset \mathbb{R}^2$. Thus the projection $E \xrightarrow{\pi} S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with

$$E := \{(e^{it}, v) \in S^1 \times \mathbb{R}^2 \mid a(t) \cdot v = v\} \quad \text{and} \quad \pi(e^{it}, v) = e^{it}$$

is a family of real vector spaces of rank one. It is in fact a line bundle. For the fibre $E_{e^{it}} = E(t)$ is the kernel of

$$1 - a(t) = \begin{pmatrix} 1-\cos t & -\sin t \\ -\sin t & 1+\cos t \end{pmatrix},$$

and as long as t is not a multiple of 2π this line is spanned by the vector $(\sin t,\, 1-\cos t)$. Therefore the assignements

$$\begin{array}{ccc} (e^{it}, \lambda) & \longmapsto & \left(e^{it}, \lambda \cdot \begin{pmatrix} \sin t \\ 1-\cos t \end{pmatrix}\right) \\ \left(e^{it}, \dfrac{1}{1-\cos t} \cdot v\right) & \longleftarrow & \left(e^{it}, \begin{pmatrix} u \\ v \end{pmatrix}\right) \end{array}$$

define mutually inverse isomorphisms of families $(S^1\setminus\{1\}) \times \mathbb{R} \to E|(S^1\setminus\{1\})$. Using $(1+\cos t,\, \sin t)$ rather than $(\sin t,\, 1-\cos t)$ as a spanning vector we similarly obtain an isomorphism between the restrictions over $S^1\setminus\{-1\}$. We thus have shown that E is a locally trivial family over S^1.

Notes (1) Let E be a vector bundle over X. In view of local triviality every section $s: X \to E$ can be locally[2] expressed as a function from X to a fixed vector space. The set of all (global) sections of E, denoted ΓE, becomes itself a vector space under addition and scalar multiplication of their values. The image of a section s is an embedded copy of X in E which in term determines s, and therefore the term of section is sometimes applied to $s(X)$ rather than s. In particular there is the *zero section* $s_0: X \to E$ which assigns to each $x \in X$ the zero vector in E_x. It provides a canonical way to identify X with the subspace $s_0(X) \subset E$.

(2) Let $X\times V$ and $X\times W$ be two product bundles over X. There is a bijective correspondence between bundle homomorphisms $h: X\times V \to X\times W$ and (continuous) mappings $\tilde{h}: X \to \operatorname{Hom}(V, W)$ into the space of linear maps, namely (mildly abusing the notation)

$$\begin{array}{ccc} h & \longmapsto & (x \mapsto h_x) \\ \big((x, v) \mapsto (x, \tilde{h}(x)(v))\big) & \longleftarrow & \tilde{h}. \end{array}$$

(3) Keeping this set-up we let $\operatorname{Hom}^*(V, W) \subset \operatorname{Hom}(V, W)$ denote the open subset of all isomorphisms from V to W. Consider a bijective bundle homomorphism $h: X\times V \to X\times W$: then for every $x \in X$ the restriction $h_x: V \to W$ is an isomorphism and $\tilde{h}$ maps into $\operatorname{Hom}^*(V, W)$. Since the inversion map $\operatorname{Hom}^*(V, W) \ni g \mapsto g^{-1} \in \operatorname{Hom}^*(W, V)$ is continuous, so is its composition with $\tilde{h}$, which is the map $X \ni x \mapsto h_x^{-1} \in \operatorname{Hom}(W, V)$. The corresponding bundle map is nothing but $h^{-1}: X\times W \to X\times V$, and we conclude that continuity of h^{-1} is automatic once the bundle homomorphism h is known to be bijective. This being a local assertion it remains true if $X\times V$ and $X\times W$ are replaced by arbitrary bundles over X.

(4) Similarly it follows from the openness of $\operatorname{Hom}^*(V, W) \subset \operatorname{Hom}(V, W)$ that for an arbitrary bundle homomorphism $h: E \to F$ the points x such that h_x is an isomorphism, form an open subset of X.

[2] In the context of bundles locality always refers to the base, never to the total space.

2 Projective Spaces

Definition Let $n \in \mathbb{N}$. The *n-dimensional real projective space* $\mathbb{R}P^n$ is the quotient space of $\mathbb{R}^{n+1}\setminus\{0\}$ by the equivalence relation

$$x \sim y \quad :\Longleftrightarrow \quad \lambda x = y \text{ for some } \lambda \in \mathbb{R}^*.$$

Alternatively $\mathbb{R}P^n$ is obtained from the sphere $S^n = \{x \in \mathbb{R}^{n+1} \,\big|\, |x| = 1\}$ identifying pairs of opposite points: $-x \sim x$. In either case we write the point of $\mathbb{R}P^n$ represented by $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ as

$$[x] = [x_0 : x_1 : \cdots : x_n] \in \mathbb{R}P^n$$

to emphasize the fact that not the values but the *ratios* between the $n+1$ numbers x_j make up the point $[x]$. Intuitively it is best to think of $\mathbb{R}P^n$ as the space of lines (one-dimensional vector subspaces) in $\mathbb{R}^{n+1}$.

While the first definition carries over literally to a definition of the complex projective space $\mathbb{C}P^n$ the alternative one becomes more involved; it presents $\mathbb{C}P^n$ as the quotient space of the sphere $S^{2n+1} = \{z \in \mathbb{C}^{n+1} \,\big|\, |z| = 1\}$ with respect to the equivalence relation

$$w \sim z \quad :\Longleftrightarrow \quad \lambda w = z \text{ for some } \lambda \in S^1 \subset \mathbb{C}.$$

Notes (1) All projective spaces are compact Hausdorff spaces. To make explicit calculations in them one uses the fact that the sets

$$X_k := \{[x] \in \mathbb{R}P^n \,\big|\, x_k \neq 0\} \quad \text{for } k = 0, \ldots, n$$

form a finite open cover of $\mathbb{R}P^n$, and that for each k the mapping $h_k \colon X_k \to \mathbb{R}^n$ acting by[1]

$$\begin{array}{ccc} [x_0 : \cdots : x_k : \cdots : x_n] & \longmapsto & \dfrac{1}{x_k} \cdot (x_0 \ldots, \widehat{x_k}, \ldots, x_n) \\ [x_0 : \cdots : 1 : \cdots : x_n] & \longleftarrow & (x_0 \ldots, \widehat{x_k}, \ldots, x_n) \end{array}$$

is a homeomorphism. Of course the complex case is analogous.

(2) Since $\mathbb{R}P^0$ and $\mathbb{C}P^0$ are one-point spaces the first possibly interesting case is that of $n = 1$. In the real case the mapping $S^1 \ni z \mapsto z^2 \in S^1$, which takes equal values on antipodal points, induces a homeomorphism $\mathbb{R}P^1 \approx S^1$. A similar homeomorphism

$$h \colon \mathbb{C}P^1 \approx S^2 = \{(w, t) \in \mathbb{C} \times \mathbb{R} \,\big|\, |w|^2 + t^2 = 1\}$$

identifying $\mathbb{C}P^1$ with the Riemann sphere involves stereographic projection and acts by the assignments

$$\begin{array}{ccc} [z_0 : z_1] & \longmapsto & \dfrac{1}{|z_0|^2 + |z_1|^2} \cdot \begin{pmatrix} 2\overline{z}_0 z_1 \\ |z_0|^2 - |z_1|^2 \end{pmatrix} \\ \big[\overline{w} : (1-t)\big] = \big[(1+t) : w\big] & \longleftarrow & \begin{pmatrix} w \\ t \end{pmatrix}. \end{array}$$

Definition We consider the map $T \xrightarrow{\pi} \mathbb{R}P^n$ with

$$T := \{([x], v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \,\big|\, v \in \mathbb{R}x\} \quad \text{and} \quad \pi([x], v) = [x].$$

[1] We use the convention that terms covered by a hat are to be omitted.

Since the fibre $T_{[x]} = \pi^{-1}\{[x]\}$ is just $\{[x]\}\times\mathbb{R}x$ we have defined a family of vector spaces over $\mathbb{R}P^n$, and in fact a line bundle, which is called the *tautological bundle* on $\mathbb{R}P^n$. Indeed the formula

$$X_0 \times \mathbb{R} \ni \big([1 : x_1 : \cdots : x_n],\, \lambda\big) \longmapsto \big([1 : x_1 \cdots : x_n],\, \lambda(1, x_1 \ldots, x_n)\big) \in T$$

trivialises the family over the open subset $X_0 \subset \mathbb{R}P^n$, and permuting the 0th with the other coordinates we cover $\mathbb{R}P^n$ by such local trivialisations. The name of this bundle expresses the fact that the fibre over a point of $\mathbb{R}P^n$ is the point itself, read as a line in $\mathbb{R}P^{n+1}$.

Theorem Further representations of $\mathbb{R}P^n$ as a quotient space include the following.

- Let $T \to \mathbb{R}P^{n-1}$ be the tautological bundle and let

$$D := \big\{([x], v) \in T \,\big|\, |v| \leq 1\big\} \quad \text{and} \quad S := \big\{([x], v) \in T \,\big|\, |v| = 1\big\}$$

denote the unit disk and sphere "bundles" in it. The space $D^n \cup_h D$ obtained by gluing the unit disk $D^n = \big\{x \in \mathbb{R}^n \,\big|\, |x| \leq 1\big\}$ to D via the homeomorphism

$$D^n \supset S^{n-1} \ni v \overset{h}{\longmapsto} ([v], v) \in S \subset D$$

is homeomorphic to $\mathbb{R}P^n$.

- Consider the map

$$S^{n-1} \ni x \overset{\varphi}{\longmapsto} [x] \in \mathbb{P}^{n-1}.$$

The quotient space of $D^n + \mathbb{R}P^{n-1}$ with respect to the equivalence relation generated by

$$D^n \ni x \sim \varphi(x) \in \mathbb{R}P^{n-1}$$

is said to be built from $\mathbb{R}P^{n-1}$ by *attaching an n-cell via* φ, and usually written $D^n \cup_\varphi \mathbb{R}P^{n-1}$. It is likewise homeomorphic to $\mathbb{R}P^n$.

- Indeed in the previous construction D^n clearly maps *onto* the quotient, and we thus may as well write the latter as a quotient space of just D^n, identifying opposite points on the boundary $S^{n-1} \subset D^n$.

Further Notes (3) The quotient mapping

$$S^n \ni z \overset{q}{\longmapsto} [z] \in \mathbb{R}P^n$$

is a two-fold covering projection. For $n > 0$ it must be non-trivial since S^n is connected. From a different point of view we see a presentation of $\mathbb{R}P^n$ as the space of orbits of S^n with respect to the natural action

$$\{\pm 1\} \times S^n \longrightarrow S^n$$

of the group $\{\pm 1\}$ on the n-sphere.

(4) In the complex case the quotient mapping

$$S^{2n+1} \ni z \overset{q}{\longmapsto} [z] \in \mathbb{C}P^n$$

is called the *Hopf mapping* or *Hopf fibration*. Each of its fibres is a homeomorphic copy of the circle S^1, and indeed it presents $\mathbb{C}P^n$ as the orbit space of $S^{2n+1} \subset \mathbb{C}^{n+1}$ by the scalar action

$$S^1 \times S^{2n+1} \longrightarrow S^{2n+1}$$

of the circle group $S^1 \subset \mathbb{C}^*$.

3 Linear Algebra of Vector Bundles

Construction We let $\mathcal{T}$ be a covariant functor from the category of finite dimensional real vector spaces into itself and assume that $\mathcal{T}$ is continuous in the sense that for any two objects V and W the mapping

$$\mathcal{T}\colon \mathrm{Hom}(V,W) \longrightarrow \mathrm{Hom}(\mathcal{T}V,\mathcal{T}W)$$

is continuous. An example is the functor $\mathcal{T}$ which assigns to V the vector space $\mathcal{T}V = \mathrm{Hom}(A,V)$ where A is a fixed real vector space of finite dimension.

We wish to extend $\mathcal{T}$ to a functor of vector bundles: given a vector bundle $E \to X$ we shall construct a new vector bundle $\mathcal{T}E \to X$ over X such that for each $x \in X$ one has

$$(\mathcal{T}E)_x = \mathcal{T}E_x,$$

while to each homomorphism of bundles $E \xrightarrow{h} F$ we will assign a homomorphism $\mathcal{T}h \xrightarrow{\mathcal{T}h} \mathcal{T}F$ such that

$$(\mathcal{T}h)_x = \mathcal{T}(h_x).$$

Since these requirements already determine $\mathcal{T}E$ as a set (the disjoint union of the $\mathcal{T}E_x$), and $\mathcal{T}h$ as a fibre-wise linear mapping of sets we only need to specify the correct topology on $\mathcal{T}E$. We do this in three steps.

In case $E = X\times V \to X$ is a product bundle we give $\mathcal{T}E = X\times\mathcal{T}V$ the product topology. If $F = X\times W$ is another product bundle and $h\colon E \to F$ a homomorphism then the corresponding map $\tilde{h}\colon X \to \mathrm{Hom}(V,W)$ is continuous, and so is the composition

$$X \longrightarrow \mathrm{Hom}(V,W) \xrightarrow{\mathcal{T}} \mathrm{Hom}(\mathcal{T}V,\mathcal{T}W),$$

by continuity of $\mathcal{T}$. Since this composition corresponds to the set mapping $\mathcal{T}E \xrightarrow{\mathcal{T}h} \mathcal{T}F$ we have proven that the latter is continuous too. This completes the treatment of product bundles.

More generally we now assume that E is any trivial bundle. We choose a trivialisation $h\colon X\times V \simeq E$ and use the bijection $\mathcal{T}h\colon \mathcal{T}(X\times V) \to \mathcal{T}E$ to transfer the topology to $\mathcal{T}E$. If $k\colon X\times W \simeq E$ is a second trivialisation then $k^{-1}\circ h\colon X\times V \to X\times W$ is a bundle isomorphism, and from the first step we know that then $(\mathcal{T}k)^{-1}\circ\mathcal{T}h = \mathcal{T}(k^{-1}\circ h)$ is a bundle isomorphism, in particular a homeomorphism. This proves that the topology on $\mathcal{T}E$ is well-defined. The continuity of the mapping $\mathcal{T}h\colon \mathcal{T}E \to \mathcal{T}F$ induced by a bundle homomorphism $h\colon E \to F$ is obvious. We finally observe that the topologies we have put on $X\times\mathcal{T}V$ and $\mathcal{T}E$ are clearly compatible with restriction to a subspace $S \subset X$, so that the notation $\mathcal{T}E|S$ is unambiguous. This completes the discussion of trivial bundles.

Let now $E \to X$ be an arbitrary bundle. For the open sets of $\mathcal{T}E$ we take all sets $V \subset \mathcal{T}E$ such that the intersection $V \cap \mathcal{T}E|U$ is open in $\mathcal{T}E|U$ whenever $U \subset X$ is open and E is trivial over U. It is easily seen that in order to test V for openness the condition need only be checked for a collection of such U that cover X. Again a bundle homomorphism $h\colon E \to F$ induces a *continuous* mapping $\mathcal{T}h\colon \mathcal{T}E \to \mathcal{T}F$, and the topology on $\mathcal{T}E$ is compatible with restriction to arbitrary subspaces $S \subset X$. This completes the construction.

Rather than with mere restrictions, the new-defined functor $\mathcal{T}$ is also compatible with pull-backs: there is a natural isomorphism

$$f^*\mathcal{T}E \simeq \mathcal{T}f^*E$$

for every mapping f from another topological space into X.

Suitable Functors for this construction include, similarly, contravariant ones as well as functors of several finite vector space variables, such as

- the contravariant functor assigning to V its dual space $V^\vee$,
- the functor assigning to a pair (V, W) of vector spaces its direct sum $V \oplus W$ — we might as well write the direct product $V \times W$, but the sum is preferred by tradition, and the resulting bundles called *Whitney sums* of vector bundles;
- the bivariant functor that assigns to V and W the space of linear mappings $\mathrm{Hom}(V, W)$,
- the covariant functor that sends V and W to the tensor product $V \otimes W$ — for which $\mathrm{Hom}(V^\vee, W)$ is a valid substitute in case you are not familiar with the tensor product,
- the functor assigning to V its d-th symmetric power $\mathrm{Sym}^d V$,
- the functor assigning to V its d-th alternating or exterior power $\Lambda^d V$,

and many others.

Notes (1) The canonical isomorphy $\mathrm{Hom}(V, W) \simeq V^\vee \otimes W$ allows to replace all bundles of homorphisms by tensor products if desired — or vice versa:

$$\mathrm{Hom}(E, F) \simeq E^\vee \otimes F \simeq \mathrm{Hom}(F^\vee, E^\vee).$$

If L is a line bundle then

$$L \otimes L^\vee \simeq \mathrm{Hom}(L, L) = \mathrm{End}\, L$$

is the trivial line bundle since endomorphisms of a one-dimensional vector space are just scalars: thus the isomorphism classes of line bundles over a fixed base X form a commutative group $\mathrm{Vect}_1 X$ under the tensor product. This group acts on the additive semi-group $\mathrm{Vect}\, X$ of isomorphism classes of all vector bundles on X by

$$[L] \cdot [E] = [L \otimes E],$$

an action which clearly preserves the rank function.

(2) The correspondence between bundle homomophisms $X \times V \to X \times W$ and mappings $X \to \mathrm{Hom}(V, W)$ now globalises to a canonical linear correspondence between homomorphisms $E \to F$ of bundles over X, and sections of the bundle $\mathrm{Hom}(E, F) \to X$.

Definition A *subbundle* of a vector bundle $F \xrightarrow{\pi} X$ is a subspace $S \subset F$ which makes $S \xrightarrow{\pi|S} X$ a vector bundle in its own right — notably this includes the condition that for each $x \in X$ the fibre $S_x = S \cap F_x \subset F_x$ is a vector subspace.

Lemma Let $h: E \to F$ be an injective homomorphism of vector bundles. Then $h(E) \subset F$ is a subbundle. Every subbundle $S \subset F$ over X is locally near $x \in X$ isomorphic to the inclusion $X \times S_x \subset X \times F_x$ induced by that of the vector spaces $S_x \subset F_x$.

Corollary If $S \subset E$ is a subbundle then the fibre-wise quotient $E/S := \bigcup_{x \in X} E_x/S_x$, equipped with the quotient topology, also is a vector bundle over X; it is called the *quotient bundle*.

Proposition Let $h: E \to F$ be a homomorphism of vector bundles such that the function $\mathrm{rank}_h: X \to \mathbb{N}$ defined by $x \mapsto \mathrm{rank}\, h_x$ is locally constant. Then the fibre-wise defined sets

$$\mathrm{kernel}\, h \subset E \quad \text{and} \quad \mathrm{image}\, h \subset F$$

are subbundles, and a fortiori $\mathrm{coker}\, h = F/\,\mathrm{image}\, h$ is a quotient bundle of F.

Examples (1) Let $W \subset \mathbb{R}^n$ be an open subset, $f: W \to \mathbb{R}^p$ a differentiable function, and $b \in \mathbb{R}^p$ a regular value of f: thus at every point x of

$$X := f^{-1}\{b\} = \big\{x \in W \;\big|\; f(x) = b\big\}$$

the differential $T_xf\colon \mathbb{R}^n \to \mathbb{R}^p$ is surjective, and $X \subset W$ is a differentiable submanifold. The set

$$TX := \bigcup_{x\in X} \{x\}\times \operatorname{kernel} T_xf = \big\{(x,v) \in X\times\mathbb{R}^p \;\big|\; T_xf(v) = 0\big\} \longrightarrow X$$

is a subbundle of the product bundle $X\times\mathbb{R}^n \to X$, and called the *tangent bundle* of X: indeed the differential of f defines the bundle homomorphism

$$X \times \mathbb{R}^n \ni (x, v) \longmapsto \big(x, T_xf(v)\big) \in X \times \mathbb{R}^p,$$

which by assumption is surjective and has TX as its kernel by definition. The quotient bundle $(X\times\mathbb{R}^n)/TX$ is called the *normal bundle* of X in W, and in this situation is always trivial since the bundle homomorphism that defines TX induces an isomorphism $(X\times\mathbb{R}^n)/TX \simeq X\times\mathbb{R}^p$.

One of the simplest particular cases is that of the function $\mathbb{R}^n \ni x \overset{f}{\longmapsto} |x|^2 \in \mathbb{R}$ with $b = 1$. Here $X = S^{n-1}$ is the unit sphere, and its tangent space at x — the fibre over x of the tangent bundle — is

$$T_xS^{n-1} = (TS^{n-1})_x = \big\{(x, v) \in S^{n-1}\times\mathbb{R}^n \;\big|\; x \perp v\big\}\,.$$

(2) Taking the quotient of TS^{n-1} by the involutive action $(x, v) \mapsto (-x, -v)$ results in a vector bundle on $\mathbb{R}P^{n-1}$ which in the theory of differentiable manifolds is identified with the tangent bundle $T(\mathbb{R}P^{n-1})$ of this projective space.

Definition If $E \to X$ is a complex vector bundle we let $\operatorname{Herm} E \to X$ denote the (real!) vector bundle whose fibre over x is the space of Hermitian forms $E_x \times E_x \to \mathbb{C}$ (conjugate-linear in the first variable). A *metric* of E is a section of $\operatorname{Herm} E$ which is positive definite at each point of X; its value on $(v, w) \in E_x\times E_x$ is often written $\langle v, w\rangle_x$ or just $\langle v, w\rangle$.

Proposition Assume that $E \to X$ admits a metric. Then for every subbundle $S \subset E$ there exists a complementary subbundle $Q \subset E$, so that in particular $E/S \simeq Q$.

Examples (1) The product bundle $X\times\mathbb{R}^n$ certainly carries the standard euclidean metric, and by the real version of the last proposition the tangent bundle $TX \subset X\times\mathbb{R}^n$ of a submanifold $X \subset W$ as above admits a complement $N \subset X\times\mathbb{R}^n$, so that

$$TX \oplus N = X\times\mathbb{R}^n.$$

On the other hand $N \simeq (X\times\mathbb{R}^n)/TX \simeq X\times\mathbb{R}^p$ must be trivial, and we record as a remarkable fact that the Whitney sum of TX (usually non-trivial as we will see) and a trivial bundle (of sufficiently large rank) is itself trivial. In the case of the sphere the complement N is the line bundle

$$N = \big\{(x, v) \in S^{n-1}\times\mathbb{R}^n \;\big|\; v \in \mathbb{R}x\big\}\,,$$

trivialised by $S^{n-1}\times\mathbb{R} \ni (x, \lambda) \mapsto (x, x\cdot\lambda) \in N$. Putting things together we obtain the isomorphism

$$\begin{array}{ccc} TS^{n-1} \oplus (S^{n-1}\times\mathbb{R}) & \simeq & S^{n-1}\times\mathbb{R}^n \\ (x,\, v\oplus\lambda) & \mapsto & (x,\, v + x\cdot\lambda) \end{array}$$

of bundles over S^{n-1}.

(2) The analogue for bundles over $\mathbb{R}P^{n-1}$ requires the tautological bundle as a tensor factor on the left hand side; it is the isomorphism

$$T \otimes \big(T(\mathbb{R}P^{n-1}) \oplus (\mathbb{R}P^{n-1}\times\mathbb{R})\big) \;\simeq\; \mathbb{R}P^{n-1}\times\mathbb{R}^n$$

given over the representative $x \in S^{n-1}$ of $[x] \in \mathbb{R}P^{n-1}$ by the assignment

$$\big(x,\, u\otimes(v\oplus\lambda)\big) \longmapsto \Big(x,\, \frac{u}{x}\cdot(v+\lambda x)\Big) = \Big(x,\, \frac{u}{x}\cdot v + u\cdot\lambda\Big)\,.$$

4 Compact Base Spaces

From now on we only consider vector bundles over compact[1] base spaces. The bundles themselves will be complex vector bundles unless stated otherwise.

Proposition Every vector bundle $E \to X$ admits a metric.

Proposition Let $E \to X$ be a vector bundle and $S \subset X$ a closed subspace. Then every section $s \in \Gamma(E|S)$ extends to a section $t \in \Gamma E$.

Lemma Let $E \to X$ and $F \to X$ be vector bundles and $S \subset X$ a closed subspace. If $E|S \xrightarrow{f} F|S$ is an isomorphism of bundles then there exist an open set $U \subset X$ with $S \subset U$ and an extension $E|U \xrightarrow{g} F|U$ of f which is an isomorphism.

Theorem Let $E \to Y$ be a vector bundle, X a compact space, and $f\colon I\times X \to Y$ be a homotopy[2] from $f_0\colon X \to Y$ to $f_1\colon X \to Y$. Then

$$f_0^* E \simeq f_1^* E.$$

Notation We let $\operatorname{Vect} X$ denote the set of isomorphism classes of vector bundles over X: this is a semi-ring[3] under the operations of Whitney sum and tensor product. It always contains the disjoint union

$$\operatorname{Vect} X = \bigcup_{d=0}^{\infty} \operatorname{Vect}_d X$$

where $\operatorname{Vect}_d X$ comprises the classes of vector bundles of rank d, but is stricly larger if X is disconnected. Since every map $f\colon X \to Y$ induces maps f^*, more precisely

$$\operatorname{Vect}_d f\colon \operatorname{Vect}_d Y \to \operatorname{Vect}_d X \quad \text{and} \quad \operatorname{Vect} f\colon \operatorname{Vect} Y \to \operatorname{Vect} X,$$

assigning to a class of bundles that of the pull-back bundle, we are dealing with contravariant functors from the category of compact spaces to that of sets, in the last case even to that of semi-rings. The result of the theorem allows to read these functors as defined on the homotopy category where continuous maps are replaced by their homotopy classes.

Corollary (1) Every bundle $E \to I\times X$ is isomorphic to the pull-back of the restriction $\operatorname{pr}^*(E\,|\,\{0\}\times X)$.

(2) If $X \xrightarrow{f} Y$ is a homotopy equivalence — that is, an isomorphism in the homotopy category — then the induced semi-ring homomorphism $f^*\colon \operatorname{Vect} Y \simeq \operatorname{Vect} X$ is an isomorphism. In particular, if X is contractible then the rank function sets up an identification $\operatorname{Vect} X = \operatorname{Vect}\{*\} = \mathbb{N}$.

[1] Compactness shall include the Hausdorff property. All such spaces X are normal, and thus obey Urysohn's and Tietze's theorems. Furthermore every finite open cover of X admits a subordinate partition of unity.

[2] As usual in homotopy theory, $I = [0,1]$ is shorthand for the unit interval.

[3] This notion, which seems to be rarely used in mathematics, refers to an algebraic structure that satisfies the standard ring axioms with the exception that addition is required to make it a semi-group rather than a group.

Note Let X be compact, $S \subset X$ a closed subspace, and $X \xrightarrow{q} X/S$ the quotient mapping that collapses S to the point $S/S \in X/S$. If $F \to X/S$ is any vector bundle over X/S the induced bundle $q^*F \to X$ not only is trivial over the subspace S but we even obtain a particular trivialisation

$$S \times \mathbb{C}^d \simeq S \times F_{S/S} = q^*F|S$$

once we have chosen a base of the single vector space $F_{S/S}$.

Collapsing Construction We reverse this process: Let X and S be as before, $E \xrightarrow{\pi} X$ be a vector bundle, and $h: S \times \mathbb{C}^d \simeq E|S$ a trivialisation of E over S. We form the quotient space of E with respect to the equivalence relation

$$v \sim w \quad :\Longleftrightarrow \quad \{v, w\} \subset E|S \text{ and } \operatorname{pr} \circ h^{-1}(v) = \operatorname{pr} \circ h^{-1}(w)$$

where $\operatorname{pr}: S \times \mathbb{C}^d \to \mathbb{C}^d$ is projection to the fibre. The result is a family of vector spaces $E/h \to X/S$, and in fact a vector bundle over X/S. For by the lemma above, h extends to a local trivialisation $h: U \times \mathbb{C}^d \simeq E|U$ over some neighbourhood $U \subset X$ of S, and this extension in turn drops[4] to a trivialisation $U/S \times \mathbb{C}^d \simeq (E/h)|(U/S)$ over the neighbourhood U/S of S/S. On the other hand local triviality of E/h over $X \setminus S$ is clear since no identifications are made over this open subspace.

Let now h_0 and h_1 be homotopic trivialisations of E over S: this means that there exists a trivialisation

$$I \times S \times \mathbb{C}^d \xrightarrow{h} I \times (E|S)$$

of the bundle $I \times (E|S) \to I \times S$ which over $\{0\} \times S$ and $\{1\} \times S$ reduces to h_0 and h_1 respectively. Using h as a gluing isomorphism we may form the bundle $(I \times E)/h \to (I \times X)/(I \times S)$, and pulling back by the quotient mapping $I \times (X/S) \xrightarrow{p} (I \times X)/(I \times S)$ obtain a vector bundle

$$p^*\big((I \times E)/h\big) \longrightarrow I \times (X/S)$$

which over $\{0\} \times X/S$ restricts to E/h_0, and over $\{1\} \times X/S$ to E/h_1. By the theorem we conclude that $E/h_0 \simeq E/h_1$.

Summarising, our construction establishes a bijection between isomorphims classes of vector bundles over X/S, and isomorphism classes of pairs $(E, [h])$ comprising a bundle E over X and a homotopy class of trivialisations h of $E|S$.

Proposition Let $S \subset X$ be a contractible closed subspace. Then the quotient mapping $q: X \to X/S$ induces an isomorphism of semi-rings

$$q^*: \operatorname{Vect} X/S \simeq \operatorname{Vect} X.$$

Gluing Construction Assume the following data are given: a decomposition

$$X = X_1 \cup X_2, \quad X_1 \cap X_2 = S$$

of the space X, vector bundles $E_1 \to X_1$ and $E_2 \to X_2$, and an isomorphism of bundles

$$E_1|S \xrightarrow{h} E_2|S.$$

Note that X may be identified with the quotient space of $X_1 + X_2$ obtained by making $x \in S \subset X_1$ equal to $x \in S \subset X_2$. Similarly the quotient space of $E_1 + E_2$ with respect to the equivalence relation generated by

$$E_1 \supset E_1|S \ni v \sim h(v) \in E_2|S \subset E_2$$

[4] Here and elsewhere we tacitly apply results on the compatibility of product and quotient topologies.

may be formed, it is written $E_1 \cup_h E_2$ and is a family of vector spaces over X. Again this family turns out to be a vector bundle:

Given a point $x \in S$ we choose a closed neighbourhood $V_1 \subset X_1$ over which we find a trivialisation

$$h_1 \colon V_1 \times \mathbb{C}^d \simeq E_1|V_1.$$

Composing the restrictions of h_1 and h over $V_1 \cap S$ we obtain a trivialisation

$$h_2 \colon (V_1 \cap S) \times \mathbb{C}^d \simeq E_2|(V_1 \cap S)$$

of E_2 over $V_1 \cap S$, which we extend at once over some neighbourhood $V_2 \subset X_2$. Then $V := V_1 \cup V_2$ is a neighbourhood of x in X, and we glue h_1 and h_2 to obtain the required trivialisation

$$h_1 \cup_{h|V} h_2 \colon V \times \mathbb{C}^d \simeq (E_1 \cup_h E_2)|V$$

of $E_1 \cup_h E_2$ over V. — Local triviality of $E_1 \cup_h E_2$ at all points of the open set $X \backslash S$ follows at once from the local triviality of E_1 and E_2.

Let us record a few obvious properties of the gluing construction:

- If $E_1 \to X_1$ and $E_2 \to X_2$ are the restrictions of an existing bundle $E \to X$ then the construction with $h = \mathrm{id}_{E|S}$ simply recovers the latter up to canonical isomorphism.
- If two sets of gluing data $(E_1 \to X_1, h, E_2 \to X_2)$ and $(E_1' \to X_1, h', E_2 \to X_2')$ are related by isomorphisms $g_1 \colon E_1 \simeq E_1'$ and $g_2 \colon E_2 \simeq E_2'$ such that

$$\begin{array}{ccc} E_1|S & \xrightarrow{g_1} & E_1'|S \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ E_2|S & \xrightarrow{g_2} & E_2'|S \end{array}$$

commutes then g_1 and g_2 induce an isomorphism $E_1 \cup_h E_2 \simeq E_1' \cup_{h'} E_2'$.
- The construction is compatible with algebraic operations on bundles:

$$(E_1 \cup_h E_2) \oplus (E_1' \cup_{h'} E_2') = (E_1 \oplus E_1') \cup_{h \oplus h'} (E_2 \oplus E_2')$$
$$(E_1 \cup_h E_2) \otimes (E_1' \cup_{h'} E_2') = (E_1 \otimes E_1') \cup_{h \otimes h'} (E_2 \otimes E_2')$$

An important fact is that gluing by homotopic isomorphisms $E_1|S \simeq E_2|S$ gives isomorphic results, as follows. A homotopy of the type in question is an isomorphism

$$H \colon (\mathrm{pr}^* E_1)|(I \times S) \;\simeq\; (\mathrm{pr}^* E_2)|(I \times S)$$

of bundles over $I \times S$, where $\mathrm{pr} \colon I \times X \to X$ is the cartesian projection. For each $t \in I$ we let

$$X \ni x \overset{j_t}{\longmapsto} (t, x) \in I \times X \quad \text{and} \quad E_1|S \ni v \overset{h_t}{\longmapsto} H(t, v) \in E_2|S$$

denote the embedding and the gluing isomorphism at time t, so that

$$E_1 \cup_{h_t} E_2 = j_t^* \big(\mathrm{pr}^* E_1 \cup_H \mathrm{pr}^* E_2\big).$$

The claim now follows from the homotopy invariance of the induced bundle.

We finally note that the gluing isomorphism $E_1|S \xrightarrow{h} E_2|S$ may, of course, be equivalently described by the corresponding section $\tilde{h} \colon S \to \mathrm{Hom}^*(E_1|S, E_2|S)$, which we will call a *gluing function*. From this point of view homotopy of gluing isomorphisms corresponds to homotopy of gluing functions in the standard sense.

Definition The *suspension* of a (not necessarily compact) topological space X is the quotient ΣX of $I\times X$ obtained by collapsing the subspaces $\{0\}\times X$ and $\{1\}\times X$ to a single point each.

Theorem Let X be compact. For each $d\in\mathbb{N}$ the set of rank d complex vector bundles over ΣX is in canonical bijection with the set of homotopy classes[5] of mappings $X\to GL(d,\mathbb{C})$:

$$\mathrm{Vect}_d\,\Sigma X \simeq [X, GL(d,\mathbb{C})]\,.$$

Examples (1) The set $[S^0, GL(d,\mathbb{C})]$ has a single element since $GL(d,\mathbb{C})$ is path-connected. Since the suspension $\Sigma S^0 \approx S^1$ is a circle we conclude that every complex vector bundle over S^1 is trivial.

(2) The choice $X=S^1$ makes $\Sigma X = \Sigma S^1$ a 2-sphere, say via the obvious homeomorphism

$$\Sigma S^1 \ni (t,z) \mapsto \left(\sqrt{1-(2t-1)^2}\cdot z,\ 2t-1\right) \in S^2,$$

and for $d=1$ the homotopy set

$$[X, GL(d,\mathbb{C})] = [S^1,\mathbb{C}^*] = \pi_1(S^1)\simeq\mathbb{Z}$$

comes down to the fundamental group of $\mathbb{C}^*$, which is infinite cyclic. If we identify S^2 with $\mathbb{C}P^1$ as in Section 2 Note (2) then the homotopy class of $S^1\ni z\mapsto z^{-1}\in\mathbb{C}^*$ corresponds to the tautological bundle $T\to\mathbb{C}P^1$.

Indeed in terms of this identification the cones

$$C_-X = \left\{[t,x]\in\Sigma X \,\middle|\, t\le\tfrac12\right\} \quad\text{and}\quad C_+X = \left\{[t,x]\in\Sigma X \,\middle|\, t\ge\tfrac12\right\}$$

become

$$\mathbb{C}P^1_- = \left\{[w:z]\in\mathbb{C}P^1 \,\middle|\, |w|\le|z|\right\} \quad\text{and}\quad \mathbb{C}P^1_+ = \left\{[[w:z]\in\mathbb{C}P^1 \,\middle|\, |w|\ge|z|\right\}$$

respectively, and the construction glues the trivial bundles $\mathbb{C}P^1_-\times\mathbb{C}$ and $\mathbb{C}P^1_+\times\mathbb{C}$ via

$$\mathbb{C}P^1_-\times\mathbb{C}\ni\big([1:z]\,,\,\lambda\big)\simeq\big([1:z]\,,\,z^{-1}\cdot\lambda\big)\in\mathbb{C}P^1_+\times\mathbb{C}\quad\text{if } |z|=1.$$

Thus the assignment

$$\begin{array}{rcl}\mathbb{C}P^1_-\times\mathbb{C}\ni\big([w:1]\,,\,\lambda\big) & \longmapsto & \left([w:1]\,,\,\lambda\cdot\begin{pmatrix}w\\1\end{pmatrix}\right)\in T\\ \mathbb{C}P^1_+\times\mathbb{C}\ni\big([1:z]\,,\,\lambda\big) & \longmapsto & \left([1:z]\,,\,\lambda\cdot\begin{pmatrix}1\\z\end{pmatrix}\right)\in T\end{array}$$

yields a well-defined isomorphism of bundles over $\mathbb{C}P^1$.

(3) The bijection between $\mathrm{Vect}_1\,\Sigma X$ and $[X,\mathbb{C}^*]$ is at once seen to be a group isomorphism, so that more generally the gluing function $z\mapsto z^{-k}$ produces the k-th tensor power $T^k = T\otimes\cdots\otimes T$ of the tautological bundle on S^2. Note that while from the previous example we learnt that T is non-trivial we now see that it is not isomorphic to its dual $T^{\check{}} = T^{-1}$.

(4) For $n>2$ the set $[S^{n-1},\mathbb{C}^*]$ consists of just one homotopy class, so that S^2 is the only sphere that carries non-trivial complex line bundles.

(5) In the theory of Lie groups it is well-known that the inclusion

$$GL(n,\mathbb{C})\ni g\mapsto g\oplus 1 = \begin{pmatrix} g & \\ & 1\end{pmatrix}\in GL(n{+}1,\mathbb{C})$$

[5] Throughout we use square brackets to indicate sets of homotopy classes.

induces an isomorphism of fundamental groups for each $n \geq 1$. Together with the previous example this implies that every complex vector bundle on S^2 is isomorphic to the Whitney sum of a unique power of the tautological bundle T, and an arbitrary trivial bundle.

Lemma Let $E \to X$ be a vector bundle. Then ΓE contains a subspace V of finite dimension such that the evaluation map

$$X \times V \ni (x, v) \longmapsto v(x) \in E$$

is surjective.

Note Given $x \in X$ and $e \in E_x$ we know that the section over $\{x\}$ with value e may be extended to a global section of E: thus the full evaluation amp $X \times \Gamma E \to \Gamma E$ certainly is surjective — but as the same argument shows ΓE is, with the few obvious exceptions, of infinite dimension.

Theorem For every vector bundle $E \to X$ there exists a vector bundle F over X such that $E \oplus F$ is a trivial bundle.

5 Projective and Flag Bundles

Definition Let $E \xrightarrow{\pi} X$ be a vector bundle, and let $E' = E \backslash X$ be the complement of the zero section. The quotient space of E' with respect to the equivalence relation

$$v \sim w \quad :\Longleftrightarrow \quad \pi(v) = \pi(w) \text{ and } \lambda v = w \text{ for some } \lambda \in \mathbb{C}^*$$

is written $P(E)$, and together with the projection $P(E) \ni [v] \stackrel{\rho}{\longmapsto} \pi(v) \in X$, is called the *projective bundle* of E.

Notes (1) It is clear how to translate the defining properties of vector bundles to such projective bundles: Firstly for each $x \in X$ the fibre $P(E)_x$ of the projective bundle over x has a well-defined structure of an abstract projective space. Furthermore the projection is locally trivial: X may be covered by open sets U with homeomorphisms h that make the diagram

$$\begin{array}{ccc} U \times \mathbb{C}P^{d-1} & \xrightarrow{\quad h \quad} & P(E)|U \\ & {\scriptstyle \text{pr}} \searrow \quad \swarrow {\scriptstyle \rho|U} & \\ & U & \end{array}$$

commutative, and over each $x \in U$ restrict to a projectivity $\mathbb{C}P^d \simeq P(E)_x$.

(2) While the notion of isomorphism between projective bundles is the obvious one it does not generalise to one of arbitrary homomorphism as only injective homomorphisms of vector spaces may be projectivised.

(3) If a line bundle L acts on E by the tensor product the vector bundle $L \otimes E$ has no reason to be isomorphic to E. Nevertheless a canonical isomorphism $P(E) \simeq P(L \otimes E)$ is induced from the assignment that sends $v \in E_x$ to $u \otimes v \in L_x \otimes E_x$ for an arbitrary non-zero choice of $u \in L_x$. Indeed the class of $u \otimes v$ is well-defined, and continuity follows from the fact that locally u may be realised as the value of a continuous section of L.

(4) The pairs $([x], v)$ with $v \in \mathbb{C}x$ form a one-dimensional subbundle of the vector bundle $\rho^* E$, which of course is called the *tautological bundle* $T \to P(E)$.

Definition The dual bundle $H := T^{\check{}} \to P(E)$ is called the *hyperplane bundle*.

Notes (5) While of course T and H mutually determine each other we will often prefer H over T in order to stay compatible with algebraic geometry, where there are strong reasons to consider H rather than T as the basic object. A good way to memorise the difference is this observation in the simplest case $E = \{*\} \times V \to \{*\} = X$: every linear function $f\colon V \to \mathbb{C}$ yields a section of H assigning to $[x] \in P(V)$ the linear form $T_{[x]} = \mathbb{C}x \ni v \mapsto f(v) \in \mathbb{C}$. This section is not just continuous but even holomorphic — by contrast T has no non-trivial holomorphic section.

The somewhat misleading term of hyperplane bundle for what after all is a line bundle also comes form algebraic geometry, it is due to the fact that the zero sets of the holomorphic sections are just the hyperplanes in $P(V)$. Algebraic geometers' preferred notation for H is $\mathcal{O}(1)$.

(6) Let us repeat our construction with the quotient bundle $\rho^* E/T \to P(E)$, which everywhere has rank one less than $\rho^* E$. A point of $P(\rho^* E/T) \to P(E) \to X$ over $x \in X$ specifies first a line L in E_x

and then another line in E_x/L or, equivalently, a plane in E_x that contains L: this is the beginning of a *flag* in E_x.

Definition Let Let $E \xrightarrow{\pi} X$ be a vector bundle of constant rank d. We inductively define vector bundles $E_j \xrightarrow{\pi_j} X_j$ and projective bundles $X_j \xrightarrow{\rho_j} X_{j-1}$

$$\begin{array}{ccccccccc} E_d & & E_{d-1} & & \cdots & & E_1 & & E_0 \\ \downarrow{\scriptstyle \pi_d} & & \downarrow{\scriptstyle \pi_{d-1}} & & & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} \\ X_d & \xrightarrow{\rho_d} & X_{d-1} & \xrightarrow{\rho_{d-1}} & \cdots & \xrightarrow{\rho_2} & X_1 & \xrightarrow{\rho_1} & X_0 \end{array}$$

starting with $E \to X$ as $E_0 \xrightarrow{\pi_0} X_0$, and putting

$$X_j = P(E_{j-1}) \xrightarrow{\rho_j} X_{j-1} \quad \text{and} \quad E_j = (\rho_j^* E_{j-1})/T_j \quad \text{for } j \geq 1$$

where $T_j \xrightarrow{\tau_j} X_j$ is the tautological bundle of E_{j-1}. Note that the rank of E_j is $d-j$, so that in particular $X_d = P(E_{d-1}) \xrightarrow{\rho_d} X_{d-1}$ is a homeomorphism and E_d the zero bundle — we nevertheless retain these objects in order to have a systematic notation.

The composition $\rho_1 \circ \cdots \circ \rho_d: X_d \to X_0$ is called the *flag bundle* $F(E) \xrightarrow{\rho} X$ of E since points of its fibre over $x \in X$ correspond to flags

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{d-1} \subset V_d = E_x \quad \text{with } \dim V_j = j$$

of vector subspaces of E_x. In case $E = \{*\} \times V \to \{*\}$ is a bundle over a point $F(E)$ is written $F(V)$ and called the *flag space* of the vector space V, and more precisely *flag manifold* or *flag variety* when additional structures are taken into account that have no relevance here.

Note (7) Pulling back all the tautological bundles $T_j \subset E_j$ to $F(E)$ we obtain line bundles

$$L_j = (\rho_{j+1} \circ \cdots \circ \rho_d)^* T_j \longrightarrow F(E) \quad \text{for } j = 1, \ldots, d.$$

Using now that the base X is compact we may realise each E_j, which by definition is a quotient bundle of $\rho_j^* E_{j-1}$, by a subbundle complementary to T_j. This makes T_{j+1} a subbundle of the pull-back of E to X_{j+1}, and a fortiori L_j a subbundle of $\rho^* E$. It is clear form the construction that we thus obtain a direct sum decomposition

$$L_1 \oplus \cdots \oplus L_d = \rho^* E.$$

Example Let $L \to X$ be a line bundle, and let $\mathbf{1} \to X$ denote the product line bundle $X \times \mathbb{C} \to X$. The fibre over $x \in X$ of the projective bundle $P(\mathbf{1} \oplus L) \xrightarrow{\rho} X$ is the projective line $P(\mathbb{C} \oplus L_x)$; in geometric terms this is L_x — a copy of the complex plane — compactified by a point at infinity:

$$P(\mathbb{C} \oplus L_x) = \big\{[w:z] \,\big|\, 0 \neq w \in \mathbb{C} \text{ and } z \in L_x\big\} \cup \{[0:1]\} = L_x \cup \{\infty\} \approx S^2.$$

Globally the special points $0 \in L_x$ and ∞ correspond to distinguished sections of the projective bundle $s_0: X \to P(\mathbf{1} \oplus L)$ and $s_\infty: X \to P(\mathbf{1} \oplus L)$ defined by

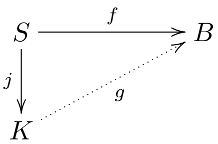
$$x \xmapsto{s_0} [1:0] \quad \text{and} \quad x \xmapsto{s_\infty} [0:v] \text{ with any non-zero } v \in L_x;$$

they are continuous by local triviality. We find it convenient to call them the zero section and the section at infinity. Note that the restrictions of the tautological bundle $T \to P(\mathbf{1} \oplus L)$ to these sections are

$$s_0^* T = \mathbf{1} \quad \text{and} \quad s_\infty^* T = L.$$

6 K-Theory

Construction Let S be an abelian semi-group. A pair consisting of an abelian group K and a homomorphism of semi-groups $j: S \to K$ is called a *Grothendieck group* of S if, given any semi-group homomorphism $f: S \to B$ into another abelian group B there exists a unique homomorphism g that lets the diagram



commute. The standard reasoning yields the uniqueness of (K, j), and the Grothendieck group of S is usually written $K(S)$.

A simple construction of $K(S)$ is by taking the cartesian product $S \times S$, which likewise is a semi-group, and forming cosets with respect to the diagonal $\Delta S \subset S \times S$: thus two pairs (x, y) and (x', y') are identified if

$$(x+z,\ y+z) = (x'+z',\ y'+z') \quad \text{holds for some } z, z' \in S.$$

The quotient set $K(S)$ inherits a semi-group structure, and indeed is a group since $[x, y]$ has the additive inverse $[y, x]$. The assignment $x \mapsto [x, 0]$ defines j, so that in particular $[x, y] = j(x) - j(y)$.

If S happens to be a semi-ring then the multiplication extends to $K(S)$ via

$$[x, y] \cdot [x', y'] = \big[xx'+yy',\ xy'+yx'\big]$$

and makes it a true ring.

In any case K is a covariant functor: every homomorphism of semi-groups or -rings $S \to T$ induces a homomorphism of groups or rings $K(S) \to K(T)$.

Examples (1) For the semi-group $\mathbb{N}$ this reproduces the well-known construction of the ring of integers from the semi-ring of natural numbers. In this case $j: \mathbb{N} \to \mathbb{Z}$ is injective, since the additive cancellation law holds in $\mathbb{N}$: the assumption $x+z = y+z$ implies $x = y$. The defining property of $\mathbb{Z} = K(\mathbb{N})$ thus simply requires that every semi-group homomorphism form $\mathbb{N}$ into a group extends as a homomorphism to $\mathbb{Z}$. This more suggestive language is also used in other cases even when j fails to be injective.

(2) The construction applies to our favourite semi-ring $\operatorname{Vect} X$ of isomorphism classes of complex vector bundles over X, and leads to the ring

$$KX := K(\operatorname{Vect} X)$$

which for lack of a more imaginative name is referred to as *K of* the topological space X. If $E \to X$ is a vector bundle we use $[E]$ to denote not only the isomorphism class of E but also its image $j([E]) \in KX$. The class of the rank d product bundle $\mathbf{d} \to X$ is simply written $d \in KX$.

By definition every element of KX is a difference $[E]-[F]$ of classes of vector bundles E and F on X: such a difference is often referred to as a *virtual bundle*. Using compactness of X we know that there exists a bundle G such that $F \oplus G$ is trivial, say $F \oplus G \simeq \mathbf{d}$. Thus the equation

$$[E]-[F] = \big([E]+[G]\big) - \big([F]+[G]\big) = \big([E]+[G]\big) - [\mathbf{d}] = \big([E]+[G]\big) - [\mathbf{d}]$$

shows that in fact every virtual bundle can be written as the difference of the class of a vector bundle and a trivial class.

The homomorphism $j\colon \operatorname{Vect} X \to KX$ need not be injective: it may happen that two non-isomorphic vector bundles E and F produce isomorphic Whitney sums $E \oplus G \simeq F \oplus G$ for some bundle G, so that $[E] = [F] \in KX$. Such bundles are called *stably isomorphic*. As we know from Section 4 we can always find yet another vector bundle H such that $G \oplus H$ is trivial, and the isomorphy above then implies $E \oplus G \oplus H \simeq F \oplus G \oplus H$. Thus E and F are stably isomorphic if and only if

$$E \oplus \mathbf{d} \simeq F \oplus \mathbf{d} \quad \text{for some, and thus every sufficiently large } d \in \mathbb{N}.$$

Note finally that K is a contravariant functor from compact topological spaces to rings: a map $f\colon X \to Y$ induces a ring homomorphism $Kf\colon KY \to KX$, a notation often simplified to f^* — and even $y \mapsto y|X$ in the particular case of an inclusion mapping $f\colon X \to Y$. Like every ring homomorphism f^* defines a scalar multiplication

$$KY \times KX \ni (y, x) \longmapsto f^*y \cdot x \in KX$$

that makes KX a module over KY and, together with the latter's ring structure, even a KY-algebra.

Periodicity Theorem Let $L \to X$ be a line bundle X over the compact space X, and let $P(\mathbf{1} \oplus L) \xrightarrow{\rho} X$ be the projective bundle. Then writing $l = [L] \in KX$ we have the identity of algebras over KX

$$K\big(P(\mathbf{1} \oplus L)\big) = KX\,[h] \big/ \big((h-1)(l\,h-1)\big)$$

in the sense that the algebra on the right is the quotient of the polynomial algebra in one determinate H by the ideal generated by the polynomial $(H-1)(l\,H-1) = 1 - H - l\,H + l\,H^2$, and that the left hand side is obtained by substituting h for H.

Note Being represented by a line bundle, $l \in KX$ always is a unit, so that the ideal is likewise generated by the unitary polynomial $(H-1)(H-l^{-1}) = H^2 - (1+l^{-1})\cdot H + l^{-1} \in KX[H]$. Thus

$$K\big(P(\mathbf{1} \oplus L)\big) = KX \oplus KX \cdot h$$

is the direct sum of two copies of KX as an additive group, with the multiplication determined by the rule $h^2 = -l^{-1} + (1+l^{-1})\cdot h$.

Example The choice $X = \{*\}$ yields $KS^2 = \mathbb{Z}[h]/(h-1)^2$. In particular the identity $h^2+1 = 2h$ shows that the bundles $H \otimes H \oplus \mathbf{1}$ and $H \oplus H$ on $\mathbb{C}P^1 = S^2$ are stably isomorphic.

The proof of the Periodicity Theorem requires careful preparation. We fix X and L as required, and abbreviate the projective bundle writing

$$P := P(\mathbf{1} \oplus L) \xrightarrow{\rho} X$$

as well as $X_0 \subset P$ and $X_\infty \subset P$ for the sections of ρ at zero and infinity. As a technical tool we now choose a metric on L, and use it to define

$$P_0 := \big\{[w : z] \,\big|\, |w| \geq |z|\big\} \xrightarrow{\sigma_0} X \quad \text{and} \quad P_\infty := \big\{[w : z] \,\big|\, |w| \leq |z|\big\} \xrightarrow{\sigma_\infty} X\,;$$

the indicated projections to X make them what one would call two-disk bundles over X while their intersection

$$S := P_0 \cap P_\infty \xrightarrow{\sigma} X$$

rather qualifies for the name of circle bundle. We thus have a decomposition

$$P = P_0 \cup P_\infty \quad \text{with} \quad P_0 \cap P_\infty = S$$

and record two facts, both to be read up to bundle isomorphism:

- Since σ_0 is a homotopy equivalence every vector bundle on P_0 has the form σ_0^*E for a unique bundle $E \to X$, and similarly for bundles over P_∞.
- Therefore every bundle E on P is of the form

$$E = \sigma_\infty^* E_\infty \cup_f \sigma_0^* E_0,$$

and if we normalise by the requirements $E_0 = E|X_0$ and $E_\infty|X_\infty$ then the homotopy class of the gluing function

$$f\colon S \to \mathrm{Hom}^*(\sigma^* E_\infty, \sigma^* E_0)$$

is uniquely determined by E. Indeed automorphisms of the bundle $\sigma_0^* E_0 \to P_0$ correspond to sections $g \in \Gamma\sigma_0^* \mathrm{Hom}(E_0, E_0)$, and if such a section g, say

$$g([1:z]) = \big([1:z], g'([1:z])\big)$$

restricts to the identity over X_0 then it is itself homotopic to the identity via the homotopy

$$I \times P_0 \ni (t, [1:z]) \longmapsto \big([1:z], g'([1:tz])\big) \in \sigma_0^* \mathrm{Hom}(E_0, E_0).$$

If conversely bundles $E_0 \to X$ and $E_\infty \to X$, and a gluing function f are given, we will write the resulting vector bundle $\sigma_\infty^* E_\infty \cup_f \sigma_0^* E_0$ simply as $\mathcal{V}(E_\infty, f, E_0)$.

Examples (1) If $L = \mathbf{1}$ is the product bundle we have $P = \mathbb{C}P^1 \times X$, and given bundles E_0 and E_∞ over X any choice of a section $a\colon X \to \mathrm{Hom}^*(E_\infty, E_0)$ — which of course forces E_∞ and E_0 to be isomorphic — gives a gluing function

$$S = S^1 \times X \ni (z, x) \overset{f}{\longmapsto} a(x) \in \mathrm{Hom}^*(\sigma^* E_\infty, \sigma^* E_0)_{(z,x)}$$

that does not depend on the coordinate $z = [1:z] \in S^1 \subset \mathbb{C}P^1$. The assignments

$$\begin{aligned} \sigma_\infty^* E_\infty &\ni \big([w:1], x, v\big) \longmapsto \big([w:1],\ x,\ a(x)\cdot v\big) \in \rho^* E_0 \\ \sigma_0^* E_0 &\ni \big([1:z], x, v\big) \longmapsto \big([1:z],\ x,\ \quad v \quad\big) \in \rho^* E_0 \end{aligned}$$

then define an isomorphism $\mathcal{V}(E_\infty, f, E_0) \simeq \rho^* E_\infty$.

(2) Let us modify the gluing function of the previous example to

$$S = S^1 \times X \ni (z, x) \overset{f}{\longmapsto} z^{-1}\, a(x) \in \mathrm{Hom}^*(\sigma^* E_\infty, \sigma^* E_0)_{(z,x)}$$

where $z \in S^1$ simply acts by scalar multiplication. Recalling Example (2) of Section 4 we see that this introduces a tensor factor $T \to P$; indeed the formulae

$$\begin{aligned} \sigma_\infty^* E_\infty &\ni \big([w:1], x, v\big) \longmapsto \left([w:1],\ x,\ \begin{pmatrix} w \\ 1 \end{pmatrix} \otimes a(x)\, v\right) \in T \otimes \rho^* E_0 \\ \sigma_0^* E_0 &\ni \big([1:z], x, v\big) \longmapsto \left([1:z],\ x,\ \begin{pmatrix} 1 \\ z \end{pmatrix} \otimes \quad v \quad\right) \in T \otimes \rho^* E_0 \end{aligned}$$

now combine and give an isomorphism $\mathcal{V}(E_\infty, f, E_0) \simeq T \otimes \rho^* E_\infty$.

While the first example at once carries over to the case of a general line bundle $L \to X$, in the second the fact that the coordinate z is not a well-defined function on S seems to pose a problem — but only at first, for z is perfectly well-defined as a section

$$S \ni [1:z] \longmapsto \big([1:z], z\big) \in \sigma^* L$$

of the pull-back bundle $\sigma^*L \to S$. As we do want the gluing function f to remain a section of the bundle $\operatorname{Hom}^*(\sigma^*E_\infty, \sigma^*E_0)$ we are now forced to take for a a section $X \to \operatorname{Hom}^*(E_\infty, L \otimes E_0)$ so that the formula

$$S \ni [1:z] \overset{f}{\longmapsto} z^{-1}\, a\big(\sigma([1:z])\big) \in \operatorname{Hom}^*\big(\sigma^*E_\infty, \sigma^*L^{-1} \otimes \sigma^*(L \otimes E_0)\big) = \operatorname{Hom}^*(\sigma^*E_\infty, \sigma^*E_0)$$

makes sense.

Example (3) We let $L \to X$ be an arbitrary line bundle but choose $E_0 = \mathbf{1}$ trivial and $E_\infty = L$, which makes

$$z^{-1} \in \Gamma(\sigma^*L^{-1}) = \operatorname{Hom}(\mathbf{1}, \sigma^*L^{-1}) = \operatorname{Hom}(\sigma^*L, \mathbf{1}) = \operatorname{Hom}(\sigma^*E_\infty, \sigma^*E_0)$$

a suitable gluing function. The calculation of Example (2) shows that the resulting line bundle $\mathcal{V}(L, z^{-1}, \mathbf{1})$ is the tautological bundle over P, and since the multiplication of gluing functions corresponds to the tensor product of bundles we may record this fact as

$$\mathcal{V}(L^{-1}, z, \mathbf{1}) \simeq H.$$

More generally, gluing functions may involve arbitrary powers of z: given an exponent $k \in \mathbb{Z}$ and any section $a_k \in \Gamma \operatorname{Hom}(L^k \otimes E_\infty, E_0)$ the formula

$$S \ni [1:z] \overset{f}{\longmapsto} z^k\, a_k\big(\sigma([1:z])\big) \in \operatorname{Hom}(\sigma^*(L^k \otimes E_\infty), \sigma^*L^k \otimes \sigma^*E_0) = \operatorname{Hom}(\sigma^*E_\infty, \sigma^*E_0)$$

defines a section which we write $a_k\, z^k \in \Gamma \operatorname{Hom}(\sigma^*E_\infty, \sigma^*E_0)$. If a_k was a gluing function then so is $a_k\, z^k$, and the resulting vector bundles on P are related by

$$\mathcal{V}(E_\infty, z^k a_k, E_0) \simeq H^k \otimes \mathcal{V}(L^k \otimes E_\infty, a_k, E_0)$$

Reviewing the statement of the Periodicity Theorem in the light of this construction one challenge emerges: Given an arbitrary vector bundle on P, and thereby an arbitrary gluing function $f\colon S \to \operatorname{Hom}^*(\sigma^*E_\infty, \sigma^*E_0)$ we must somehow reduce f to gluing functions which are as simple as those of the two examples. As a first step to this we would like to represent f by a Laurent series $\sum_{k\in\mathbb{Z}} a_k\, z^k$ — a wish that finds ready fulfilment in classical Fourier analysis.

7 Fourier Series

Let $C^0(S^1)$ denote the space of complex-valued continuous functions on the circle. This is a unitary space under the hermitian form
$$\langle f,g\rangle = \frac{1}{2\pi i}\oint_{S^1}\overline{f(\zeta)}\,g(\zeta)\,\frac{d\zeta}{\zeta},$$
and it is seen at a glimpse that the functions z^k for $k\in\mathbb{Z}$ form an orthonormal system in $C^0(S^1)$:
$$\langle z^j,z^k\rangle = \frac{1}{2\pi i}\oint\zeta^{-j}\,\zeta^k\,\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\oint\zeta^{k-j-1}\,d\zeta = \begin{cases}1 & \text{if } j=k, \text{ and}\\ 0 & \text{else.}\end{cases}$$

Definition An infinite series $\displaystyle\sum_{k=-\infty}^{\infty} a_k\,z^k$ with coefficients $a_k\in\mathbb{C}$ is called a *Fourier series*.

Note If you associate Fourier series rather with sines and cosines you may want to substitute $z=e^{it}$ with a periodic real variable t, and even split $z^k = e^{ikt} = \cos kt + i\sin kt$ into real and imaginary parts. But be advised to do so only in direst emergency unless you are fond of tiresome calculations.

Can every $f\in C^0(S^1)$ be written as a convergent Fourier series? A good candidate of such a series is obtained by computing the projections of f with respect to the orthonormal system of the z^k, known as the Fourier coefficients of f
$$\hat{f}_k = \langle z^k,f\rangle = \frac{1}{2\pi i}\oint\zeta^{-k}\,f(\zeta)\,\frac{d\zeta}{\zeta}.$$
While the corresponding Fourier series $\displaystyle\sum_{k=-\infty}^{\infty}\hat{f}_k\,z^k$ is, by definition, the best approximation to f in the sense of the hermitian norm, in general it fails to converge even point-wise to f. But it does not fail too badly:

Fejér's Theorem For every $f\in C^0(S^1)$ the sequence $(c_n)_{n=0}^{\infty}$ of *Cesàro means* defined as
$$c_n(z) = \frac{1}{n+1}\sum_{k=0}^{n}\Big(\sum_{j=-k}^{k}\hat{f}_j\,z^j\Big)$$
converges uniformly to f.

Proof We will describe the operator that takes the function f to the n-th term of the Fejér sequence in terms of the *Fejér kernel*
$$F_n(z) = \frac{1}{n+1}\sum_{k=0}^{n}\sum_{j=-k}^{k} z^j,$$
which itself is a function in $C^0(S^1)$. We state and prove three properties:

- $\displaystyle\frac{1}{2\pi i}\oint F_n(\zeta)\,\frac{d\zeta}{\zeta} = 1$ for all $n\in\mathbb{N}$. Indeed we calculate that

$$\oint F_n(\zeta)\,\frac{d\zeta}{\zeta} = \frac{1}{n+1}\oint\sum_{k=0}^{n}\sum_{j=-k}^{k}\zeta^j\,\frac{d\zeta}{\zeta} = \frac{1}{n+1}\sum_{k=0}^{n} 2\pi i = 2\pi i.$$

- $F_n(e^{it}) = \dfrac{1}{n+1}\left(\dfrac{\sin\frac{n+1}{2}t}{\sin\frac{1}{2}t}\right)^2$ for all $t\in\mathbb{R}$; in particular $F_n \geq 0$. This but requires to evaluate the geometric series:

$$\begin{aligned}
F_n(z) &= \frac{1}{n+1}\sum_{k=0}^{n}\sum_{j=-k}^{k} z^j \\
&= \frac{1}{n+1}\sum_{k=0}^{n} z^{-k}\,\frac{z^{2k+1}-1}{z-1} \\
&= \frac{1}{n+1}\,\frac{1}{z-1}\sum_{k=0}^{n}\left(z^{k+1}-z^{-k}\right) \\
&= \frac{1}{n+1}\,\frac{1}{z-1}\left(z-z^{-n}\right)\frac{z^{n+1}-1}{z-1} \\
&= \frac{1}{n+1}\,z^{-n}\left(\frac{z^{n+1}-1}{z-1}\right)^2 \\
&= \frac{1}{n+1}\left(\frac{\sqrt{z}^{\,n+1}-\sqrt{z}^{\,-(n+1)}}{\sqrt{z}-\sqrt{z}^{\,-1}}\right)^2
\end{aligned}$$

- For every $\delta>0$ one has $\lim\limits_{n\to\infty}\displaystyle\int F_n(\zeta)\,\frac{d\zeta}{\zeta}=0$ where integration is restricted to all $\zeta=e^{it}$ such that $t\in[\delta,2\pi-\delta]$. There indeed we have the estimate

$$F_n(e^{it}) \leq \frac{1}{n+1}\,\frac{1}{\sin\frac{1}{2}t} \leq \frac{1}{n+1}\,\frac{1}{\sin\frac{1}{2}\delta}$$

implying that F_n converges to zero uniformly on the domain of integration.

Turning to the proof proper of Fejér's Theorem, consider a given $f\in C^0(S^1)$. We compute

$$\begin{aligned}
\sum_{j=-k}^{k}\hat{f}_j\,z^j &= \frac{1}{2\pi i}\sum_{j=-k}^{k}\oint \zeta^{-j}\,f(\zeta)\,\frac{d\zeta}{\zeta}\,z^j \\
&= \frac{1}{2\pi i}\sum_{j=-k}^{k}\oint (z/\zeta)^j\,f(\zeta)\,\frac{d\zeta}{\zeta} \\
&= \frac{1}{2\pi i}\sum_{j=-k}^{k}\oint \eta^j\,f(z/\eta)\,\frac{d\eta}{\eta}\,,
\end{aligned}$$

and substituting the definition of the Fejér kernel obtain

$$c_n(z) = \frac{1}{n+1}\sum_{k=0}^{n}\sum_{j=-k}^{k}\hat{f}_j\,z^j = \frac{1}{2\pi i}\oint F_n(\eta)\,f(z/\eta)\,\frac{d\eta}{\eta}\,.$$

Using the stated properties we estimate

$$\begin{aligned}
|c_n(z)-f(z)| &= \left|\frac{1}{2\pi i}\oint F_n(\eta)\,f(z/\eta)\,\frac{d\eta}{\eta} - \frac{1}{2\pi i}\oint F_n(\eta)\,f(z)\,\frac{d\eta}{\eta}\right| \\
&\leq \frac{1}{2\pi i}\oint F_n(\eta)\,\big|f(z/\eta)-f(z)\big|\,\frac{d\eta}{\eta}\,.
\end{aligned}$$

Given any $\varepsilon>0$ we now choose $\delta>0$ so that

$$\big|f(e^{is})-f(e^{it})\big|<\varepsilon \quad \text{for all } s,t\in[0,2\pi] \text{ with } |s-t|\leq\delta;$$

this is possible since f is uniformly continuous. We correspondingly split the last integral; again using properties of the Fejér kernels we conclude

$$\int_{-\delta}^{\delta} F_n(e^{it})\,\big|f(z/e^{it})-f(z)\big|\,dt \le \int_{-\delta}^{\delta} F_n(e^{it})\,\varepsilon\,dt \le \varepsilon\int_0^{2\pi} F_n(e^{it})\,dt = \varepsilon$$

and

$$\int_{\delta}^{2\pi-\delta} F_n(e^{it})\,\big|f(z/e^{it})-f(z)\big|\,dt \le \max_{t\in[0,2\pi]}\big|f(z/e^{it})-f(z)\big|\cdot\int_{\delta}^{2\pi-\delta} F_n(e^{it})\,dt < \varepsilon$$

for all sufficiently large $n\in\mathbb{N}$. This completes the proof of Fejér's Theorem.

We must generalise Fejér's Theorem to a setting sufficiently general in order to accommodate gluing functions $f\colon S\to \operatorname{Hom}^*(\sigma^*E_\infty,\sigma^*E_0)$. The nature of the bundle $\operatorname{Hom}(E_\infty,E_0)$ being quite irrelevant here we replace it by a general vector bundle $F\to X$, for which we choose a metric.

Fejér's Theorem for Bundles Let X be a compact space, $L\to X$ a line bundle with metric, giving rise to the sphere bundle $S\xrightarrow{\sigma} X$. Let further $F\to X$ be a vector bundle with metric. Then every $f\in\Gamma(\sigma^*F)$ has well-defined Fourier coefficients

$$\hat f_k = \langle z^k, f\rangle \in \Gamma(L^{-k}\otimes F),$$

and the sequence in $\Gamma(\sigma^*F)$ of Cesàro means $(c_n)_{n=0}^\infty$, defined as before by

$$c_n(z) = \frac{1}{n+1}\sum_{k=0}^{n}\Big(\sum_{j=-k}^{k}\hat f_j\, z^j\Big)$$

converges uniformly to f.

Adaptions of the Proof The Fourier coefficients are well-defined since the measure of integration $d\zeta/\zeta$ does not change under multiplication by a constant. Note that the metric on F gives a meaning to uniform convergence of sections, a meaning which in fact does not depend on the particular choice of that metric.

The main difficulty that remains is that the topology of X need not be induced by a metric and that therefore there is no notion of uniform continuity of f. On the other hand compactness of X makes the statement local in X. We may thus assume that the bundles $L=X\times\mathbb{C}\to X$ and $F=X\times\mathbb{C}^d\to X$ are product bundles and that the metric of F is the standard one.

The statement is thereby reduced to one for a scalar function $f\colon S^1\times X\to\mathbb{C}$ and thus to the classical case but for the presence of the parameter space X. By inspection we see that the proof given above works fine if we interpret uniform continuity as a version involving a parameter, as in the conclusion of the

Proposition Let X be compact and $f\in C^0(S^1\times X)$ be given. Then for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$|f(w,x)-f(z,x)|<\varepsilon \quad \text{holds for all } w,z\in S^1 \text{ with } |w-z|<\delta \text{ and all } x\in X.$$

Proof Given $\varepsilon>0$ we choose for each $(c,a)\in S^1\times X$ a real number $\delta_{ca}>0$ and an open neighbourhood $V_{ca}\subset X$ such that

$$|f(z,x)-f(c,a)|<\varepsilon \quad \text{for all } z\in S^1 \text{ with } |z-c|<2\delta_{ca} \text{ and all } x\in V_{ca}.$$

Using compactness we pick a finite set $\Lambda\subset S^1\times X$ such that the sets $U_{\delta_{ca}}(c,a)\times V_{ca}$ with $(c,a)\in\Lambda$ cover $S^1\times X$. We put $\delta=\min\{\delta_{ca}\,|\,(c,a)\in\Lambda\}$.

Consider now any points $w, z \in S^1$ with $|w-z| < \delta$ and any $x \in X$. We choose $(c,a) \in \Lambda$ such that $(z,x) \in U_{\delta_{ca}}(c,a)\times V_{ca}$; since $|w-z| < \delta$ and $|z-c| < \delta_{ca}$ we have $|w-c| < 2\delta_{ca}$ and therefore both

$$|f(w,x)-f(c,a)| < \varepsilon \quad \text{and} \quad |f(z,x)-f(c,a)| < \varepsilon.$$

We conclude $|f(w,x)-f(z,x)| < 2\varepsilon$, and thus complete the proof.

Lemma 1 (in the proof of the Periodicity Theorem) Let E and F be vector bundles over X, and let $f\colon S \to \mathrm{Hom}^*(\sigma^*E, \sigma^*F)$ be a gluing function. For every $n \in \mathbb{N}$ we consider the n-th Cesàro mean of the Fourier series of f

$$c_n(z) = \sum_{k=-n}^{n} a_k\, z^k\colon S \to \mathrm{Hom}^*(\sigma^*E, \sigma^*F).$$

Then the sequence of Laurent polynomials $(c_n)_{n=0}^{\infty}$ has the property that for all sufficently large $n \in \mathbb{N}$ the linear homotopies

$$t \longmapsto (1-t)\cdot c_n + t\cdot c_{n+1} \quad \text{and} \quad t \longmapsto (1-t)\cdot c_n + t\cdot f$$

are homotopies of gluing functions.

Proof Choose a metric on the bundle $\mathrm{Hom}(E,F)$. The function $d\colon S \to [0,\infty)$ which assigns to $s \in S$ the distance from $f(s) \in \mathrm{Hom}(\sigma^*E, \sigma^*F)_s$ to the closed subset

$$\mathrm{Hom}(\sigma^*E, \sigma^*F)_s \backslash \mathrm{Hom}^*(\sigma^*E, \sigma^*F)_s \subset \mathrm{Hom}(\sigma^*E, \sigma^*F)_s$$

is everywhere positive and continuous; we let $\varepsilon > 0$ be its smallest value.

By Fejér's Theorem for all sufficiently large $n \in \mathbb{N}$ we have $|c_n(s)-f(s)| < \varepsilon$ for all $s \in S$. This implies not only that c_n maps into $\mathrm{Hom}^*(\sigma^*E, \sigma^*F)$ but also that the stated homotopies do so at any time $t \in I$.

Corollary Every vector bundle on P is isomorphic to a bundle $\mathcal{V}(E,p,F)$ with a gluing function

$$c(z) = \sum_{k=-n}^{n} a_k\, z^k\colon S \to \mathrm{Hom}^*(\sigma^*E, \sigma^*F)$$

which is a Laurent polynomial.

8 Polynomial and Linear Gluing Functions

The main result of the previous section suggests to study Laurent rather than general gluing functions. At the end of Section 6 we have already noted the isomorphy

$$\mathcal{V}(E, z^k f, F) \simeq H^k \otimes \mathcal{V}(L^k \otimes E, f, F)$$

for gluing functions $f\colon S \to \operatorname{Hom}^*\big(\sigma^*(L^k \otimes E), \sigma^* F\big)$: we thus know the effect of multiplication by a power of z and may restrict our attention even further to gluing functions that are polynomial in z.

Notation Let E and F be vector bundles over X and let for some $n \in \mathbb{N}$

$$p(z) = \sum_{k=0}^{n} a_k\, z^k \colon S \to \operatorname{Hom}^*(\sigma^* E, \sigma^* F)$$

be a polynomial gluing function of degree at most n. We introduce the auxiliary bundle

$$E_n = \bigoplus_{k=1}^{n} L^k \otimes E$$

and define the section $\operatorname{lin}_n p \in \Gamma \operatorname{Hom}\big((\sigma^*(E \oplus E_n), \sigma^*(F \oplus E_n)\big)$ by the matrix

$$\operatorname{lin}_n p(z) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ -z & 1 & & & \\ & -z & 1 & & \\ & & \ddots & \ddots & \\ & & & -z & 1 \end{pmatrix};$$

as the notation suggests this is a *linearisation* of p in the sense that it has degree at most one in z. It is compatible with forming direct sums, as well as tensor products with a further bundle $D \to X$:

$$\operatorname{lin}_n(p \oplus p') = \operatorname{lin}_n p \oplus \operatorname{lin}_n p' \quad \text{and} \quad \operatorname{lin}_n(\operatorname{id}_{\sigma^* D} \otimes p) = \operatorname{id}_{\sigma^* D} \otimes \operatorname{lin}_n p\,.$$

Lemma 2 The linearisation $\operatorname{lin}_n p$ is a gluing function, and there is an isomorphism

$$\mathcal{V}(E, p, F) \oplus \rho^* E_n \simeq \mathcal{V}(E \oplus E_n, \operatorname{lin}_n p, F \oplus E_n).$$

Proof We define polynomials $q_0, \dots, q_n \in \Gamma \operatorname{Hom}\big(\sigma^*(E \oplus E_n), \sigma^*(F \oplus E_n)\big)$ inductively by

$$q_0 = p \quad \text{and} \quad z \cdot q_k(z) = q_{k-1}(z) - q_{k-1}(0) \text{ for } k = 1, \dots, n$$

so that explicitly $q_k(z) = \sum_{j=0}^{n-k} a_{k+j} z^j$. Since all values of p are isomorphisms the matrix identity

$$\operatorname{lin}_n p(z) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ -z & 1 & & & \\ & -z & 1 & & \\ & & \ddots & \ddots & \\ & & & -z & 1 \end{pmatrix} = \begin{pmatrix} 1 & q_1 & q_2 & \dots & q_n \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} p & & & & \\ -z & 1 & & & \\ & -z & 1 & & \\ & & \ddots & \ddots & \\ & & & -z & 1 \end{pmatrix}$$

shows that the same is true of $\mathrm{lin}_n p$. Furthermore, applying a time factor t to all off-diagonal terms on the right hand side we define a homotopy of gluing functions that joins $\mathrm{lin}_n p$ to $p \oplus 1_{E_n}$. This yields the stated isomorphy.

Lemma 3 Let E and F be vector bundles, and $p: S \to \mathrm{Hom}^*(\sigma^* E, \sigma^* F)$ a polynomial gluing function of degree not exceeding n. Then there are homotopies

$$\mathrm{lin}_{n+1} p \simeq \mathrm{lin}_n p \oplus 1_{L^{n+1} \otimes E}$$

of gluing functions $S \to \mathrm{Hom}^*\big(\sigma^*(E \oplus E_n) \oplus \sigma^*(L^{n+1} \otimes E), \sigma^*(F \oplus E_n) \oplus \sigma^*(L^{n+1} \otimes E)\big)$, and

$$\mathrm{lin}_{n+1}\, z \cdot p(z) \simeq z \oplus \mathrm{lin}_n\, p(z)$$

of gluing functions $S \to \mathrm{Hom}^*\big((\sigma^*(L^{-1} \otimes E) \oplus \sigma^*(E \oplus E_n), \sigma^* E \oplus \sigma^*(F \oplus E_n)\big)$.

Proof By definition we have

$$\mathrm{lin}_{n+1}\, p(z) = \left(\begin{array}{ccccc|c} & & \mathrm{lin}_n\, p(z) & & & 0 \\ \hline 0 & \dots & \dots & 0 & -z & 1 \end{array}\right)$$

and obtain the first homotopy by just applying a time factor to the last entry $-z$. In the same way we make disappear the 1 from the first line of

$$\mathrm{lin}_{n+1}\, z \cdot p(z) = \begin{pmatrix} -z & 1 & & & \\ 0 & a_0 & a_1 & \dots & a_n \\ & -z & 1 & & \\ & & \ddots & \ddots & \\ & & & -z & 1 \end{pmatrix}$$

while a rotation by $180°$ turns the remaining term $-z$ into $+z$ within the homotopy class.

Corollary Under the assumptions of Lemma 3 there are bundle isomorphisms

$$\mathcal{V}(E \oplus E_{n+1}, \mathrm{lin}_{n+1} p, F \oplus E_{n+1}) \simeq \mathcal{V}(E \oplus E_n, \mathrm{lin}_n p, F \oplus E_n) \oplus \rho^*(L^{n+1} \otimes E)$$

$$\mathcal{V}\big((L^{-1} \otimes (E \oplus E_{n+1}), \mathrm{lin}_{n+1}\, z \cdot p(z), F \oplus\, L^{-1} \otimes E_{n+1}\big) \simeq H \otimes \rho^* E \oplus \mathcal{V}(E \oplus E_n, \mathrm{lin}_n p, F \oplus E_n).$$

Examples (1) The simplest choice, that of a "constant" gluing polynomial $p: X \to \mathrm{Hom}^*(E, F)$ and of $n = 0$, reduces the last isomorphism to

$$\mathcal{V}(L^{-1} \otimes E \oplus E, \mathrm{lin}_1\, pz, F \oplus E) \simeq H \otimes \rho^* E \oplus \mathcal{V}(E, \mathrm{lin}_0 p, F)$$

or, using Lemma 2 and substituing F via the isomorphism $E \xrightarrow{p} F$,

$$\mathcal{V}(L^{-1} \otimes E, z, E\big) \oplus \rho^* E \simeq H \otimes \rho^* E \oplus \rho^* E,$$

which tells us no more than we already know.

(2) By contrast choose $z: L^{-1} \to \mathbf{1}$ for the polynomial p and put $n = 1$. The second isomorphy of the corollary reads

$$\mathcal{V}(L^{-2} \oplus L^{-1} \oplus \mathbf{1}, \mathrm{lin}_2 z^2, \mathbf{1} \oplus L^{-1} \oplus \mathbf{1}) \simeq H \otimes \rho^* L^{-1} \oplus \mathcal{V}(L^{-1} \oplus \mathbf{1}, \mathrm{lin}_1 z, \mathbf{1} \oplus \mathbf{1}),$$

and via Lemma 2 we obtain

$$\mathcal{V}(L^{-2}, z^2, \mathbf{1}) \oplus \rho^* L^{-1} \oplus \mathbf{1} \simeq H \otimes \rho^* L^{-1} \oplus \mathcal{V}(L^{-1}, z, \mathbf{1}) \oplus \mathbf{1}$$

and thus in terms of the structure of KP as a KX-module, the important identity $h^2+l^{-1} = l^{-1}\,h+h$ or

$$(l\,h-1)(h-1) = 0 \in KP.$$

We now analyse linear gluing functions $l(z) = az+b\colon S \to \mathrm{Hom}^*(\sigma^*E, \sigma^*F)$. We will make use of the fact that the formula defining l extends to make it a section $l \in \Gamma(\sigma_0^*E, \sigma_0^*F)$ over the disk bundle $P_0 \to X$, while similarly $\frac{1}{z}l(z)$ is a section in $\Gamma(\sigma_\infty^*E, \sigma_\infty^*F)$. Note that over $X_\infty \subset P_\infty$ the latter restricts to a. The values of these sections off S no longer need to be isomorphisms.

Proposition Let V be a finite-dimensional complex vector space, and assume that the endomorphism $f\colon V \to V$ has no eigenvalue on the unit circle. Then the endomorphism

$$q_f := \frac{1}{2\pi i}\oint_{S^1} \frac{d\zeta}{\zeta - f} \in \mathrm{End}\,V$$

is the projector onto the sum of those generalised eigenspaces of f which belong to eigenvalues inside the unit circle.

Proof For each $\zeta \in S^1$ the operator $\zeta - f$ is invertible, its inverse commutes with f, and so does q_f. Therefore the generalised eigenspaces of f are stable under q_f, and it suffices to prove the proposition under the additional hypothesis that f has just one eigenvalue λ.

In the case of $|\lambda| > 1$ the operator $(\zeta - f)^{-1}$ is a holomorphic function of $\zeta \in D^2 \subset \mathbb{C}$, and the integral must vanish. On the other hand for $|\lambda| < 1$ it is holomorphic outside D^2 but for a pole at infinity, and the substitution $\zeta = \eta^{-1}$ yields

$$q_f = \frac{1}{2\pi i}\oint \frac{d\zeta}{\zeta - f} = \frac{1}{2\pi i}\oint \frac{1}{\eta^{-1} - f}\,\frac{d\eta}{\eta^2} = \frac{1}{2\pi i}\oint \frac{1}{1-\eta f}\,\frac{d\eta}{\eta} = 1 \in \mathrm{End}\,V.$$

Let E and F be vector bundles, and $l\colon S \to \mathrm{Hom}^*(\sigma^*E, \sigma^*F)$ a linear gluing function. We define endomorphisms $q_l\colon E \to E$ and $r_l\colon F \to F$ by the integrals

$$q_l = \frac{1}{2\pi i}\oint l^{-1}dl = \frac{1}{2\pi i}\oint l(\zeta)^{-1}l'(\zeta)\,d\zeta \quad \text{and} \quad r_l = \frac{1}{2\pi i}\oint dl\,l^{-1} = \frac{1}{2\pi i}\oint l'(\zeta)l(\zeta)^{-1}\,d\zeta$$

taken over each fibre of the sphere bundle $S \xrightarrow{\sigma} X$. We have put ζ as an argument of l' in order to indicate the variable of differentiation though of course here $l'(\zeta) = a$ does not depend on ζ. This will no longer true once we transform to a different variable of integration as we will do in a minute.

Lemma 4 For every linear gluing function l the endomorphisms q_l and r_l are projection operators, and they satisfy

$$l \circ q_l = r_l \circ l.$$

There results a decomposition of l

$$E = E_+ \oplus E_- \xrightarrow{l_+ \oplus l_-} F_+ \oplus F_- = F$$

with $E_+ = \mathrm{image}\,q_l$ and $E_- = \mathrm{kernel}\,q_l$ complementary subbundles of $E \to X$, while $F_+ = \mathrm{image}\,r_l$ and $F_- = \mathrm{kernel}\,r_l$ are complementary subbundles of $F \to X$. Furthermore l_+ is an isomorphism over P_∞, and l_- an isomorphism over P_0.

Proof Assuming first that X is a one-point space we thoroughly normalise the situation: $L = \mathbb{C}$, and as l is an isomorphism over S^1 we may pick a real number $t > 1$ such that $l(t)\colon E \to F$ is an isomorphism too: we now use it to identify the vector spaces E and F. The Moebius transformation

$$\zeta \longmapsto \eta := \frac{1 - t\,\zeta}{\zeta - t}$$

preserves the unit disc, sends t to ∞, and ∞ to $-t$, so that in terms of the new coordinate η the gluing function — again written l by a classical abuse of language — becomes the rational function

$$l(\eta) = \frac{\eta - f}{\eta + t} \quad \text{for some endomorphism } f\colon E \to E.$$

Evaluating the integrals we obtain

$$q_l = \frac{1}{2\pi i}\oint l^{-1} dl = \frac{1}{2\pi i}\oint \left(\frac{\eta - f}{\eta + t}\right)^{-1} \frac{d}{d\eta}\frac{\eta - f}{\eta + t}\; d\eta = \frac{1}{2\pi i}\oint \frac{d\eta}{\eta - f} - \frac{1}{2\pi i}\oint \frac{d\eta}{\eta + t} = \frac{1}{2\pi i}\oint \frac{d\eta}{\eta - f}$$

since t is outside the unit circle, and the same result for r_l. Thus both q_l and r_l conincide with the projector q_f of the proposition, and all assertions can now be read off from the latter.

For general base spaces X we apply the special case to each fibre and need but add the observation that the ranks of the projectors q_l and r_l must be locally constant functions on X.

Corollary If under the assumptions of Lemma 4 we write out $l_+ = a_+\, z + b_+$ and $l_- = a_-\, z + b_-$ then the function

$$I \times S \ni (t, z) \longmapsto (a_+\, z + t\, b_+) \oplus (t\, a_-\, z + b_-) \in \operatorname{Hom}(\sigma^* E_+, \sigma^* F_+) \oplus \operatorname{Hom}(\sigma^* E_-, \sigma^* F_-)$$

defines a homotopy of linear gluing functions between l and $a_+ z \oplus b_-$. We have an isomorphism of bundles on P

$$\mathcal{V}(E, l, F) \simeq H \otimes \rho^*(L \otimes E_+) \;\oplus\; \rho^* E_- \;.$$

Proof By Lemma 4 we know that for every $t \in I$ and every $z \in S^1$ all values

$$a_+\, z + t\, b_+ \in \operatorname{Hom}(\sigma^* E_+, \sigma^* F_+) \quad \text{and} \quad t\, a_-\, z + b_- \in \operatorname{Hom}(\sigma^* E_-, \sigma^* F_-)$$

are invertible, so that we have written down a homotopy of gluing functions indeed; we conclude that

$$\mathcal{V}(E, l, F) \simeq \mathcal{V}(E_+, a_+ z, F_+) \oplus \mathcal{V}(E_-, b_-, F_-)\,.$$

In particular $a_+\colon L \otimes E_+ \to F_+$ and $b_-\colon E_- \to F_-$ are isomorphisms of bundles over X, and we thus may substitute

$$\mathcal{V}(E_+, a_+\, z, F_+) \simeq \mathcal{V}(E_+, z, L \otimes E_+)$$
$$\mathcal{V}(E_-, b_-, F_-) \simeq \mathcal{V}(E_-, 1, E_-)\,.$$

Notation Given vector bundles E and F over X, and a polynomial gluing function $p\colon S \to \operatorname{Hom}^*(\sigma^* E, \sigma^* F)$ of degree at most n we let

$$\mathcal{V}_n(E, p, F) \longrightarrow X$$

denote the subbundle $(E \oplus E_n)_+ \subset E \oplus E_n \to X$ associated to the linearisation $\operatorname{lin}_n p$ by Lemma 4.

Note that $\mathcal{V}_n(E, p, F)$ is compatible with direct sums and with taking tensor products with a fixed bundle $D \to X$:

$$\mathcal{V}_n(E \oplus E', p \oplus p', F \oplus F') \simeq \mathcal{V}_n(E, p, F) \oplus \mathcal{V}_n(E', p', F')$$
$$\mathcal{V}_n(D \otimes E, \operatorname{id} \otimes p, D \otimes F) \simeq D \otimes \mathcal{V}_n(E, p, F).$$

Lemma 5 Let E and F be vector bundles over X, and a polynomial gluing function p of degree at most n. Then, written in terms of the module structure KP over KX, the identity

$$[\mathcal{V}(E, p, F)] = [E] + \big[\mathcal{V}_n(E, p, F)\big] \cdot (l\, h - 1)$$

holds, which expresses the class of $\mathcal{V}(E, p, F)$ in KP in terms of $[\mathcal{V}_n(E, p, F)] \in KX$.

Proof The corollary to Lemma 4 tells us

$$\mathcal{V}(E\oplus E_n, \mathrm{lin}_n p, F\oplus E_n) \simeq H\otimes\rho^*(L\otimes\mathcal{V}_n(E,p,F))\oplus\rho^*\big((E\oplus E_n)/\mathcal{V}_n(E,p,F)\big).$$

On the other hand we have the obvious isomorphism of bundles over X

$$E\oplus E_n \simeq \mathcal{V}_n(E,p,F)\oplus(E\oplus E_n)/\mathcal{V}_n(E,p,F)\,,$$

and taking the difference of classes in KP we obtain the identity

$$\big[\mathcal{V}(E\oplus E_n,\mathrm{lin}_n p, F\oplus E_n)\big]-\big[\rho^*(E\oplus E_n)\big]=\big[H\otimes\rho^*(L\otimes\mathcal{V}_n(E,p,F))\big]-\big[\rho^*\mathcal{V}_n(E,p,F)\big]\,.$$

By Lemma 2 the left hand side is $[\mathcal{V}(E,p,F)]-[\rho^*E]$, and we conclude

$$\big[\mathcal{V}(E,p,F)\big]=[E]+\big[\mathcal{V}_n(E,p,F)\big]\cdot(l\,h-1)\,.$$

Lemma 6 Assume that p_0 and p_1 are homotopic as polynomial gluing functions of degree at most n. Then they define isomorphic bundles

$$\mathcal{V}_n(E,p_0,F)\simeq\mathcal{V}_n(E,p_1,F).$$

For every polynomial gluing function p of degree at most n we have isomorphisms

$$\mathcal{V}_{n+1}(E,p,F)\simeq\mathcal{V}_n(E,p,F)$$
$$\mathcal{V}_{n+1}(L^{-1}\otimes E, z\cdot p(z),F)\simeq L^{-1}\otimes E\,\oplus\,\mathcal{V}_n(E,p,F).$$

Proof Applying Lemma 4 over $I\times X$ in place of X yields a vector bundle that restricts to $\mathcal{V}_n(E,p_0,F)$ over $\{0\}\times X$, and to $\mathcal{V}_n(E,p_1,F)$ over $\{1\}\times X$: this proves the first statement.

On the other hand the isomorphism $\mathcal{V}_{n+1}(E,p,F)\simeq\mathcal{V}_n(E,p,F)$ follows from the first homotopy of Lemma 3 since the term involving the constant gluing polynomial $1_{L^{n+1}\otimes E}$ makes no contribution to $(L^{n+1}\otimes E)_+$. Similarly the second homotopy of Lemma 3 implies

$$\mathcal{V}_{n+1}(L^{-1}\otimes E,z\cdot p(z),F)\simeq(L^{-1}\otimes E,z,E)_+\oplus\mathcal{V}_n(E,p,F)=L^{-1}\otimes E\,\oplus\,\mathcal{V}_n(E,p,F)\,.$$

We are now ready to assemble everything and prove the Periodicity Theorem. For this final part we let H denote an indeterminate over the ring KX. As we have seen in Example (2), substitution of $h\in KP$ for H defines a ring homomorphism

$$KX\,[H]\,/\big((H-1)(lH-1)\big)\xrightarrow{\ \varphi\ }KP\,.$$

We proceed to construct an additive homomorphism $KP\xrightarrow{\ \psi\ }KX[H]/((H-1)(lH-1))$ which will turn out to be the inverse of φ.

Let $G\to P$ be a given vector bundle and choose a gluing function f for it, so that $E\simeq\mathcal{V}(E,f,F)$. We choose $n\in\mathbb{N}$ as large as required by Lemma 1, and put $p_n=z^n c_n(z)$: this is a polynomial gluing function of degree at most $2n$ and yields the bundle

$$\mathcal{V}(L^{-n}\otimes E,p_n,F)\simeq H^n\otimes\mathcal{V}(E,c_n,F)\,.$$

We note that in view of the relation $H+lH-lH^2=1$ the congruence class of H in $KX\,[H]\,/\big((H-1)(lH-1)\big)$ is a unit, and define $\psi(f,n)\in KX\,[H]\,/\big((H-1)(lH-1)\big)$ by

$$H^n\cdot\psi(f,n)=l^{-n}\,[E]+\big[\mathcal{V}_{2n}(L^{-n}\otimes E,p,F)\big](lH-1)\,.$$

If we increase n by one, we know from Lemma 1 that p_{n+1} and $z\,p_n$ are homotopic gluing functions of degree at most $2n{+}2$, so that Lemma 6 gives an isomorphism

$$\mathcal{V}_{2n+2}(L^{-n-1}\otimes E,p_{n+1},F)\simeq\mathcal{V}_{2n+2}(L^{-n-1}\otimes E,z\,p_n,F)$$

as well as

$$\mathcal{V}_{2n+1}(L^{-n}\otimes E, p_n, F) \simeq \mathcal{V}_{2n}(L^{-n}\otimes E, p_n, F)$$
$$\mathcal{V}_{2n+2}(L^{-n-1}\otimes E, z\,p_n, F) \simeq L^{-n-1}\otimes E \,\oplus\, \mathcal{V}_{2n+1}(L^{-n}\otimes E, p_n, F)\,.$$

We conclude that

$$\begin{aligned}
H^{n+1}\cdot\psi(f,n{+}1) &= l^{-n-1}\,[E] + \big[\mathcal{V}_{2n+2}(L^{-n-1}\otimes E, p_{n+1}, F)\big](lH-1)\\
&= l^{-n-1}\,[E] + \big[\mathcal{V}_{2n+2}(L^{-n-1}\otimes E, z\,p_n, F)\big](lH-1)\\
&= l^{-n-1}\,[E] + \Big(l^{-n-1}\,[E] + \big[\mathcal{V}_{2n+1}(L^{-n}\otimes E, p_n, F)\big]\Big)(lH-1)\\
&= l^{-n-1}\,[E] + l^{-n-1}\,[E]\,(lH-1) + \big[\mathcal{V}_{2n}(L^{-n}\otimes E, p_n, F)\big](lH-1)\\
&= H\cdot l^{-n}\,[E] + H\big[\mathcal{V}_{2n}(L^{-n}\otimes E, p_n, F)\big](lH-1)\\
&= H\cdot H^n\cdot\psi(f,n)\,,
\end{aligned}$$

using the congruence $H \equiv 1 \bmod (lH{-}1)$. Therefore $\psi(f,n)$ does not depend on the choice of n, and we may write it as $\psi(f)$.

Let now g be another gluing function for the bundle G. Then f and g are homotopic, and choosing a homotopy $I\times S \to \mathrm{Hom}^*(\sigma^*E, \sigma^*F)$ and applying Lemma 1 with the base space $I\times X$ we obtain a new $n\in\mathbb{N}$ and a homotopy between the two polynomials p_n associated to f and g. Then Lemma 6 assures that $\psi(f) = \psi(g)$, so that this value in fact only depends on the isomorphism class of G, and may be written $\psi([G])$. We thus have constructed a well-defined mapping

$$\psi\colon \mathrm{Vect}\,P \longrightarrow KX\,[H]\,/\big((H-1)(lH-1)\big).$$

Since ψ clearly is a homomorphism of semi-groups it extends to a unique group homomorphism

$$\psi\colon KP \longrightarrow KX\,[H]\,/\big((H-1)(lH-1)\big).$$

We know from the construction that ψ even is a homomorphism of KX-modules, as of course is φ.

It remains to verify that ψ is inverse to φ as claimed. As the KX-module $KX\,[H]\,/\big((H{-}1)(lH{-}1)\big)$ is spanned by 1 and H^{-1} the composition $\psi\circ\varphi$ need only be evaluated on these two elements. While $\psi\circ\varphi(1) = 1$ is obvious we calculate according to the definitions

$$(\psi\circ\varphi)(H^{-1}) = \psi(h^{-1}) = \psi\big(\mathcal{V}(L, z^{-1}, \mathbf{1})\big)$$

and thus

$$H\otimes(\psi\circ\varphi)(H^{-1}) = l^{-1}[L] + \big[\mathcal{V}_2(L^{-1}\otimes L, z\cdot z^{-1}, \mathbf{1})\big] = 1 + \big[\mathcal{V}_0(\mathbf{1}, \mathrm{id}, \mathbf{1})\big] = 1\,.$$

The other composition need but be applied to a vector bundle $\mathcal{V}(E, f, F) \to P$, where we may assume that f is a Laurent polynomial. Then for $n \gg 0$ the Cesàro mean c_n coincides with f, and we have a valid identity

$$H^n\cdot\psi([\mathcal{V}(E,f,F)]) = l^{-n}\,[E] + \big[\mathcal{V}_{2n}(L^{-n}\otimes E, z^n f, F)\big](lH-1)$$

in $KX\,[H]\,/\big((H-1)(lH-1)\big)$. Using the fact that φ is a ring homomorphism Lemma 5 now implies

$$h^n\cdot(\varphi\circ\psi)([\mathcal{V}(E,f,F)]) = [L^{-n}\otimes E] + \big[\mathcal{V}_{2n}(L^{-n}\otimes E, z^n f, F)\big](lh{-}1) = [\mathcal{V}(L^{-n}\otimes E, z^n f, F)] = h^n\cdot[\mathcal{V}(E,f,F)]\,.$$

This completes the proof of the Periodicity Theorem.

9 Cohomological Properties

So far we have worked, on the topological side, with the category **Cp** of compact spaces and continuous maps. In the context of homotopy and cohomology it is often convenient to have various related categories at hand. They include the category $\mathbf{Cp}^\circ$ of *pointed spaces*, that is pairs (X, a) comprising a compact space X and a point $a \in X$ called its *base point* — and often implied by the context and then dropped from the notation. A wider framework is the category $\mathbf{Cp(2)}$ of compact *pairs*, whose objects are pairs[1] (X, A) consisting of a compact space X and a closed subspace $A \subset X$. Morphisms $(X, a) \to (Y, b)$ and $(X, A) \to (Y, B)$ in these categories are maps that send a to b and A into B. Apart from the obvious inclusion and forgetful functors $\mathbf{Cp}^\circ \longrightarrow \mathbf{Cp(2)} \longrightarrow \mathbf{Cp}$ there are more interesting functors

$$\mathbf{Cp} \longrightarrow \mathbf{Cp(2)} \longrightarrow \mathbf{Cp}^\circ$$

which send a space X to the pair $(X, \emptyset)$, and a general pair (X, A) to the quotient $(X/A, A/A)$: their composition would "collapse" the empty subspace to a base point and thus send X to the pointed space $X^+ = \{*\} + X$.

Definition The *one-point union* or *wedge* of two pointed spaces (X, a) and (Y, b) is the quotient space

$$X \vee Y := (X + Y)/(a \sim b)\,,$$

and their *smash product*[2] is the space

$$X \wedge Y := (X \times Y)/(X \vee Y)$$

obtained by collapsing the subspace $X \vee Y = X \times \{b\} \,\cup\, \{a\} \times Y$ to the new base point.

Examples (1) We denote the boundary $\{0, 1\} \subset [0, 1] = I$ by ∂I, and more generally by ∂I^d the boundary of the d-dimensional unit cube I^d, comprising all points with at least one component in ∂I. We identify the quotient $I^d/\partial I^d$ with S^n passing from I^d to the one-point compactification of $\mathbb{R}^d$ with $t \mapsto \tan \pi(t - \frac{1}{2})$ applied to each component, and then to S^d via the stereographic projection. The charm of this description of S^d is that $S^d \wedge S^e = S^{d+e}$ holds by the very definitions.

(2) Let (X, a) be a pointed space. The smash product $S^1 \wedge X$ may be read as the quotient obtained from the suspension ΣX by collapsing the contractible subspace $I \times \{a\}$: it is therefore called the *reduced suspension* of the pointed space (X, a). We will from now on write ΣX for the reduced version.

(3) The smash product is easily seen to be associative[3], and for the iterated suspension we therefore have a canonical homeomorphism $\Sigma^d X \approx S^d \wedge X$.

Definition The *reduced cone* of a pointed space X is the smash product

$$CX := (I, \{0\}) \wedge X\,;$$

[1] In topology space pairs are usually understood in this restricted sense: the second component is required to be a subspace of the first.

[2] The name refers to the way it is constructed — the smash product is not the abstract product in the category of pointed spaces.

[3] This would not be true for general non-compact spaces.

from now on CX will have this meaning. If $f\colon X \to Y$ is a map of pointed spaces the *mapping cone* of f is the quotient

$$Cf := (CX + Y)/\sim$$

with respect to the identification

$$CX \ni [1, x] \sim f(x) \in Y.$$

Notes (1) The canonical mapping $f^1\colon Y \to CX+Y \to Cf$ always is a topological embedding, and it is convenient to identify Y with its image $f^1(Y) \subset Cf$.

(2) If f is an embedding then so is the canonical map $CX \to CX+Y \to Cf$. Thus the mapping cone Cf contains the reduced cone over $f(X) \subset Y$, which of course is a contractible subspace. The mapping cone may therefore be thought of as a homotopy analogue of $Y/f(X)$ — which avoids the brutality of simply collapsing $f(X)$ to a point. Collapsing the subspace $CX \subset Cf$ eventually does result in a quotient Cf/CX homeomorphic to $Y/f(X)$.

(3) Starting with any pointed map $f = f\colon X \to Y$ the construction of the mapping cone Cf can be re-applied to the embedding $f^1\colon Y \to Cf$, and thus iterated writing $f = f^0$ and defining f^{k+1} as the embedding $Cf^{k-1} \subset Cf^k$:

$$X \xrightarrow{f} Y \xrightarrow{f^1} Cf \xrightarrow{f^2} \quad \cdots \quad \xrightarrow{f^k} Cf^{k-1} \xrightarrow{f^{k+1}} Cf^k \longrightarrow \quad \cdots$$

Proposition For every pointed map $f\colon X \to Y$ there is a canonical homeomorphism $Cf^1/CY \approx \Sigma X$ which applied to f, f^1, f^2 ... fits into the diagram

$$\begin{array}{ccccccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{f^1} & Cf & \xrightarrow{f^2} & Cf^1 & \xrightarrow{f^3} & Cf^2 & \xrightarrow{f^4} & Cf^3 & \xrightarrow{f^5} & Cf^4 \longrightarrow \cdots \\
 & & & & & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
 & & & & & & Cf^1/CY & & Cf^2/Cf & & Cf^3/Cf^1 & & Cf^4/Cf^3 \quad \cdots \\
 & & & & & & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \approx} \\
 & & & & & & \Sigma X & \xrightarrow{\Sigma' f} & \Sigma Y & \xrightarrow{\Sigma' f^1} & \Sigma Cf & \xrightarrow{\Sigma' f^2} & \Sigma Cf^1 \longrightarrow \cdots
\end{array}$$

which is commutative up to base-point preserving homotopy. All quotient mappings are marked q, and the reflected suspension $\Sigma' h$ of any map h sends the class $[t, y]$ to $[1-t, h(y)]$. Let us use the homeomorphisms in this diagram to identify spaces; the diagram may then be repeatedly extended to yield an infinite sequence

$$\begin{array}{ccccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{f^1} & Cf & \xrightarrow{f^2} & Cf^1 & & & & \\
 & & & & & & \downarrow{\scriptstyle q} & & & & \\
 & & & & & & \Sigma X & \xrightarrow{\Sigma' f} & \Sigma Y & \xrightarrow{\Sigma' f^1} & \Sigma Cf \longrightarrow \Sigma Cf^1 \\
 & & & & & & & & & & \qquad\qquad \downarrow{\scriptstyle q} \\
 & & & & & & & & & & \qquad\qquad \Sigma^2 X \xrightarrow{\Sigma'^2 f} \Sigma^2 Y \xrightarrow{\Sigma'^2 f^1} \cdots
\end{array}$$

where each q is an identification mapping that collapses an embedded reduced cone to the base point.

Proof The mapping cone Cf^1 is a quotient of $CX+CY$, and the promised homeomorphism $Cf^1/CY \approx \Sigma X$ simply sends $[s, x] \in CX$ to $[s, x] \in \Sigma X$.

As to the diagram

$$\begin{array}{ccc}
Cf^1 & \xrightarrow{f^3} & Cf^2 \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
Cf^1/CY & & Cf^2/Cf \\
\downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \approx} \\
\Sigma X & \xrightarrow{\Sigma' f} & \Sigma Y
\end{array}$$

the mapping $I \times Cf^1 \to \Sigma Y$ given by

$$\begin{aligned}
I \times CX \ni (\tau, [s, x]) &\longmapsto [1 - \tau s, f(x)] \in \Sigma Y \\
I \times CY \ni (\tau, [t, y]) &\longmapsto [(1-\tau)\, t\,, y\,] \quad \in \Sigma Y
\end{aligned}$$

is a pointed homotopy from the upper right to the lower left hand composition. This implies the commutativity of the large diagram as stated, and thereby completes the proof.

Definiton The *reduced K-theory* of a pointed space (X, a) is the kernel $\tilde{K}(X, a)$ of the ring homomorphism

$$KX \longrightarrow K\{a\} = \mathbb{Z}$$

induced by the inclusion $\{a\} \to X$. The *K-theory of a pair* (X, A) is defined as

$$K(X, A) = \tilde{K}(X/A)\,.$$

Both notions clearly are functorial for pointed maps respectively pairs of maps.

Proposition Assume that the pointed map $f: X \to Y$ is a topological embedding. Then the induced sequences in K-theory

$$\tilde{K}X \xleftarrow{f^*} \tilde{K}Y \xleftarrow{(f^1)^*} \tilde{K}Cf \quad \text{and} \quad KX \xleftarrow{f^*} KY \xleftarrow{(f^1)^*} \tilde{K}Cf$$

are *exact*: $\operatorname{kernel} f^* = \operatorname{image}(f^1)^*$.

Proof From the commutative diagram

$$\begin{array}{ccccc}
\tilde{K}X & \xleftarrow{f^*} & \tilde{K}Y & \xleftarrow{(f^1)^*} & \tilde{K}Cf \\
\downarrow & & \downarrow & & \| \\
KX & \xleftarrow{f^*} & KY & \xleftarrow{(f^1)^*} & \tilde{K}Cf \\
\downarrow & & \downarrow & & \\
K\{*\} & = & K\{*\} & &
\end{array}$$

with exact columns we read off that both the relevant image and kernel are contained in the reduced part, and that therefore exactness of either sequence is equivalent to that of the other. We will prove exactness of the unreduced version.

Identify X with the subspace $f(X) \subset Y$, and let $q: Y \to Y/X$ denote the quotient map. As noted before the canonical mapping

$$Cf \longrightarrow Cf/CX \approx Y/X$$

is an identification map, and since CX is contractible we know from Section 4 that an isomorphism $KCf \simeq K(Y/X)$ is induced. Substituting $\tilde{K}Cf$ we rewrite the unreduced sequence as

$$KX \xleftarrow{f^*} KY \xleftarrow{q^*} \tilde{K}(Y/X)\,.$$

As to exactness, note first that the composition $q\circ f$ is constant, so that the induced composition[4] f^*q^* in K-theory factors through $\tilde K\{*\}=\{0\}$,

$$\tilde KX \xleftarrow{f^*} \tilde K\{*\} \xleftarrow{q^*} \tilde K(Y/X)\,.$$

and thus must be the zero homomorphism. This proves the inclusion image $q^*\subset$ kernel j^*.

Conversely let $y\in KY$ be such that $f^*(y)=0$. We may represent $y=[E]-d$ as the difference between the class of a vector bundle $E\to Y$ and a trivial class. The property $f^*(y)=0$ means that the restriction $E|X\to X$ is a stably trivial bundle and that E has constant rank d on X; at the cost of increasing d we may even assume that $E|X$ is truly trivial. We choose any trivialisation and apply the collapsing construction from Section 4: the result is a vector bundle $F\to Y/X$ with $q^*F\simeq E$. This implies

$$q^*([F]-d)=[E]-d=y,$$

and since $\operatorname{rank}_F(X)=\operatorname{rank}_E|X=d$ we have $[F]-d\in\tilde K(Y/X)$ and therefore $y\in$ image q^*. This proves kernel $j^*\subset$ image q^*.

Corollary There is a functorial sequence

$$\tilde KX \xleftarrow{f^*} \tilde KY \xleftarrow{(f^1)^*} \tilde KCf \xleftarrow{(f^2)^*} \tilde K\Sigma X \xleftarrow{(\Sigma' f)^*} \tilde K\Sigma Y \xleftarrow{(\Sigma' f^1)^*} \tilde K\Sigma Cf \longleftarrow \cdots$$
$$\cdots \xleftarrow{(\Sigma' f^2)^*} \tilde K\Sigma^2 X \xleftarrow{(\Sigma'^2 f)^*} \tilde K\Sigma^2 Y \xleftarrow{(\Sigma'^2 f^1)^*} \tilde K\Sigma^2 Cf \longleftarrow \cdots$$

which is exact at every position from $\tilde KCf$ to the right, and also at $\tilde KY$ if f is an embedding.

Proof The vertically drawn identification mappings of the previous long step diagram collapse contractible subspaces and therefore induce isomorphisms in K-theory. It remains to note that the constructions of suspension and mapping cone commute up to canonical homeomorphism: $\Sigma Ch=C\Sigma h$ for every pointed map h.

Definition Given a pair (X,A), a map $r\colon X\to A$ with $r|A=\mathrm{id}_A$ is called a *retraction*, and A called a *retract* of X if such a retraction exists.

Proposition Every retraction $r\colon X\to A$ defines canonical splittings of additive groups

$$KX=KA\oplus K(X,A)\quad\text{and}\quad \tilde KX=\tilde KA\oplus K(X,A),$$

the latter if r is a retraction of pointed spaces.

Proof The case of empty A being trivial it suffices to consider the reduced version. We let $j\colon A\to X$ denote the inclusion and note that $\Sigma r\colon \Sigma X\to\Sigma A$ likewise is a retraction. Using $\tilde KCj=\tilde K(X/A)=K(X,A)$ we read from the exact sequence of the corollary the piece

$$\tilde KA \underset{r^*}{\overset{j^*}{\rightleftarrows}} \tilde KX \longleftarrow K(X,A) \longleftarrow \tilde K\Sigma A \underset{(\Sigma' r)^*}{\overset{(\Sigma' j)^*}{\rightleftarrows}} \tilde K\Sigma X\,;$$

the homomorphisms induced by r split it as indicated. The identity $j^*r^*=\mathrm{id}$ forces j^* and $(\Sigma' j)^*$ to be surjective, so that we may extract the short exact sequence

$$0\longleftarrow \tilde KA \underset{r^*}{\overset{j^*}{\rightleftarrows}} \tilde KX \longleftarrow K(X,A)\longleftarrow 0$$

which is still split.

[4] It is harmless, and often makes for easier reading to suppress the composition symbol.

Corollary There is a functorial additive isomorphism $\tilde K(X\times Y)\simeq \tilde KX\oplus\tilde K(X\wedge Y)\oplus\tilde KY$.

Proof Clearly $X=X\times\{*\}$ is a retract of $X\times Y$, and $Y=\{*\}\times Y$ one of $(X\times Y)/(X\times\{*\})$. Applying the last proposition twice we obtain

$$\tilde K(X\times Y)\simeq \tilde KX\oplus\tilde K\big((X\times Y)/(X\times\{*\})\big)\simeq\tilde KX\oplus\tilde K(X\wedge Y)\oplus\tilde KY\,.$$

Applications (1) Let (X,a) be a pointed space. Applied to the retract $\{a\}\subset X$ the proposition yields the canonical decomposition

$$KX=K\{a\}\oplus K(X,\{a\})=K\{a\}\oplus\tilde KX=\mathbb Z\oplus\tilde KX$$

of KX. Explicitly, it comes down to writing a virtual bundle on X as the sum of a trivial virtual bundle and one with rank zero at the point a.

(2) Let $h\in K(S^2\times X)$ denote the class of the hyperplane bundle as usual. Multiplication by $h-1$ induces an additive isomorphism $\tilde KX\simeq\tilde K\Sigma^2X$.

Proof of (2) The Periodicity Theorem provides an additive isomorphism

$$KX\oplus KX\longrightarrow K(S^2\times X)$$

that sends (w,x) to $w\cdot 1+(h-1)\,x$. The claim is that the inverse image of

$$\tilde K\Sigma^2X=\{0\}\oplus\tilde K\Sigma^2X\oplus\{0\}\subset\tilde KS^2\oplus\tilde K\Sigma^2X\oplus\tilde KX=\tilde K(S^2\times X)\subset K(S^2\times X)$$

is exactly $\{0\}\oplus\tilde KX$.

Consider an arbitrary $y=w\cdot1+(h-1)\,x\in K(S^2\times X)$. The statement that $y\in\tilde K\Sigma^2X$ can be broken down into three successive conditions, as follows. The first is that $y\in\tilde K(S^2\times X)$, or equivalently $w\cdot1\in\tilde K(S^2\times X)$, for $(h-1)\,x$ is a virtual bundle of degree zero. Now the projection of y to $\tilde KX$ makes sense and is just w, so that the second condition is $w=0$. The third and final condition requires that the projection of y to $\tilde KS^2$ is zero: this projection being $\operatorname{rank}_x(*)\,(h-1)\in\tilde KS^2$ the condition is equivalent to $\operatorname{rank}_x(*)=0$ and thus to $x\in\tilde KX$.

Theorem For every pointed map $f\colon X\to Y$ there is a functorial periodic sequence of K-theory

$$\begin{array}{ccccc}
\tilde KX & \xleftarrow{\;f^*\;} & \tilde KY & \xleftarrow{\;(f^1)^*\;} & \tilde KCf\\
{\scriptstyle\delta}\downarrow & & & & \uparrow{\scriptstyle\delta}\\
\tilde K\Sigma Cf & \xrightarrow[(\Sigma f^1)^*]{} & \tilde K\Sigma Y & \xrightarrow[\Sigma f^*]{} & \tilde K\Sigma X
\end{array}$$

which is exact throughout; the arrows marked δ are called *coboundary* operators.

Proof This is the previous exact sequence where all double suspensions have been substituted. Note that for any pointed map h the double reflected suspension Σ'^2h is homotopic to Σ^2h by a rotation of S^2; therefore the induced homomorphisms in K-theory coincide. Exactness at the suspension free positions is insured since they may be replaced by double suspensions.

Examples (1) Based on the observation that $\tilde KS^0=K\{*\}=\mathbb Z$ the isomorphisms $\tilde KX\simeq\tilde K\Sigma^2X$ now imply that

$$\tilde KS^n\simeq\mathbb Z\quad\text{for all even } n\in\mathbb N.$$

Since from the examples of Section 4 we know $\operatorname{Vect}S^1=\mathbb N$ we conclude that by contrast

$$\tilde KS^n=\{0\}\quad\text{for all odd } n\in\mathbb N.$$

(2) Every real invertible matrix $g \in GL(n,\mathbb{R})$ naturally acts on the sphere S^n — the one-point compactification of $\mathbb{R}^n$ — and thus also on the n-fold suspension $\Sigma^n X = S^n \wedge X$ of any given space X. The induced isomorphism on $\tilde K\Sigma^n X$ is $\pm\,\mathrm{id}$ according to whether g preserves or reverses orientation.

Definiton A *cell complex*[5] is a topological space with a filtration by subspaces

$$X^{-1} \subset X^0 \subset \quad \cdots \quad \subset X^{j-1} \subset X^j \subset \quad \cdots \quad \subset X^{n-1} \subset X^n = X$$

such that $X^{-1} = \emptyset$, and for each $j \geq 0$ the space X^j is obtained form X^{j-1} by attaching a finite number of j-cells. The subspace $X^j \subset X$ is the j-*skeleton* of X, and in case $X^{n-1} \neq X^n$ we call n the *dimension* of X.

Application Let X be a cell complex and assume that all cells of X have even dimension. Then

$$KX \simeq \bigoplus_{c\in C} \mathbb{Z}c$$

is isomorphic to the free abelian group on the set C of cells while $\tilde K\Sigma X = \{0\}$ is trivial.

Proof If X is empty the conclusion is obvious, and so is the value of KX if $\dim X = 0$, that is, if X is discrete. For such X we also know from Section 4 that $\mathrm{Vect}_d\,\Sigma X \simeq [X, GL(d,\mathbb{C})]$ comprises just the trivial bundle for each $d \in \mathbb{N}$, and conclude that $\tilde K\Sigma X = \{0\}$ as claimed.

Proceeding by induction we may assume that the conclusion holds for a cell complex X and must prove it for the new cell complex $D^n \cup_\varphi X$, where $n > 0$ is even and $\varphi\colon S^{n-1} \to X$ an attaching map.

By definition $Y := D^n \cup_\varphi X$ is the unreduced version of the mapping cone of $\varphi\colon S^{n-1} \to X$, and if we choose any base point $* \in S^{n-1}$ then passing to the reduced cone $C\varphi$ means collapsing the embedded interval $I \cdot \{*\} \subset D^n \cup_\varphi X$ — a modification that does not affect K-theory. We thus may work with $C\varphi$ rather than $D^n \cup_\varphi X$. Here the exact sequence of the theorem runs

$$\begin{array}{ccccc} \tilde KS^{n-1} & \longleftarrow & \tilde KX & \longleftarrow & \tilde KC\varphi \\ \delta\big\downarrow & & & & \big\uparrow\delta \\ \tilde K\Sigma C\varphi & \longrightarrow & \tilde K\Sigma X & \longrightarrow & \tilde K\Sigma S^{n-1} \end{array}$$

or

$$\begin{array}{ccccc} 0 & \longleftarrow & \tilde KX & \longleftarrow & \tilde KC\varphi \\ \delta\big\downarrow & & & & \big\uparrow\delta \\ \tilde K\Sigma C\varphi & \longrightarrow & 0 & \longrightarrow & \tilde KS^n\,. \end{array}$$

We conclude that $\tilde K\Sigma C\varphi = \{0\}$, and observe that the short exact sequence

$$0 \longrightarrow \tilde KS^n \longrightarrow \tilde KC\varphi \longrightarrow \tilde KX \longrightarrow 0$$

must split since $\tilde KX$ is a free $\mathbb{Z}$-module: this proves the assertion.

Example (3) We have seen in Section 2 that $\mathbb{R}P^n$ can be obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell. This also works for $\mathbb{C}P^n$: the attaching map

$$S^{2n-1} \ni z = (z_1,\ldots,z_n) \overset{\varphi}{\longmapsto} [z_1 : \cdots : z_n] \in \mathbb{C}P^{n-1}$$

[5] This would be called a *finite* complex in standard terminology — the finiteness restriction makes it compact as a topological space.

defines a compact space $D^{2n} \cup_\varphi \mathbb{C}P^{n-1}$, and the assignments

$$\begin{aligned} D^{2n} \ni (z_1, \dots, z_n) &\longmapsto \left[\sqrt{1-|z|^2} : z_1 : \cdots : z_n\right] \in \mathbb{C}P^n \\ \mathbb{C}P^{n-1} \ni [z_1 : \cdots : z_n] &\longmapsto [0 : z_1 : \cdots : z_n] \in \mathbb{C}P^n \end{aligned}$$

combine to give a homeomorphism $D^{2n} \cup_\varphi \mathbb{C}P^{n-1} \approx \mathbb{C}P^n$.

The construction gives $\mathbb{C}P^n$ the structure of a cell complex with exactly one j-cell for $j = 0, 2, \dots, 2n$, and we conclude that as an additive group $K\mathbb{C}P^n \simeq \mathbb{Z}^{n+1}$ while $\tilde{K}\Sigma\mathbb{C}P^n = \{0\}$.

10 Graded K-Theory and Products

Notation We define new functors on the categories $\mathbf{Cp}^\circ$ and $\mathbf{Cp(2)}$ respectively by

$$K^1X := \tilde{K}\Sigma X \quad \text{and} \quad K^1(X,A) := \tilde{K}\Sigma(X/A)\,.$$

When we want to stress the analogy with the standard functors of K-theory we now mark these by a zero superscript:

$$K^0X = KX, \quad \tilde{K}^0X = \tilde{K}X, \quad \text{and} \quad K^0(X,A) = K(X,A)\,.$$

Recall the identity $K^0(X,\emptyset) = \tilde{K}^0(X^+) = K^0X$, which holds by the very definitions. The analogue for K^1 is true though not obvious:

Lemma There is a functorial isomorphism $K^1(X,\emptyset) \simeq K^1X$ for pointed spaces X.

Proof Let $a \in X$ be the base point of X, and denote by $\tilde{X}$ the unreduced suspension of X^+, that is the quotient of $I\times X^+$ in which $\{0\}\times X^+$ and $\{1\}\times X^+$ are collapsed to one point each. The subspace $I\times\{*\}\cup I\times\{a\}$ is a retract of $\tilde{X}$ and homeomorphic to S^1, and we know from the previous section that then

$$\tilde{K}\tilde{X} = \tilde{K}S^1 \oplus \tilde{K}(\tilde{X}/S^1)\,.$$

Now collapsing the interval $I\times\{*\}$ in $\tilde{X}$ gives $\Sigma(X^+)$, so that $\tilde{K}\tilde{X} = \tilde{K}\Sigma(X^+) = K^1(X,\emptyset)$. On the other hand $\tilde{K}S^1 = \{0\}$ and $\tilde{X}/S^1 = \Sigma X$, which implies $\tilde{K}(\tilde{X}/S^1) = \tilde{K}\Sigma X = K^1X$.

In view of the lemma we identify $K^1(X,\emptyset)$ with K^1X even for X with no distinguished base point, and complete the definition by putting $K^1\emptyset = \{0\}$.

Definition The *graded K-theory* functors on the categories $\mathbf{Cp}$, $\mathbf{Cp}^\circ$, and $\mathbf{Cp(2)}$ are defined by

$$K^*X = K^0X \oplus K^1X, \quad \tilde{K}^*X = \tilde{K}^0X \oplus K^1X, \quad \text{and} \quad K^*(X,A) = K^0(X,A) \oplus K^1(X,A);$$

they take values, so far, in the category of abelian groups which are graded by $\mathbb{Z}/2$.

Note There are many ways to restate and specialise the main result of the previous section. An obvious one is the exact sequence

$$\begin{array}{ccccc}
\tilde{K}^0X & \longleftarrow & \tilde{K}^0Y & \longleftarrow & \tilde{K}^0Cf \\
{\scriptstyle\delta}\downarrow & & & & \uparrow{\scriptstyle\delta} \\
\tilde{K}^1Cf & \longrightarrow & \tilde{K}^1Y & \longrightarrow & \tilde{K}^1X
\end{array}$$

for pointed maps $f\colon X \to Y$. If (X,A) is a space pair with inclusion $j\colon A \to X$ then the choice $f = j^+\colon A^+ \subset X^+$ yields the exact sequence of the pair

$$\begin{array}{ccccc}
K^0A & \longleftarrow & K^0X & \longleftarrow & K^0(X,A) \\
{\scriptstyle\delta}\downarrow & & & & \uparrow{\scriptstyle\delta} \\
K^1(X,A) & \longrightarrow & K^1X & \longrightarrow & K^1A
\end{array}$$

as well as its reduced version

$$\begin{array}{ccccc} \tilde K^0 A & \longleftarrow & \tilde K^0 X & \longleftarrow & K^0(X,A) \\ \delta\downarrow & & & & \uparrow\delta \\ K^1(X,A) & \longrightarrow & \tilde K^1 X & \longrightarrow & \tilde K^1 A; \end{array}$$

these are the most widely known and used instances of the exact sequence. Often convenient is the more concise graded version which reduces to an exact triangle

$$\begin{array}{ccc} K^*A & \longleftarrow & K^*X \\ & {}_\delta\searrow \quad \nearrow & \\ & K^*(X,A) & \end{array}$$

involving one homomorphism δ of degree one while the unmarked ones are geometric and have degree zero.

Lemma There is a functorial isomorphism $K^*X \simeq K(S^1 \times X)$ of (non-graded) abelian groups.

Proof This is obvious if $X = \emptyset$. For non-empty X we choose an arbitrary base point and consider the decomposition $\tilde K(S^1 \times X) \simeq \tilde K S^0 \oplus \tilde K(S^1 \wedge X) \oplus \tilde K X$. The first term vanishes and the second is K^1X. Adding the trivial bundles on both sides we obtain the result.

We return to the study of the K-theory product $KX \times KX \to KX$. It admits an equivalent *external* version

$$KX \times KY \ni (x,y) \longmapsto x \times y := \mathrm{pr}^* x \cdot \mathrm{pr}^* y \in K(X \times Y);$$

indeed the internal one is recovered using the diagonal $d: X \to X \times X$ as $x \cdot y = d^*(x \times y)$.

Proposition Let X and Y be pointed spaces. Then the external product $KX \times KY \to K(X \times Y)$ restricts to a product

$$\tilde K X \times \tilde K Y \longrightarrow \tilde K(X \wedge Y)$$

in reduced K-theory.

Proof If two virtual bundles on X and Y restrict to zero over the base points then their product is zero over $X \times \{*\} \cup \{*\} \times Y$. Thus the cartesian projections of $\tilde K(X \times Y) = \tilde K X \oplus \tilde K(X \wedge Y) \oplus \tilde K Y$ to $\tilde K X$ and $\tilde K Y$ send the product to zero.

Corollary There is a natural external product for space pairs

$$K(X,A) \times K(Y,B) \xrightarrow{\times} K\big((X,A) \times (Y,B)\big)$$

where the product of two pairs is defined as

$$(X,A) \times (Y,B) = (X \times Y,\ X \times B \cup A \times Y)\,.$$

Together with the diagonal mapping $d: (X, X_1 \cup X_2) \to (X, X_1) \times (X, X_2)$ it yields an internal version

$$K(X, X_1) \times K(X, X_2) \longrightarrow K(X, X_1 \cup X_2)$$

which applies to spaces X with two distinguished subspaces X_1 and X_2. It generalises, and is compatible with the original product $KX \times KX \to KX$ in the commutative diagram

$$\begin{array}{ccc} K(X,X_1) \times K(X,X_2) & \longrightarrow & K(X, X_1 \cup X_2) \\ \downarrow & & \downarrow \\ KX \times KX & \longrightarrow & KX \end{array}$$

of products and restrictions.

Definition For every space pair (X, A) consider the bilinear mappings

$$\begin{aligned}
\tilde{K}(X/A) \times \tilde{K}(X/A) &\longrightarrow \tilde{K}(X/A) \\
\tilde{K}(X/A) \times \tilde{K}\Sigma(X/A) &\longrightarrow \tilde{K}\big((X/A) \wedge \Sigma(X/A)\big) \xrightarrow{d^*} \tilde{K}\Sigma(X/A) \\
\tilde{K}\Sigma(X/A) \times \tilde{K}\Sigma(X/A) &\longrightarrow \tilde{K}\big(\Sigma(X/A) \wedge \Sigma(X/A)\big) \xrightarrow{e^*} \tilde{K}\Sigma^2(X/A) \xleftarrow{\simeq} \tilde{K}(X/A)
\end{aligned}$$

built with the partial diagonals $d\colon \Sigma(X/A) \to (X/A)\wedge\Sigma(X/A)$ and $e\colon \Sigma^2(X/A) \to \Sigma(X/A)\wedge\Sigma(X/A)$ and the suspension isomorphism. These mappings combine to give

$$K^*(X, A) = K^0(X, A) \oplus K^1(X, A) = \tilde{K}(X/A) \oplus \tilde{K}\Sigma(X/A)$$

the structure of a graded commutative ring:

$$x \cdot y = (-1)^{\deg x \cdot \deg y}\, y \cdot x \quad \text{for homogeneous elements } x, y \in K^*(X, A).$$

Similarly, K^*A and $K^*(X, A)$ are graded modules over K^*X, and the homomorphisms of the exact triangle

$$\begin{array}{ccc}
K^*A & \longleftarrow & K^*X \\
 & {\scriptstyle\delta}\searrow \quad \nearrow & \\
 & K^*(X, A) &
\end{array}$$

turn out to be K^*X-linear.

Proposition Let X be a compact base space, and let $L_1, \ldots, L_d$ be line bundles over X. As usual let $h \in KP(L_1 \oplus \cdots \oplus L_d)$ denote the class of the hyperplane bundle. Then

$$KP(L_1 \oplus \cdots \oplus L_d) = KX\,[h] \Big/ \prod_{i=1}^{d} \big([L_i]\, h - 1\big)$$

as a KX-algebra in the sense that h behaves like an indeterminate which is subject to the stated relation.

Proof Without loss of generality we may assume that $L_1 = \mathbf{1}$, since multiplying each bundle L_i by L_1^{-1} replaces h by $[L_1]h$ and leaves the projective bundle unchanged:

$$P(L_1 \oplus L_2 \oplus \cdots \oplus L_d) = P(\mathbf{1} \oplus L_1^{-1}\otimes L_2 \oplus \cdots \oplus L_1^{-1}\otimes L_d).$$

We will argue by induction on d, the case of $d = 0$ being trivial. For $d > 0$ we suppose that $L_1 = \mathbf{1}$ and abbreviate

$$P = P(L_1 \oplus \cdots \oplus L_d) \quad \text{and} \quad P' = P(L_2 \oplus \cdots \oplus L_d);$$

note that the restriction of the hyperplane bundle H of P over $P' \subset P$ is the same as the hyperplane bundle H' of P'. The projective bundles $P = P(\mathbf{1} \oplus L_2 \oplus \cdots \oplus L_d) \to X$ and $P(\mathbf{1} \oplus H'^{\smallsmile}) \to P'$ contain a copy of their base spaces as the sections

$$X \approx P(\mathbf{1} \oplus \mathbf{0}) \subset P \quad \text{respectively} \quad P' \approx P(\mathbf{1} \oplus \mathbf{0}) \subset P(\mathbf{1} \oplus H'^{\smallsmile}).$$

We will treat them as embedded subspaces $X \subset P$ and $P' \subset P(\mathbf{1}\oplus H'^{\smallsmile})$; note that both are retracts. Let $q\colon P(\mathbf{1} \oplus H'^{\smallsmile}) \to P$ be the map

$$P(\mathbf{1} \oplus H'^{\smallsmile}) \ni [v_1 : \cdots : v_d]_{[w_2 : \cdots : w_d]} \xmapsto{q} [v_1 : \cdots : v_d] \in P$$

and note that in the point on the left hand side $(v_2, \ldots, v_d)$ must be a (possibly zero) multiple of $(w_2, \ldots, w_d) \neq 0$. Together with the projection $P' \to X$ the map q makes a commutative diagram

$$\begin{array}{ccc} P' & \longrightarrow & X \\ \downarrow & & \downarrow \\ P(\mathbf{1} \oplus H'^{\check{}}) & \xrightarrow{q} & P \end{array}$$

inducing a homeomorphism $P(\mathbf{1} \oplus H'^{\check{}})/P' \approx P/X$. Furthermore the definition of q shows that $G := q^* H$ is the hyperplane bundle on $P(\mathbf{1} \oplus H'^{\check{}})$.

From the Periodicity Theorem we know that

$$KP(\mathbf{1} \oplus H'^{\check{}}) = KP'\,[g] \,/\, \big((g-1)(g-h')\big)$$

where $g = [G] \in KP(\mathbf{1} \oplus H'^{\check{}})$ and $h' = [H'] \in KP'$. Since $G|P'$ is the trivial line bundle, the second term in the decomposition

$$KP(\mathbf{1} \oplus H'^{\check{}}) = KP' \oplus K\big(P(\mathbf{1} \oplus H'^{\check{}}), P'\big)$$

is the KP'-submodule freely generated by the element $g-1$: note the valid identity

$$g \cdot x = h' \cdot x \quad \text{for all } x \in K\big(P(\mathbf{1} \oplus H'^{\check{}}), P'\big)$$

on this submodule. Translating to the pair (P, X) we obtain that $K(P, X)$ is a free KP'-module with generator $h-1$, and

$$h \cdot x = h' \cdot x \quad \text{for all } x \in K(P, X)\,.$$

We now use the inductive assumption that the proposition holds for $d-1$ in place of d and thus may assume the isomorphism

$$KP' = KX\,[h'] \,/\, \prod_{i=2}^{d} \big([L_i]\, h' - 1\big)\,.$$

In terms of an indeterminate U we have the commutative diagram of KX-module isomorphisms

$$\begin{array}{ccc} KX\,[U] \,/\, \prod_{i=2}^{d}\big([L_i]\,U - 1\big) & \xrightarrow{U \mapsto h'} & KP' \\ \downarrow{\scriptstyle \cdot(U-1)} & & \downarrow{\scriptstyle \cdot(h-1)} \\ (U-1)KX\,[U] \,/ (U-1) \prod_{i=2}^{d}\big([L_i]\,U - 1\big) & \xrightarrow{U \mapsto h} & K(P, X)\,. \end{array}$$

In view of the fact that X is a retract of P, and the resulting decomposition $KP \simeq KX \oplus K(P, X)$, complementing the bottom row by the term KX on both sides gives the KX-algebra isomorphism $KX\,[U] \,/\, \prod_{i=1}^{d}\big([L_i]\,U - 1\big) \simeq KP$. This completes the induction step.

Corollary $K^*\mathbb{C}P^n = \mathbb{Z}[h]/(h-1)^{n+1}$ is the truncated polynomial algebra.

Addendum The conclusion of the proposition holds likewise for graded K-theory:

$$K^*P(L_1 \oplus \cdots \oplus L_d) = K^*X\,[h] \,/\, \prod_{i=1}^{d} \big([L_i]\, h - 1\big)$$

Proof In terms of the projection $S^1 \times X \xrightarrow{\text{pr}} X$ we have $S^1 \times P(L_1 \oplus \cdots \oplus L_d) = P(\text{pr}^* L_1 \oplus \cdots \oplus \text{pr}^* L_d)$ and therefore

$$K^*P(L_1 \oplus \cdots \oplus L_d) \simeq KP(\text{pr}^* L_1 \oplus \cdots \oplus \text{pr}^* L_d)\,;$$

it thus suffices to apply the proposition to $S^1 \times X$ in place of X.

Theorem Let $P \xrightarrow{\rho} X$ be a map, and let $p_1, \ldots, p_d \in K^*P$ be homogeneous elements. Assume that every $x \in X$ admits a neighbourhood Y such that for all compact subsets $A \subset Y$ the homomorphism of KA-modules

$$(K^*A)^d = K^*A \oplus \cdots \oplus K^*A \ni (x_1, \ldots, x_d) \overset{\varphi_A}{\longmapsto} \sum_{i=1}^{d} x_i p_i \in K^*\rho^{-1}A$$

is bijective. Then the map

$$(K^*X)^d \xrightarrow{\varphi_X} K^*P$$

is an isomorphism of KX-modules. Note that φ_X is homogeneous of weight zero if we shift the degrees of the i-th term of the direct sum by $\deg p_i \in \{0, 1\}$.

Proof For the purposes of the proof let us call an open subset $Y \subset X$ good if φ_A is bijective for all compact $A \subset Y$. By assumption is covered by good subsets, and we choose a finite subcover. We will show that if Y and Z are good subsets of X then so is their union $Y \cup Z$: this clearly will prove the theorem.

Assuming $Y \subset X$ good we first note that by the Five Lemma for any compact pair (A, B) with $A \subset Y$ not only φ_A and φ_B but also the relative version φ_{AB} is an isomorphism:

$$\begin{array}{ccccccccc}
(K^*A)^d & \longrightarrow & (K^*B)^d & \xrightarrow{\delta^d} & K^*(A,B)^d & \longrightarrow & (K^*A)^d & \longrightarrow & (K^*B)^d \\
\downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle \varphi_B} & & \downarrow{\scriptstyle \varphi_{AB}} & & \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle \varphi_B} \\
K^*\rho^{-1}A & \longrightarrow & K^*\rho^{-1}B & \xrightarrow{\delta} & K^*\rho^{-1}(A,B) & \longrightarrow & K^*\rho^{-1}A & \longrightarrow & K^*\rho^{-1}B
\end{array}$$

Let Z be another good subset of X and let $A \subset Y \cup Z$ be compact. We may write $A = B \cup C$ with compact subsets $B \subset Y$ and $C \subset Z$ — using that X is normal we find a compact neighbourhood V of $A \backslash Z$ in Y and put $B = A \cap V$ and $C = A \backslash V^\circ$. Since Y is good $\varphi_{B,B\cap C}$ is an isomorphism, and in view of the homeomorphism $B/(B \cap C) \approx (B \cup C)/C = A/C$ this means that φ_{AC} is an isomorphism. On the other hand φ_C is an isomorphism since Z is good, and by another a¡pplication of the Five Lemma we conclude that φ_A is an isomorphism too.

Proposition Let $E \to X$ be a vector bundle of rank d, and let $h \in K^0P(E)$ denote the class of the hyperplane bundle. Then the homomorphism of K^*X-modules

$$(K^*X)^d = K^*X \oplus \cdots \oplus K^*X \ni (x_1, \ldots, x_{d-1}) \longmapsto \sum_{i=0}^{d-1} x_i h^i \in K^*P(E)$$

is bijective.

Proof Over some neighbourhood of any $x \in X$ the bundle E is locally trivial and a fortiori a Whitney sum of line bundles. Therefore the previous proposition assures that the hypothesis of the theorem is satisfied, and from the theorem we directly obtain the result.

Splitting Principle Let $E \to X$ be a vector bundle of rank d. Then the flag bundle $F := F(E) \xrightarrow{\rho} X$ has the properties that

- the homomorphism $\rho^*: K^*X \to K^*F$ is injective, and
- the pull-back ρ^*E is isomorphic to Whitney sum of d line bundles.

Proof The first property follows from the last proposition since F is an iterated projective bundle, and the second is a known fact from the construction of flag bundles in Section 5.

Theorem For every vector bundle $E \to X$ of degree d put $\lambda_{-1}[E](h) = \sum_{i=0}^{d} (-1)^i [\Lambda^i E]\, h^i \in K^0 X\,[h]$. Then

$$K^* P(E) = K^* X\,[h] \,/\, \big(\lambda_{-1}[E](h)\big)$$

with h, the hyperplane class on the left hand side, playing the usual role of an indeterminate subject to the relation $\lambda_{-1}[E]$ on the right.

Proof Let t be an indeterminate over K^*X. By the last proposition the K^*X-algebra homomorphism

$$K^* X\,[t] \longrightarrow K^* P(E)$$

that substitutes h for t is surjective, and its kernel is the principal ideal generated by a monic polynomial of degree d. It therefore suffices to prove that the identity $\lambda_{-1}[E](h) = 0$ holds in $K^* P(E)$.

We wish to apply the Splitting Principle. Denoting the bundle projections by $P(E) \xrightarrow{\pi} X$ and $F(E) \xrightarrow{\rho} X$ we identify the canonically homeomorphic spaces $F(\pi^* E)$ and $P(\rho^* E)$ and obtain the commutative diagram

$$\begin{array}{ccc} P(\rho^* E) & \longrightarrow & P(E) \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ F(E) & \xrightarrow[\rho]{} & X \end{array}$$

whose horizontal arrows induce injections in K-theory. We therefore may without loss of generality assume that $E = L_1 \oplus \cdots \oplus L_d$ is a sum of line bundles, and in view of our former results it just remains to verify the identity

$$\lambda_{-1}[E](h) = \prod_{i=1}^{d} \big(1 - [L_i]\, h\big)$$

in this case:

Lemma For any d line bundles $L_1, \ldots, L_d$ over a common base space X the polynomial identity

$$\sum_{i=0}^{d} (-1)^i \left[\Lambda^i (L_1 \oplus \cdots \oplus L_d)\right] t^i = \prod_{i=1}^{d} \big(1 - [L_i]\, t\big)$$

holds in $KX\,[t]$.

Proof This is certainly true for $d = 0$, and proved in general by induction on d: for $d > 0$ the canonical isomorphisms

$$\Lambda^i (L_1 \oplus \cdots \oplus L_d) \simeq \Lambda^i (L_1 \oplus \cdots \oplus L_{d-1}) \;\oplus\; \Lambda^{i-1} (L_1 \oplus \cdots \oplus L_{d-1}) \otimes L_d$$

from linear algebra, and the induction hypothesis immediately yield

$$\sum_{i=0}^{d} (-1)^i \left[\Lambda^i (L_1 \oplus \cdots \oplus L_d)\right] t^i = \prod_{i=1}^{d-1} \big(1 - [L_i]\, t\big) \cdot \big(1 - [L_d]\, t\big)\,.$$

Theorem For every vector bundle $E \to X$ of degree d let $l_1, \ldots, l_d \in KF(E)$ be the classes of the canonical line bundles L_i on the flag bundle $F(E) \to X$. Then

$$K^* F(E) = K^* X\,[l_1, \ldots, l_d] \,/ \Sigma$$

where the ideal of relations

$$\Sigma = \big(\sigma_1(l) - [E],\ \sigma_2(l) - [\Lambda^2 E],\ \ldots,\ \sigma_d(l) - [\Lambda^d E]\big)$$

is generated by expressions involving the elementary symmetric polynomials

$$\sigma_i(l) = \sigma_i(l_1, \ldots, l_d) \in \mathbb{Z}\,[l_1, \ldots, l_d]\ .$$

Proof Repeatedly applying the previous theorem we arrive at the representation

$$K^*F(E) = K^*X\,[l_1, \ldots, l_d]\,/\tilde{\Sigma}$$

with an ideal $\tilde{\Sigma} \subset K^*X\,[l_1, \ldots, l_d]$ that must contain Σ since the identities

$$\sigma_i(l_1, \ldots, l_d) = \big[\Lambda^i(L_1 \oplus \cdots \oplus L_d)\big] = [\Lambda^i E] \quad \text{for } i = 1, \ldots, d$$

hold in $K^*F(E)$. In order to prove the opposite inclusion $\tilde{\Sigma} \subset \Sigma$ we need a more explicit description of $\tilde{\Sigma}$. For $d = 0$ we have $\tilde{\Sigma} = \{0\}$ and there is nothing to show, while for $d > 0$ we interpret $F(E)$ as the flag bundle of $E/L_1 \to P(E)$ and, by induction, obtain the formula

$$K^*F(E) = K^*P(E)[l_2 \ldots, l_d]\Big/\Big(\sigma_1(l')-[E/L_1],\ \sigma_2(l')-[\Lambda^2 E/L_1],\ \ldots,\ \sigma_{d-1}(l')-[\Lambda^{d-1}E/L_1]\Big)$$

with $l' = (l_2, \ldots, l_d)$ and

$$K^*P(E) = K^*X[l_1]\Big/\ \big(\lambda_{-1}[E](l_1^{-1})\big)\,,$$

recalling that $L_1 \to F(E)$ is pulled back form the tautological bundle of $P(E)$. Calculating modulo Σ we see that

$$\lambda_{-1}[E](l_1^{-1}) = \sum_{i=0}^{d} (-1)^i [\Lambda^i E]\, l_1^{-i} \equiv \sum_{i=0}^{d} (-1)^i \sigma_i(l)\, l_1^{-i} \bmod \Sigma\,,$$

but the last expression is

$$l_1^{-d} \cdot \sum_{i=0}^{d} (-1)^i \sigma_i(l)\, l_1^{d-i} = l_1^{-d} \cdot \prod_{i=1}^{d} (l_1 - l_i) = 0\,.$$

We further obtain successively for $i = 1, \ldots, d-1$ that

$$\sigma_i(l') - [\Lambda^i E/L_1] = \sigma_i(l) - [\Lambda^i E] - l_1 \cdot \sigma_{i-1}(l') + [L_1 \otimes \Lambda^{i-1} E/L_1] \equiv 0 \bmod \Sigma$$

as well. This proves the inclusion $\tilde{\Sigma} \subset \Sigma$ and thereby concludes the proof.

Further results on the computation of the K-functor include the following, not mentioned in class.

Theorem Let X be a space such that K^*X is torsion free as an abelian group, and let Y be a cell complex. Then the K-theory product induces an isomorphism

$$K^*X \otimes K^*Y \simeq K^*(X \times Y)\,.$$

Remarks The point of excluding torsion from K^*X is that then taking the tensor product over $\mathbb{Z}$ with K^*X preserves exact sequences, in particular the K-theory sequence of any pair $(\tilde{Y}, Y)$. By induction of the number of cells of Y this reduces the theorem to the case where Y is a sphere — compare Problem 36.

Theorem The K-theory of the real projective spaces is

$$\tilde{K}^0\mathbb{R}P^{2n} \simeq \tilde{K}^0\mathbb{R}P^{2n+1} \simeq \mathbb{Z}/2^n$$
$$K^1\mathbb{R}P^{2n} = \{0\} \text{ and } K^1\mathbb{R}P^{2n+1} \simeq \mathbb{Z}$$

for all $n \in \mathbb{N}$.

Remarks Not surprisingly the K-theory of complex vector bundles is less suitable for a direct approach to the real projective spaces. But it is remarkable that K-theory generalises quite easily to a so-called *equivariant* theory which applies to spaces with an action of a finite group rather than a mere topological space. The calculation of the groups shown above in Atiyah's book is fairly simple and based on $\mathbb{Z}/2$-equivariant K-thery.

11 Operations I

Definition An *operation* in K-theory is a natural self-transformation of the functor K, considered as defined on **Cp** with values in the category of sets. Explicitly an operation η assigns to each compact space X a mapping $\eta_X : KX \to KX$ such that for every map $f : X \to Y$ the diagram

$$\begin{array}{ccc} KX & \xrightarrow{\eta_X} & KX \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ KY & \xrightarrow{\eta_Y} & KY \end{array}$$

commutes. We do not require η_X to respect the ring operations of KX.

Note Every operation is determined by its action on vector bundles of constant rank. Indeed, if $E \to X$ is a general vector bundle then the function $\mathrm{rank}_E : X \to \mathbb{N}$ defines a decomposition $X = \sum_{d=0}^{\infty} X_d$ of X as a (but formally infinite) topological sum, and the diagram

$$\begin{array}{ccc} KX & \xrightarrow{\eta_X} & KX \\ \simeq\downarrow & & \downarrow\simeq \\ \prod_{d=0}^{\infty} KX_d & \xrightarrow{\prod \eta_{X_d}} & \prod_{d=0}^{\infty} KX_d \end{array}$$

induced by the restriction homomorphisms commutes.

Examples (1) Sending $x \in KX$ to $4x^5 + 6 \in KX$ is an operation but not an interesting one since it can be — in fact is — expressed completely in terms of the ring structure. If we know KX as a ring no additional information on X can be gathered from this operation.

(2) A host of operations is obtained from the linear algebra of vector bundles. For instance passing form a vector bundle E to its dual $E\check{}$ defines a semi-group homomorphism $\mathrm{Vect}\, X \to \mathrm{Vect}\, X$ and thus an operation

$$KX \ni x \longmapsto \overline{x} \in KX$$

in K-theory. It sends the generator $h-1 \in \tilde{K}S^2$ to $h^{-1}-1 = (2-h)-1 = 1-h$. Since the n-fold exterior product $(h-1)\times\cdots\times(h-1)$ generates the group $\tilde{K}S^{2n} \simeq \mathbb{Z}$ we see that the dualising operation acts on $\tilde{K}S^{2n}$ as multiplication by $(-1)^n$. In particular it helps to distinguish between the spaces S^2 and S^4 which cannot be homeomorphic, even though the rings KS^2 and KS^4 are both isomorphic to $K[t]/(t^2)$.

(3) We consider the power series

$$\lambda_t[E] := \sum_{k=0}^{\infty} [\Lambda^k E]\, t^k \in KX\,[[t]]$$

in an indeterminate t over KX, with coefficients depending on the vector bundle $E \to X$; for any given bundle E whose rank is bounded by d the series reduces to a polynomial of degree at most d. Note that the special value $\lambda_{-1}[E] = \sum (-1)^k [\Lambda^k E]$ has been considered before and that

$\lambda_t[E] \in 1+(t)\cdot KX\,[[t]]$ belongs to the multiplicative group of power series with constant term 1. If F is another bundle on X then the canonical isomorphism

$$\Lambda^k(E\oplus F) \simeq \bigoplus_{i+j=k} \Lambda^i E\otimes \Lambda^j F$$

shows that $\lambda_t[E\oplus F] = \lambda_t[E]\cdot\lambda_t[E]$. By the defining property of the K-group the homomorphism $\operatorname{Vect} X \ni [E] \to 1+(t)\cdot KX\,[[t]]$ extends to a group homomorphism

$$\lambda_t\colon KX \longrightarrow 1+(t)\cdot KX\,[[t]]$$

given explicitly by

$$\lambda_t\left([E]-[F]\right) = \lambda_t[E]\cdot\lambda_t[F]^{-1}\,.$$

Thus for each $k\in\mathbb{N}$ we have, taking the coefficient of t^k, the operation

$$\lambda^k\colon KX \longrightarrow KX$$

in K-theory. To the class of a vector bundle in $\operatorname{Vect} X$ it simply applies the k-th exterior power functor.

Definition If $E\to X$ is a vector bundle we let $\operatorname{rank} E \in \operatorname{Vect} X$ be the bundle that on each connected component of X restricts to the trivial bundle of the same rank as $E|X$. This notion at once extends to virtual bundles $x\in KX$, and we define an operation Ψ^0 putting

$$\psi^0(x) = \operatorname{rank} x\,.$$

We proceed to define an infinite series of operations ψ^k: in view of $\lambda_t\in 1+(t)\cdot KX\,[[t]]$ for each $x\in KX$ there is a well-defined power series

$$\frac{d}{dt}\log\lambda_{-t}(x) = \frac{1}{\lambda_{-t}(x)}\cdot\frac{d}{dt}\lambda_{-t}(x) = \sum_{k=0}^{\infty}\big(1-\lambda_{-t}(x)\big)^k\cdot\frac{d}{dt}\lambda_{-t}(x)\,,$$

and we put

$$\psi_t(x) = \sum_{k=0}^{\infty}\psi^k(x)\,t^k := \psi^0(x) - t\,\frac{d}{dt}\log\lambda_{-t}(x) \in KX\,[[t]]\,.$$

Thus for each $k\in\mathbb{N}$ we obtain an operation

$$\psi^k\colon KX \longrightarrow KX$$

in K-theory called the k-th *Adams operation*.

Proposition The Adams operations have the following properties for all $x,y\in KX$.

- $\psi^k(x+y) = \psi^k(x)+\psi^k(y)$ for all $k\in\mathbb{N}$.
- If x is the class of a line bundle then $\psi^k(x) = x^k$.
- $\psi^k(x\cdot y) = \psi^k(x)\cdot\psi^k(y)$ for all $k\in\mathbb{N}$.
- $\psi^k\left(\psi^l(x)\right) = \psi^{kl}(x)$ for all $k,l\in\mathbb{N}$.
- If $p\in\mathbb{N}$ is prime then $\psi^p(x)\equiv x^p \bmod p$.
- If $x\in\tilde{K}S^{2n}$ then $\psi^k(x) = k^n\cdot x$ for all $k\in\mathbb{N}$.

Remarks We will see in the proof that already the first two properties determine the operations ψ^k uniquely, a fact that offers a certain compensation for the apparent lack of motivation in the definition. — The forth rule implies of course that the Adams operations commute among themselves. — Comparing properties we see that ψ^{-1} would have been a consistent notation for the operation $x\mapsto\overline{x}$ considered above.

Proof Let E and F be two vector bundles on X. We know that $\lambda_{-t}[E \oplus F] = \lambda_{-t}[E]\cdot\lambda_{-t}[E]$ and therefore $\log\lambda_{-t}[E \oplus F] = \log\lambda_{-t}[E] + \log\lambda_{-t}[E]$, so that the additivity of ψ^k for $k > 0$ follows from that of differentiation. For $k = 0$ it is obvious.

If $L \to X$ is a line bundle then $\lambda_{-t}([L]) = 1 - [L]\,t$, and we obtain

$$\sum_{k=0}^{\infty} \psi^k[L]\,t^k = 1 - t\,\frac{d}{dt}\log(1-[L]\,t) = 1 + \frac{[L]\,t}{1-[L]\,t} = \frac{1}{1-[L]\,t} = \sum_{k=0}^{\infty} [L]^k\,t^k\,.$$

This shows that $\psi^k(x) = x^k$ for the class of any line bundle $x \in KX$. If η is an arbitrary additive operation in K-theory with this property then η and ψ^k coincide on sums of line bundles, and thus on the class of every vector bundle on X, in view of the splitting principle and the note above. They must therefore altogether coincide.

The identity $\psi^k(xy) = \psi^k(x)\cdot\psi^k(y)$ is now obvious if x is the class of a line bundle. If $x_1,\dots,x_d \in KX$ and $y_1,\dots,y_e \in KX$ are such classes then using additivity we also have

$$\psi^k\Big(\sum_i x_i\cdot\sum_j y_j\Big) = \psi^k\Big(\sum_{i,j} x_i y_j\Big) = \Big(\sum_{i,j} x_i y_j\Big)^k = \Big(\sum_i x_i\Big)^k\cdot\Big(\sum_j y_j\Big)^k = \psi^k\Big(\sum_i x_i\Big)\cdot\psi^k\Big(\sum_j y_j\Big)\,.$$

Since the identity holds for sums of line bundles it holds in general, again by the splitting principle.

The same argument works to prove $\psi^k\left(\psi^l(x)\right) = \psi^{kl}(x)$, calculating

$$\psi^k\Big(\psi^l\Big(\sum_i x_i\Big)\Big) = \sum_i \psi^k\left(\psi^l(x_i)\right) = \sum_i \psi^{kl}(x) = \psi^{kl}\Big(\sum_i x_i\Big)\,.$$

Let $p \in \mathbb{N}$ be prime. If $x_1,\dots,x_d \in KX$ represent line bundles then

$$\psi^p\Big(\sum_i x_i\Big) = \sum_i x_i^p \equiv \Big(\sum_i x_i\Big)^p \bmod p\,,$$

and by the splitting principle this implies the congruence $\psi^p(x) \equiv x^p \bmod p$ for general $x \in KX$.

Finally we know that for $n > 0$ the additive group $\tilde{K}S^{2n}$ is spanned by the n-fold external product

$$(h-1)\times\cdots\times(h-1) \in K(S^2\times\cdots\times S^2)$$

where $h \in KS^2$ denotes the hyperplane class as usual. Applying ψ^k sends each factor $h-1$ to

$$\psi^k(h-1) = h^k - 1 = \Big(\sum_{i=0}^{k-1} h^k\Big)\cdot(h-1) = k\cdot(h-1)$$

where the latter identity results from the congruence $h \equiv 1 \bmod (h–1)$. We conclude that $\psi^k(x) = k^n\,x$ holds for all $x \in \tilde{K}S^{2n}$ and $n > 0$. For $n = 0$ we are dealing with bundles over the one-point space, and ψ^k acts identically for all k.

Example (4) The last stated property of the Adams operations clearly shows that two spheres of different even dimensions are not homotopy equivalent, in particular not homeomorphic. This result is immediately extended to include spheres of arbitrary dimensions using $K^1S^n = \tilde{K}^0S^{n+1}$. It also implies that $\mathbb{R}^m$ and $\mathbb{R}^n$ cannot be homeomorphic unless $m \neq n$.

12 Division Algebras

Definition Let K be a field. In this section an *algebra* over K is a vector space A over K together with a multiplication $A\times A\to A$ which is required to be K-bilinear:

$$(\lambda x+\mu y)\cdot z=\lambda(x\cdot z)+\mu(y\cdot z)\quad\text{and}\quad x\cdot(\mu y+\nu z)=\mu(x\cdot y)+\nu(x\cdot z)$$

as well as

$$(\lambda x)\cdot y=\lambda(x\cdot y)=x\cdot(\lambda y)$$

must hold for all $\lambda,\mu,\nu\in K$ and all $x,y,z\in A$. Thus neither associativity nor commutativity, nor the existence of a unit is required.

If, on the other hand, A is not the zero algebra, and for any choice of $a,b\in A$ with $a\neq 0$ the equations $a\cdot x=b$ and $y\cdot a=b$ have unique solutions $x,y\in A$ then A is called a *division algebra*.

Note Given an element a of an algebra A we denote by

$$l_a\colon A\to A\quad\text{and}\quad r_a\colon A\to A$$

the K-linear mappings $x\mapsto l_a(x)=a\,x$ and $x\mapsto r_a(x)=x\,a$: the condition for A to be a division algebra is that these maps are bijective for all non-zero $a\in A$. In case A has finite dimension over K it suffices to establish either injectivity or surjectivity of l_a and r_a.

Examples of Real Division Algebras (1) The field $\mathbb{R}$ itself is the trivial example.

(2) The field $\mathbb{C}$ is well-known to be the only finite-dimensional real division algebra which is associative and commutative.

(3) The skew field $\mathbb{H}$ of *quaternions*, discovered by Hamilton in 1843, admits various realisations, including one as a subring of $\mathrm{Mat}(2\times 2,\mathbb{C})$:

$$\mathbb{H}=\big\{\lambda+ih\in\mathrm{Mat}(2\times 2,\mathbb{C})\,\big|\,\lambda\in\mathbb{R}\text{ and }h\text{ hermitian}\big\}\,.$$

The scalar matrix λ is sometimes called the *scalar part*, and ih the *vectorial part* of the quaternion. More explicitly, in terms of the Pauli matrices

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad\text{and}\quad\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

the matrices 1, $-i\sigma_x$, $-i\sigma_y$, and $-i\sigma_z$ form a basis of $\mathbb{H}$ as a real vector space. The latter three quaternions are traditionally denoted by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ respectively, and the formulae

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{i}\,\mathbf{j}\,\mathbf{k}=-1$$

completely describe the multiplication in $\mathbb{H}$.

Associated with any quaternion $w=\lambda+ih=\lambda+\mu_x\mathbf{i}+\mu_y\mathbf{j}+\mu_z\mathbf{k}$ there is the *conjugate quaternion*

$$w^*=\lambda-ih=\lambda-\mu_x\mathbf{i}-\mu_y\mathbf{j}-\mu_z\mathbf{k}\,.$$

Conjugation is an anti-involution: $(vw)^* = w^* \cdot v^*$. It leads in turns to the norm

$$|w| = \sqrt{w^* w} = \sqrt{\lambda^2 + \mu_x^2 + \mu_y^2 + \mu_z^2}\,.$$

As in the case of complex numbers the norm is multiplicative, and the inverse of a non-zero quaternion w turns out to be

$$w^{-1} = \frac{w^*}{|w|^2}\,.$$

The impression that quaternions might be after all be not unlike complex numbers — apart from commutativity — is correct in some formal respects but quite erroneous in others. It is true that much of elementary linear algebra works well over $\mathbb{H}$ and other skew fields if the necessary care is taken. For instance in order to preserve matrix calculus one should work with right rather than left vector spaces, so that the compatibility of skew field and scalar multiplications reads $v(\lambda\mu) = (v\lambda)\mu$ for vectors v and scalars λ and μ. There also is a good theory of $\mathbb{H}$-linear group representations, and there are projective spaces over $\mathbb{H}$. On the other hand a striking feature is that every purely vectorial quaternion w of unit norm satisfies $w^2 = -1$ and together with the real number 1 spans an embedded copy of the complex field, with w playing the role of the imaginary unit: thus the skew field extension $\mathbb{H}/\mathbb{R}$ of finite degree 4 contains an infinite variety of intermediate fields which are all conjugate to each other.

(4) The algebra $\mathbb{O}$ of *octonians* was discovered by Graves shortly after Hamilton's description of the quaternions, and independently in 1845 by Cayley. Not being an associative algebra, $\mathbb{O}$ cannot be embedded in a matrix algebra, and the simplest description seems to be in terms of pairs of quaternions: $\mathbb{O} = \mathbb{H}^2$ with the multiplication

$$(u, v) \cdot (w, z) = (uw - vz^*,\, uz + vw^*)\,.$$

Again we have a conjugation $(u, v) \mapsto (u^*, -v)$ which is an anti-involution and provides the norm, which in turn allows to write down the inverse of any non-zero octonian with respect to the unit $(1, 0) \in \mathbb{O}$.

Needless to say, due to the lack of associativity calculations in $\mathbb{O}$ require extreme care.

(5) Division algebras of infinite dimension abound. They include the fields $\mathbb{R}(\!(t)\!)$ of Laurent series

$$\sum_{k=d}^{\infty} a_k\, t^k \quad \text{with } d \in \mathbb{Z} \text{ and } a_k \in \mathbb{R}$$

and $\mathbb{R}(t)$ of rational functions

$$\frac{a(t)}{b(t)} \quad \text{with polynomials } a(t) \in \mathbb{R}[t] \text{ and } 0 \neq b(t) \in \mathbb{R}[t]\,.$$

After the discoveries of Hamilton, Graves, and Cayley various questions of existence and classification of division algebras with prescribed properties have been studied. We will concentrate on the particular question of whether given a number $n \in \mathbb{N}$ there exists at least one real division algebra A with $\dim_{\mathbb{R}} A = n$. Surprisingly this problems turns out to have a purely topological answer.

As a first step towards topology we rewrite the algebra multiplication as a map between spheres. Throughout we assume that $A = \mathbb{R}^n$ as a vector space, and we consider the sphere $S^{n-1} \subset A$ with respect to the standard euclidean norm. Correcting for the fact that the multiplication of A has no reason to respect this norm — or any other for that matter — we define the map $\mu_A \colon S^{n-1} \times S^{n-1} \to S^{n-1}$ by

$$\mu_A(u, v) = \frac{1}{|u \cdot v|}\, u \cdot v\,;$$

it is well-defined since A has no zero divisors. This topological version of the algebra multiplication still has the property that for every $a \in S^{n-1}$ the mapping $S^{n-1} \ni v \mapsto \mu_A(a, v) \in S^{n-1}$ is a homeomorphism. Indeed, it is inverted sending $w \in S^{n-1}$ to $|l_a^{-1}(w)|^{-1}\, l_a^{-1}(w)$. Of course the same is true of $u \mapsto \mu_A(u, b)$ for fixed $b \in S^{n-1}$.

Proposition Let A be a finite-dimensional real division algebra. Then either A is isomorphic to $\mathbb{R}$ itself, or $\dim_{\mathbb{R}} A$ is even.

Proof If $\dim_{\mathbb{R}} A = 1$ then choosing any non-zero $a \in A$ we find an element $e \in A$ such that $a \cdot e = a$. Writing $a = \lambda e$ with $\lambda \in \mathbb{R}^*$ we conclude $\lambda e^2 = \lambda e$: thus e is idempotent, and the linear map sending $1 \in \mathbb{R}$ to $e \in A$ an isomorphism of algebras.

We now make the assumption that $\dim_{\mathbb{R}} A$ is odd, say $\dim A = 2n+1$ with $n \in \mathbb{N}$, and will show that this leads to a contradiction. We consider the homomorphism in K-theory $\mu_A^*\colon KS^{2n} \to K(S^{2n} \times S^{2n})$ that is induced by the map μ_A. In terms of the known description of KS^{2n} it is a ring homomorphism

$$\mathbb{Z}[z]/(z^2) \longrightarrow \mathbb{Z}[x]/(x^2) \otimes_{\mathbb{Z}} \mathbb{Z}[y]/(y^2) = \mathbb{Z}[x, y]/(x^2, y^2)$$

where x, y, and z are three copies of the canonical generator $(h-1) \times \cdots \times (h-1) \in \tilde{K}S^{2n}$. Fixing any $a \in S^{2n}$ the restriction homomorphism $K(S^{2n} \times S^{2n}) \to K\big(\{a\} \times S^{2n}\big)$ simply annihilates y, and the fact that $S^{2n} \ni v \mapsto \mu(a, v) \in S^{2n}$ is a homeomorphism implies that the composition

$$\mathbb{Z}[z]/(z^2) \xrightarrow{\mu_A} \mathbb{Z}[x, y]/(x^2, y^2) \longrightarrow \mathbb{Z}[x, y]/(x^2, y) = \mathbb{Z}[x]/(x^2)$$

is an isomorphism. In terms of the generator z this means that $\mu_A^*(z) \equiv \pm y \bmod (x)$, and by way of symmetry we also have $\mu_A^*(z) \equiv \pm x \bmod (y)$. We conclude that the image of z has the form

$$\mu_A^*(z) = \pm x \pm y + k\, xy \in \mathbb{Z}[x, y]/(x^2, y^2) \quad \text{for some } k \in \mathbb{Z}.$$

Since none of these elements has vanishing square this contradicts the fact that μ_A^* is a homomorphism of rings.

Thus odd dimensional division algebras are essentially ruled out. The case of a real division algebra A of even dimension $2n$ is decidedly much subtler. We first pass from the map μ_A to another mapping

$$\varphi_A\colon S^{4n-1} \longrightarrow S^{2n}$$

which imitates the Hopf fibration $S^3 \ni z \mapsto [z] \in \mathbb{C}P^1 = S^2$ from Note (4) of Section 2. It is convenient to use homeomorphisms similar to those at the beginning of Section 9 to identify the domain S^{4n-1} of φ_A with the subspace

$$(S \times D) \cup (D \times S) \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n} = \mathbb{R}^{4n}$$

where we have abbreviated $D := D^{2n}$ and $S := S^{2n-1}$. On the target side of the mapping φ_A we identify S^{2n} with the one-point compactification of $A = \mathbb{R}^{2n}$ and write $D_0 := D^{2n} \subset S^{2n}$ and $D_\infty := S^{2n} \setminus (D_0)^\circ$. The map $\varphi_A\colon S^{4n-1} \longrightarrow S^{2n}$ then is defined by the assignments

$$S \times D \ni (u, v) \longmapsto |v| \cdot \mu_A\Big(u, \frac{1}{|v|}\, v\Big) \in D_0$$

$$D \times S \ni (u, v) \longmapsto \frac{1}{|u|} \cdot \mu_A\Big(\frac{1}{|u|}\, u, v\Big) \in D_\infty\,.$$

Note that for fixed $a, b \in S$ restricting φ_A yields homeomorphisms

$$D = \{a\} \times D \xrightarrow{\approx} D_0 \quad \text{and} \quad D = D \times \{b\} \xrightarrow{\approx} D_\infty$$

which are inverted sending $z \in D_0$ to $|z| \cdot v$ and $z \in D_\infty$ to $\frac{1}{|z|} \cdot u$ if $u, v \in S$ are the unique points with $\mu_A(a, v) = \frac{1}{|z|} \cdot z$ respectively $\mu_A(u, b) = \frac{1}{|z|} \cdot z$.

Definition Let $n \in \mathbb{N}$ be a positive integer and $f\colon S^{4n-1} \to S^{2n}$ a map. The reduced K-theory exact sequence of f then runs

$$\tilde{K}S^{4n-1} \longleftarrow \tilde{K}S^{2n} \longleftarrow \tilde{K}Cf \longleftarrow K^1S^{4n-1} \longleftarrow K^1S^{2n}$$

or

$$0 \longleftarrow (y)/(y^2) \longleftarrow \tilde{K}Cf \longleftarrow (x)/(x^2) \longleftarrow 0$$

where $x \in \mathbb{Z}[x] = K\Sigma S^{4n-1}$ and $y \in \mathbb{Z}[y] = KS^{2n}$ are the canonical generators of reduced K-theory. The short exact sequence shows that $\tilde{K}Cf$ is the free abelian group generated by x and a lift $\tilde{y}$ of y, and since $y^2 = 0 \in \tilde{K}S^n$ the square of $\tilde{y}$ must be an integral multiple

$$\tilde{y}^2 = H(f)\cdot x$$

of x. The product $x\cdot\tilde{y}$ likewise is a multiple of x, say $x\cdot\tilde{y} = \lambda x$ with $\lambda \in \mathbb{Z}$, but calculating

$$0 = x \cdot H(f)\,x = x\cdot\tilde{y}^2 = (x\cdot\tilde{y})\cdot\tilde{y} = \lambda x\cdot\tilde{y} = \lambda^2 x$$

we see that λ must vanish and that $x\cdot\tilde{y} = 0$. This implies that the integer $H(f)$ does not depend on the choice of the lift $\tilde{y}$ and thus is uniquely defined by f alone — in fact even by the homotopy class of f. It is called the *Hopf invariant* of f.

Proposition Let A be a real division algebra of even vector space dimension $2n \in \mathbb{N}$. Then the corresponding map $\varphi = \varphi_A\colon S^{4n-1} \longrightarrow S^{2n}$ has Hopf invariant $H(\varphi) = \pm 1$.

Proof Recalling the abbreviations $D := D^{2n}$ and $S := S^{2n-1}$ we note that there is a well-defined mapping $\Phi\colon D\times D \to C\varphi$ that sends (u,v) to the class

$$\Big[m,\ \frac{1}{m}\,(u,v)\Big] \in C\varphi = \big(I\times(D\times D) + S^{2n}\big)\,/\sim$$

with $m = \max\{|u|,|v|\}$ the "box" norm of (u,v). As the latter takes the value 1 on the boundary $(S\times D)\cup(D\times S)$ the restriction $\Phi\big|\big((S\times D)\cup(D\times S)\big)$ is the composition of φ with the canonical embedding $S^{2n} \to C\varphi$.

We choose and fix arbitrary points $a, b \in S$. As observed above φ restricts to homeomorphisms $D = \{a\}\times D \xrightarrow{\approx} D_0$ and $D = D\times\{b\} \xrightarrow{\approx} D_\infty$, so that the curved arrows in

$$\begin{array}{ccc} (D,S) & \qquad & (D,S) \\ \big\downarrow{\scriptstyle(\mathrm{id},b)} & & \big\downarrow{\scriptstyle(a,\mathrm{id})} \\ (D,S)\times D & & D\times(D,S) \\ \big\downarrow{\scriptstyle\Phi} & & \big\downarrow{\scriptstyle\Phi} \\ (C\varphi, D_0) & & (C\varphi, D_\infty) \end{array}$$

embed the space pair (D,S) as $(D_\infty, D_0\cap D_\infty)$ respectively $(D_0, D_0\cap D_\infty)$ in $S^{2n} \subset C\varphi$. The induced diagrams in K-theory enter in the upper left hand part of the large commutative diagram

$$\begin{array}{ccc} K(D,S)\otimes K(D,S) & \xrightarrow{\ \simeq\ } & K\big((D,S)\times(D,S)\big) \\ \big\uparrow{\scriptstyle\simeq} & & \big\| \\ K\big((D,S)\times D\big)\otimes K\big(D\times(D,S)\big) & \longrightarrow & K\big((D,S)\times(D,S)\big) \\ \big\uparrow{\scriptstyle\Phi^*\otimes\Phi^*} & & {\scriptstyle\simeq}\big\uparrow{\scriptstyle\Phi^*} \\ K(C\varphi, D_0)\otimes K(C\varphi, D_\infty) & \longrightarrow & K(C\varphi, S^{2n}) \\ \big\downarrow{\scriptstyle\simeq} & & \big\downarrow \\ \tilde{K}C\varphi\otimes\tilde{K}C\varphi & \longrightarrow & \tilde{K}C\varphi \end{array}$$

with horizontals which are K-theory product maps: at the top we have an exterior product while the others are interior ones. The arrows marked as such really are isomorphisms since cartesian projection gives homotopy inverses to the embeddings (a, id) and (id, b), and collapsing the contractible subspaces D_0 and D_∞ has no effect on K-theory. On the right hand side Φ^* is an isomorphism because collapsing $S^{2n} \subset C\varphi$ to a point results in the suspension ΣS^{4n-1}.

The element $\tilde{y} \in \tilde{K}C\varphi$ whose square defines the Hopf invariant, by definition restricts to a generator of $\tilde{K}S^{2n}$, which in turn corresponds to generators of

$$\tilde{K}(S^{2n}, D_0) = \tilde{K}(D_\infty, S) \quad \text{and} \quad \tilde{K}(S^{2n}, D_\infty) = \tilde{K}(D_0, S)\,.$$

Thus the tensor square $\tilde{y} \otimes y \tilde{\in} \tilde{K}C\varphi \otimes \tilde{K}C\varphi$ corresponds to a tensor product $z' \otimes z''$ of generators of the cyclic group $K(D, S)$. It follows that $z' \cdot z'' \in K\big((D, S) \times (D, S)\big)$ also generates and so maps to a generator of $K(C\varphi, S^{2n})$. This means that $\tilde{y}^2 = \pm x \in \tilde{K}C\varphi$, and finally that $H(\varphi) = \pm 1$.

Theorem Let $n \in \mathbb{N}$ be positive and $f\colon S^{4n-1} \to S^{2n}$ a mapping. If $H(f)$ is odd then $n \in \{1, 2, 4\}$.

Proof Recalling the short exact sequence

$$0 \longleftarrow (y)/(y^2) \longleftarrow \tilde{K}Cf \longleftarrow (x)/(x^2) \longleftarrow 0$$

from the definition of the Hopf invariant let $\tilde{y} \in \tilde{K}Cf$ be a lift of y. By the properties of the Adams operations we have $\psi^k(y) = k^n \cdot y$ and therefore

$$\psi^2(\tilde{y}) = \lambda \cdot x + 2^n \cdot \tilde{y} \quad \text{and} \quad \psi^3(\tilde{y}) = \mu \cdot x + 3^n \cdot \tilde{y}$$

with integers λ and μ. Applying the Adams operations once more we obtain the competing representations

$$\psi^6(\tilde{y}) = \psi^3\big(\psi^2(\tilde{y})\big) = \psi^3(\lambda \cdot x + 2^n \cdot \tilde{y}) = (3^{2n}\lambda + 2^n\mu) \cdot x + 6^n \cdot \tilde{y}$$
$$\psi^6(\tilde{y}) = \psi^2\big(\psi^3(\tilde{y})\big) = \psi^2(\mu \cdot x + 3^n \cdot \tilde{y}) = (2^{2n}\mu + 3^n\lambda) \cdot x + 6^n \cdot \tilde{y}\,.$$

We conclude that $3^{2n}\lambda + 2^n\mu = 2^{2n}\mu + 3^n\lambda$, or

$$2^n(2^n - 1)\mu = 3^n(3^n - 1)\lambda.$$

By another property of the Adams operations

$$H(f) \cdot x = \tilde{y}^2 \equiv \psi^2(\tilde{y}) \bmod 2,$$

and since the Hopf invariant is odd and $n > 0$ the coefficient λ must be odd too. Therefore 2^n must be a divisor of $3^n - 1$.

The rest is elementary number theory. We first show that n must be a power of 2. To this end we write $n = 2^l\, m$ with m odd. Thus $2^{2^l m} = (2^m)^{2^l}$ is a divisor of

$$3^{2^l m} - 1 = (3^{2^l} - 1) \cdot \sum_{j=0}^{m-1} 3^{2^l j}\,.$$

Since the sum is odd this implies $(2^m)^{2^l} \,\big|\, (3^{2^l} - 1)$, in particular $2^m < 3$ and therefore $m = 1$, that is $n = 2^l$.

Next we show by induction that for $l > 0$ the highest power of 2 that divides $3^{2^l} - 1$ is 2^{l+2}. Indeed for $l = 1$ this is true since $2^{1+2} = 8 = 3^{2^1} - 1$. Assuming the statement for fixed l we write

$$3^{2^{l+1}} - 1 = (3^{2^l} - 1)(3^{2^l} + 1)\,.$$

Since $l \geq 1$ the second factor satisfies $3^{2^l}+1 = 9^{2^{l-1}}+1 \equiv 2 \bmod 4$, and we conclude that 2^{k+1} divides $(3^{2^{l+1}}-1)$ if and only if 2^k divides $(3^{2^l}-1)$. By the inductive hypotheses this implies that 2^{l+1} is the highest power of 2 contained in $(3^{2^{l+1}}-1)$: this completes the induction.

Summarising, as we know that $n = 2^l$ and $2^{2^l} \mid (3^{2^l}-1)$ we have either $l = 0$, or $l > 0$ and $2^l \leq l+2$, which means $l \in \{1, 2\}$. Thus finally $n = 2^l \in \{1, 2, 4\}$.

Corollary A finite-dimensional real division algebra of dimension n exists if and only if $n \in \{1, 2, 4, 8\}$.

Proof By our first proposition such an algebra A is either isomorphic to $\mathbb{R}$, or n is even. In that case by the second proposition the map μ_A has Hopf invariant 1, and by the theorem this implies $n \in \{2, 4, 8\}$. Conversely the examples show that all four dimensions are realised.

One will naturally wonder whether the use of the Adams operations in the proof of the theorem may fit into a systematic context. Indeed it does, though here we cannot go beyond the definition of an interesting new invariant.

Definition Let m and n be positive integers, and let $f\colon S^{2m+2n-1} \to S^{2n}$ be a map. As before consider the exact sequence of the mapping cone

$$0 \longleftarrow (y)/(y^2) \longleftarrow \tilde{K}Cf \longleftarrow (x)/(x^2) \longleftarrow 0$$

with generators $x \in \tilde{K}S^{2m+2n}$ and $y \in \tilde{K}S^{2n}$, and let $\tilde{y} \in \tilde{K}Cf$ be a lift of y. For every $k \in \mathbb{N}$ we have an equation

$$\psi^k(\tilde{y}) = \lambda_k \cdot x + k^n \cdot \tilde{y} \in \tilde{K}Cf$$

for some $\lambda_k \in \mathbb{Z}$. If $l \in \mathbb{N}$ is a second choice for k then using $\psi^l \circ \psi^k = \psi^{kl} = \psi^k \circ \psi^l$ we see that

$$\psi^{kl}(\tilde{y}) = \psi^l\big(\psi^k(\tilde{y})\big) = \psi^l(\lambda_k \cdot x + k^n \cdot \tilde{y}) = (l^{m+n}\lambda_k + k^n\lambda_l) \cdot x + k^n l^n \cdot \tilde{y}$$
$$\psi^{kl}(\tilde{y}) = \psi^k\big(\psi^l(\tilde{y})\big) = \psi^k(\lambda_l \cdot x + l^n \cdot \tilde{y}) = (k^{m+n}\lambda_l + l^n\lambda_k) \cdot x + k^n l^n \cdot \tilde{y}$$

and thus $k^n(k^m-1)\lambda_l = l^m(l^n-1)\lambda_k$. Therefore the rational number

$$e(f) := \frac{\lambda_k}{k^n(k^m-1)}$$

does not depend on the choice of $k > 0$. On the other hand it does depend on that of the lift $\tilde{y}$: substituting $\mu \cdot x + \tilde{y}$ as an alternative choice for $\tilde{y}$ we obtain

$$\psi^k(\mu x+\tilde{y}) = \lambda_k \cdot x + k^{m+n}\mu \cdot x + k^n \cdot \tilde{y} = \lambda_k \cdot x + (k^{m+n}-k^n)\mu \cdot x + k^n \cdot (\mu x+\tilde{y})\,.$$

Thus the integer μ is added to $e(f) \in \mathbb{Q}$, and we see that the congruence class

$$e(f) \in \mathbb{Q}/\mathbb{Z}$$

is a well-defined invariant of f alone. It is called the *e-invariant* of f — in fact of the homotopy class $[f] \in \big[S^{2m+2n-1}, S^{2n}\big]$.

13 Operations II

One might think of a very simple structure on the set KX that would specify exactly which classes can be represented by true vector bundles on X, or, better, for a given $x \in KX$ the smallest number $n \in \mathbb{N}$ that makes $x+d$ the class of a vector bundle. This idea seems far away from any algebraic notion but in fact is not, but rather related to an observation we might have made on the (few) examples of rings KX that we know like $K\mathbb{C}P^n = \mathbb{Z}[h]/(h-1)^{n+1}$: they suggest that virtual bundles of rank zero — here $h-1$ — should be given a more prominent role. We do this in a systematic way.

Notation We let

$$K_1X = \text{kernel rank} \subset KX$$

denote the kernel of the rank homomorphism rank: $KX \longrightarrow KX$. Thus K_1X is an ideal of KX.

Definition The geometric series

$$\frac{t}{1-t} = \sum_{i=1}^{\infty} t^i \in \mathbb{Z}[[t]]$$

of vanishing constant term may be substituted in $\lambda_t(x) \in KX[[t]]$ and yields the new series

$$\sum_{k=0}^{\infty} \gamma^k(x)\, t^k = \gamma_t(x) := \lambda_{\frac{t}{1-t}}(x) \in KX[[t]]$$

and thereby the K-theory operations

$$\gamma^k: KX \longrightarrow KX \quad \text{for all } k \in \mathbb{N}.$$

Notes (1) Assigning to $x \in KX$ the difference $x - \text{rank}\, x$ we obtain an additive endomorphism of KX that projects KX onto its subgroup K_1X, which therefore is a direct factor of KX. For connected non-empty spaces X it coincides with $\tilde{K}X$.

(2) The series γ_t clearly inherits from $\lambda_t(x)$ the property

$$\gamma_t(x+y) = \gamma_t(x) \cdot \gamma_t(y) \quad \text{for all } x, y \in KX.$$

(3) Each γ^k is an integral linear combination of the λ^j with $j \le k$, and since the relation $s = \frac{t}{1-t}$ is inverted by $t = \frac{s}{1+s}$ we have $\lambda_s = \gamma_{\frac{s}{1+s}}$, so that conversely each λ^k is an integral linear combination of the γ^j with $j \le k$.

(4) For the trivial bundle we have

$$\gamma_t(1) = \lambda_{\frac{t}{1-t}}(1) = 1 + \frac{t}{1-t} = \frac{1}{1-t}$$

so that for an arbitary line bundle $L \to X$ we obtain

$$\gamma_t([L]-1) = \Big(1 + [L]\,\frac{t}{1-t}\Big) \cdot \Big(\frac{1}{1-t}\Big)^{-1} = 1 + ([L]-1)\, t\,.$$

Proposition Consider any $x \in K_1X$.

- Then $\gamma^k(x) \in K_1X$ for all $k > 0$, and $\gamma^1(x) = x$.
- Let $d \in \mathbb{N}$ be the smallest number such that $x{+}d \in KX$ is represented by a vector bundle. Then $\gamma_t(x)$ is a polynomial of rank at most d.

Proof We have just seen that the first statement holds for arguments of the form $x = [L]{-}1$. If more generally $x = \sum_{i=1}^d x_i$ is such that $x_i{+}1$ is the class of a line bundle for each i, then the formula $\gamma_t(y{+}z) = \gamma_t(y)\cdot\gamma_t(z)$ shows that $\gamma^k(x) \in K_1$ in this case too, and also that $\gamma^1(x) = x$ since the coefficient $\gamma^1(x)$ is additive in x.

Let now $x \in K_1X$ be general, and assume that $x = [E]{-}d$ for some vector bundle $E \to X$. By the splitting principle we may assume that $E \simeq \bigoplus_{i=0}^d L_i$ is isomorphic to a Whitney sum af line bundles, so that $x = \sum_i ([L_i]{-}1)$. Thus the first assertion of the proposition is true for x, and the second follows from

$$\gamma_t(x) = \prod_{i=1}^d (1{+}x_i\,t) = 1 + \sum_{i=0}^d \sigma_i(x_1,\ldots,x_d)\,t^i\,.$$

Definition We formally assign to each operation γ^k the weight $k \in \mathbb{N}$, and therefore to each product

$$\gamma^{k_1}\gamma^{k_2}\,\cdots\,\gamma^{k_r}$$

(repeated factors allowed) the weight $k_1{+}k_2{+}\cdots{+}k_r$. Given a space X we put $K_0X = KX$, and for $s \in \mathbb{N}$ let $K_sX \subset KX$ be the additive subgroup generated by all products

$$\gamma^{k_1}(x)\,\gamma^{k_2}(x)\,\cdots\,\gamma^{k_r}(x) \quad \text{with } x \in K_1X$$

of weight at least s: the definition is consistent since $\gamma^1(x) = x$ for all $x \in K_1X$. The decreasing sequence

$$KX = K_0X \supset K_1X \supset \;\cdots\; \supset K_sX \supset K_{s+1}X \supset \;\cdots$$

is called the γ-*filtration* of KX.

Theorem Assume that $x \in K_1X$. There exists a number $m \in \mathbb{N}$ such that $\gamma^{k_1}(x)\,\gamma^{k_2}(x)\,\cdots\,\gamma^{k_r}(x) = 0$ for all products $\gamma^{k_1}\gamma^{k_2}\,\cdots\,\gamma^{k_r}$ of weight greater than m.

Proof We first assume that $x = [L]{-}1$ with a line bundle $L \to X$, and write $-x = [E]{-}d$ for some vector bundle $E \to X$. By the splitting principle we may assume that $E = \bigoplus_{i=1}^d L_i$ is a sum of line bundles, so that $-x = \sum_i y_i$ with $y_i = [L_i]{-}1$.

We then have $\gamma_t(x) = 1 + x\,t$ and $\gamma_t(y_i) = 1 + y_i\,t$, and therefore

$$1 = \gamma_t(x)\cdot\gamma_t(-x) = (1 + x\,t)\cdot\prod_{i=1}^d (1 + y_i\,t) = (1 + x\,t)\cdot\Big(1 + \sum_{i=1}^d \sigma_i(y_1,\ldots,y_d)\,t^i\Big).$$

Equating coefficients we successively obtain that $x^i = (-1)^i\sigma_i(y_1,\ldots,y_d)$ for all $i > 0$, in particular $x^{d+1} = 0$. This was the claim for x of this special type.

Let now $x \in K_1X$ be arbitrary. Again by the splitting principle we may assume that $x = \sum_{i=1}^d x_i$ with $x_i = [L_i]{-}1$; then

$$\gamma_t(x) = \prod_{i=1}^d (1 + x_i\,t) = 1 + \sum_{i=1}^d \sigma_i(x_1,\ldots,x_d)\,t^i$$

and thus $\gamma^i(x) = \sigma_i(x_1,\ldots,x_d)$ for all $i > 0$. Using the special case already treated, we now choose $m_i \in \mathbb{N}$ such that $x_i^{m_i+1} = 0$ for all i. The expression $\gamma^{k_1}(x)\,\gamma^{k_2}(x)\,\cdots\,\gamma^{k_r}(x) \in KX$ is a homogeneous polynomial of degree $k_1{+}k_2{+}\cdots{+}k_r$ in the x_i: if this degree exceeds $m := m_1{+}\cdots{+}m_n$ then one factor must vanish, and the whole expression is zero.

Remark The proposition does not lie as deep as the splitting principle: the proof may be based on a formal algebraic rather than a true splitting of the bundles involved.

Corollary Let X be a space and $x \in K_1X$. Then x is nilpotent.

Proof Choose $k_1 = \cdots = k_r = 1$ and recall that $\gamma^1(x) = x$.

Theorem Let X be a space such that KX is finitely generated as an abelian group. Then there exists an $m \in \mathbb{N}$ such that $K_{m+1}X = \{0\}$.

Proof Being a direct summand K_1X also is finitely generated, say by $x_1, \ldots, x_r$. We put $x_{r+j} = -x_j$ for $j = 1, \ldots, r$ and apply the previous proposition to x_j for each $j = 1, \ldots, 2r$, obtaining the number $m_j \in \mathbb{N}$; we then put $m = m_1 + \cdots + m_{2r}$. Any product of weight greater than m in the $\gamma^i(x_j)$ must, for some particular j, contain a product of weight greater than m_j in the $\gamma^i(x_j)$, and therefore must vanish.

Since every element of K_1X is the sum of a collection of x_j the result now follows from the formula $\gamma_t(x+y) = \gamma_t(x) \cdot \gamma_t(y)$.

Theorem Let $s \in \mathbb{N}$ and $x \in K_sX$. Then

$$\psi^k(x) \equiv k^s \cdot x \mod K_{s+1}X$$

for every $k \in \mathbb{N}$. Thus every ψ^k preserves the γ-filtration and acts on $K_sX/K_{s+1}X$ as multiplication by k^s.

Proof In case $s = 0$ it suffices to write out

$$\psi^k(x) - k^0\, x = \psi^k(x - \operatorname{rank} x) - (x - \operatorname{rank} x)$$

and to note that ψ^k preserves the rank function.

We now fix some $x \in K_1$ and prove that $\psi^k \circ \gamma^s(x) - k^n \cdot \gamma^s(x) \in K_{s+1}X$ for every $s > 0$. By the splitting principle we may assume $x = x_1 + \cdots + x_d$ with x_i such that $1+x_i \in K_X$ is a line bundle for each i. Then $\gamma^s(x) = \sigma_s(x_1, \ldots, x_d)$ and $\psi^k(x_i) = (1+x_i)^k - 1$ and thus

$$\begin{aligned}
\psi^k \circ \gamma^s(x) &= \psi^k\big(\sigma_s(x_1, \ldots, x_d)\big) \\
&= \sigma_s\big(\psi^k(x_1), \ldots, \psi^k(x_d)\big) \\
&= \sigma_s\big((1+x_1)^k - 1,\ \ldots,\ (1+x_d)^k - 1\big) \\
&= \sigma_s(k\, x_1, \ldots, k\, x_d) + y(x_1, \ldots, x_d) \\
&= k^s \cdot \sigma_s(x_1,\ \ldots,\ x_d) + y(x_1, \ldots, x_d)
\end{aligned}$$

for some symmetric polynomial y whose monomials all have degrees greater than s. Therefore the difference

$$\psi^k \circ \gamma^s(x) - k^s \cdot \gamma^s(x) = y(x_1, \ldots, x_d)$$

is a polynomial in elementary symmetric functions $\sigma_j(x_1, \ldots, x_d)$ and thus in $\gamma^j(x)$ with $j > s$. This means that the difference belongs to $K_{s+1}X$.

Let now $\gamma^{k_1}\gamma^{k_2} \cdots \gamma^{k_r}$ any product of weight s. Since

$$\psi^k \circ \gamma^{k_j}(x) \equiv k^{k_j} \cdot \gamma^{k_j}(x) \bmod K_{k_j+1}X \quad \text{for each } j$$

and ψ^k is a ring endomorphism we conclude that

$$\psi^k\left(\gamma^{k_1}(x)\gamma^{k_2}(x) \cdots \gamma^{k_r}(x)\right) \equiv k_j^s \cdot \gamma^{k_1}(x)\gamma^{k_2}(x) \cdots \gamma^{k_r}(x) \bmod K_{s+1}X\,.$$

This proves the theorem.

Corollary Let $m \in \mathbb{N}$ and assume that $K_{m+1}X = \{0\}$. Then for any choice of non-negative integers $k_0, k_1, \ldots, k_m$ the composition product vanishes:

$$\big(\psi^{k_m} - (k_m)^m\big) \circ \cdots \circ \big(\psi^{k_1} - (k_1)^1\big) \circ \big(\psi^{k_0} - (k_0)^0\big) = 0\,.$$

Proof Apply the theorem repeatedly.

Example Recall that the ring $K\mathbb{C}P^n = \mathbb{Z}[h]/(h-1)^{n+1}$ is generated by the class h of the hyperplane bundle. The ideal $K_1\mathbb{C}P^n = \tilde{K}\mathbb{C}P^n$ is spanned by $h-1$, so that

$$K_s\mathbb{C}P^n = \begin{cases} (h-1)^s/(h-1)^{n+1} & \text{for } s \le n \\ \{0\} & \text{for } s > n \end{cases}$$

is the γ-filtration. The k-th Adams operation acts on $(h-1)^s$ by

$$\begin{aligned}\psi^k\big((h-1)^s\big) &= (h^k-1)^s \\ &= \Big(\big(1+(h-1)\big)^k - 1\Big)^s \\ &= \Big(k\cdot(h-1) + \sum_{i=2}^{k} (-1)^i \tbinom{k}{i} (h-1)^i\Big)^s \\ &= k^s\cdot(h-1)^s + \text{ terms containing higher powers of } (h-1)\,.\end{aligned}$$

Taking the tensor product with the rational field we may form a weaker version $\mathbb{Q}\otimes_{\mathbb{Z}} KX$ of K-theory, a functor with values in the category of commutative $\mathbb{Q}$-algebras, on which the Adams operations act as endomorphisms. Assuming $K_{m+1}X = \{0\}$ and putting $k_0 = k_1 = \cdots = k_m = k$ in the corollary we obtain the identity

$$\prod_{j=0}^{m}(\psi^k - k^j) = 0 \in \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}\otimes_{\mathbb{Z}} KX)$$

in the space of linear endomorphisms. Thus for $k > 0$ the endomorphism ψ^k has a minimal polynomial with but simple factors, and the eigenvalues of ψ^k are powers of k not exceeding k^m, which in fact is still true for $k = 0$ since $\psi^0 = \text{rank}$ is a projector onto K_1X. In particular ψ^k can be diagonalised, and we let $E(k,j) \subset \mathbb{Q}\otimes_{\mathbb{Z}} KX$ be the eigenspace correspondung to k^j. If we fix j and substitute a different choice $l \in \mathbb{N}$ for k then the corollary with $k_j = l$ and $k_i = k$ for all $i \neq j$ yields

$$\big(\psi^l - l^j\big)\cdot\prod_{i\neq j}\big(\psi^k - k^i\big) = 0\,.$$

For $k > 0$ this implies that $(\psi^l - l^j)|E(k,j) = 0$ and thus $E(k,j) \subset E(l,j)$. This result gives sense to the

Definition Let X be a space and assume that there exists an $m \in \mathbb{N}$ with $K_{m+1}X = \{0\}$. Then the vector spaces

$$H^{2j}(X;\mathbb{Q}) := E(k,j) \subset \mathbb{Q}\otimes_{\mathbb{Z}} KX \quad \text{for any choice of } k > 0$$

form the *rational cohomology* of even degree of X. Making the analogous assumption on the γ-filtration of the suspension ΣX^+ we complement the definition by the part of odd degree

$$H^{2j+1}(X;\mathbb{Q}) := H^{2j+2}(\Sigma X^+;\mathbb{Q})\,.$$

Note Under the assumptions of the definition we clearly have $\mathbb{Q}\otimes_{\mathbb{Z}} K_sX = \bigoplus_{j=s}^{m} H^{2j}(X;\mathbb{Q})$, and therefore

$$H^{2j}(X;\mathbb{Q}) \simeq \mathbb{Q}\otimes_{\mathbb{Z}} (K_sX/K_{s+1}X)$$

expresses the cohomology directly in terms of the γ-filtration.

Recall that the suspension isomorphism $\tilde{K}X \simeq \tilde{K}\Sigma^2 X$ is given by external left multiplication with the generator $h-1 \in \tilde{K}S^2$. Since ψ^k acts on $\tilde{K}S^2$ as multiplication by k the suspension isomorphism sends $H^{2j}(X;\mathbb{Q}) = E(k,j)$ onto $H^{2j+2}(\Sigma X^+;\mathbb{Q}) = E(k,j{+}1)$, and by definition of the odd graded cohomology we obtain suspension isomorphisms

$$H^q(X;\mathbb{Q}) \xrightarrow{\simeq} H^{q+1}(\Sigma X^+;\mathbb{Q}) \quad \text{for all } q \in \mathbb{N}.$$

The exact triangle of rational K-theory thus unwinds to the long exact cohomology sequence known to those who are familiar with algebraic topology:

Proposition For every compact space pair (X,A) such that KX and KA are finitely generated abelian groups there is a functorial exact sequence

$$\cdots \longleftarrow H^{q+1}(X,A;\mathbb{Q}) \xleftarrow{\delta} H^q(A;\mathbb{Q}) \longleftarrow H^q(X;\mathbb{Q}) \longleftarrow H^q(X,A;\mathbb{Q}) \longleftarrow \cdots$$

of rational cohomology.

14 Equivariant Vector Bundles

Definition Let G be a finite group. An *action* of G on a topological space X is a continuous mapping

$$G \times X \ni (g, x) \longmapsto g\cdot x \in X$$

with the properties that $1\cdot x = x$ and $(gh)\cdot x = g\cdot (h\cdot x)$ holds for all $g, h \in G$ and all $x \in X$. These axioms imply at once that the assignment $g \longmapsto (x \mapsto g\cdot x)$ defines a homomorphism from G to the group of self-homeomorphisms of X — this is, in fact, an alternative description of the notion of a group action.

A pair consisting of a space X and an action of G on it is called a *G-space*. For fixed G the G-spaces form a category whose morphisms are the *equivariant* maps $f\colon X \to Y$, namely those with $f(g\cdot x) = g\cdot f(x)$ for all $g \in G$ and $x \in X$.

Of course the subobjects in G-equivariant categories are the *G-stable* subsets $S \subset X$, that is, those for which $x \in S$ implies $g\cdot x \in S$ for all $g \in G$. If $S \subset X$ is any subset then

$$G\cdot S := \big\{ g\cdot x \,\big|\, g \in G \text{ and } x \in S \big\}$$

clearly is the smallest G-stable subset of X containing S. The minimal subobjects of a given G-spaces arise as the so-called *orbits* $G\cdot x := G\cdot\{x\}$ of points $x \in X$. The orbits in X define a partition of X and thus an equivalence relation on X, and the corresponding quotient space is denoted by[1] X/G and called the space of orbits, or just the *orbit space* of the G-space X.

Examples (1) The action of $\mathbb{Z}/2$ as ± 1 on the sphere S^n has $\mathbb{R}P^n$ as its orbit space by the very definition.

(2) For fixed positive $m \in \mathbb{N}$ let $\varepsilon = e^{2\pi i/m}$ denote the primitive m-th root of unity. Then for any choice of further constants $a_0, \ldots, a_n$ the group $\mathbb{Z}/m$ acts on $\mathbb{C}P^n$ by

$$g\cdot[z_0 : z_1 : \ldots : z_n] := [\varepsilon^{a_0 g} z_0 : \varepsilon^{a_1 g} z_1 : \ldots : \varepsilon^{a_n g} z_n]\,.$$

An element $g \in \mathbb{Z}/m$ acts trivially on $\mathbb{C}P^n$ if and only if $(a_i - a_j)g = 0 \in \mathbb{Z}/m$ for all $i \neq j$, which in terms of a representing integer means that m divides $(a_i - a_j)\cdot g$, or that g is a multiple of the number

$$\frac{m}{\gcd(m, a_i - a_j)}$$

where the greatest common divisor is taken over m and all $a_i - a_j$ with $i \neq j$. A similar condition determines whether a particular point $[z_0 : z_1 : \ldots : z_n] \in \mathbb{C}P^n$ is held fixed by $g \in \mathbb{Z}/m$: now only the differences $a_i - a_j$ with $z_i \neq 0 \neq z_j$ are taken into account.

The examples suggest further notions: The subgroup

$$G_x := \big\{ g \in G \,\big|\, g\cdot x = x \big\} \subset G$$

is called the *isotropy group* of the point x. It gives rise to the map

$$G/G_x \ni g\,G_x \longmapsto g\cdot x \in Gx$$

[1] The fact that G acts on the left of X suggests $G\backslash X$ as a more consequent but awkward notation, which indeed is used when left and right actions are around simultaneously.

from the space of cosets to the orbit of x, which is bijective and thus a homeomorphism (of discrete spaces) provided that X is a Hausdorff space. A point $x \in X$ with $G_x = G$ is a *fixed point* of the G-action; the fixed points form the subspace $X^G \subset X$ on which G acts trivially. In Example (2) many different subgroups G_x can be realised choosing the weights $a_0, \ldots, a_n$ as suitable divisors of m, and points $z \in \mathbb{C}P^n$ with various vanishing and non-vanishing components. In any case the $n+1$ points of $\mathbb{C}P^n$ with just one non-zero entry are fixed points. — At the other extreme lies the case that the isotropy group $G_x = \{1\}$ is trivial for all $x \in X$: such actions, like that of Example (1), are called *free*.

Note If X is a Hausdorff or normal space then so is X/G. Indeed assume the former and let Gx and Gy be two different orbits in X. For any two elements $g, h \in G$ we find disjoint open neighbourhoods $U_{g,h} \subset X$ of gx and $V_{g,h} \subset X$ of gy. Then

$$U := \bigcap_{g,h \in G} g^{-1} U_{g,h} \quad \text{and} \quad V := \bigcap_{g,h \in G} h^{-1} V_{g,h}$$

are open neighbourhoods of x and y with $G{\cdot}U \cap G{\cdot}V = \emptyset$; they therefore project to open sets that separate the points Gx and Gy in X/G. The proof of normality is similar.

Definition Let G be a finite group and X a G-space. A *G-vector bundle* over X is a vector bundle $E \xrightarrow{\pi} X$ with an action of G on E such that

- π is equivariant and
- for every $g \in G$ and every $x \in X$ the map $E_x \ni v \mapsto g{\cdot}v \in E_{gx}$ is linear.

A *homomorphism* between G-bundles is, of course, a homomorphism of bundles which is G-equivariant.

In the same way as ordinary vector bundles over a one-point space reduce to finite-dimensional vector spaces, G-bundles over the one-point space are known in algebra as *representations* of the group G, also referred to as *G-modules*: they are just homomorphisms $G \to GL(V)$ for some (finite-dimensional complex) vector space V. Many of their properties remind of those of (ordinary) vector bundles, and we briefly recall some of their most basic ones.

It is clear that the direct sum and the tensor product of G-modules again are G-modules, with $g \in G$ acting as $g{\cdot}(x \oplus y) = gx \oplus gy$ and $g{\cdot}(x \otimes y) = gx \otimes gy$.

If $\langle\, , \rangle$ is any Hermitian metric on a G-module V then we may average over the group G to obtain a new metric

$$\langle\langle v, w \rangle\rangle := \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle$$

which is G-invariant: $\langle\langle gv, gw \rangle\rangle = \langle\langle v, w \rangle\rangle$ for all $g \in G$ and $v, w \in V$. Thus G acts on V by automorphisms which are unitary with respect to this metric. One important application concerns subrepresentations $U \subset V$: the orthogonal complement of U then is G-invariant and thus a representation of G in its own right, making $V = U \oplus U^\perp$ the direct sum of two representations. From this fact it easily follows that every G-module at all is isomorphic to a direct sum of so-called *irreducible* G-modules V: these are G-modules that have exactly two submodules — necessarily the zero submodule and V itself.

Example A $\mathbb{Z}/2$-module consists of a vector space V which carries an involution $g: V \to V$. The eigenvalues ± 1 determine the decomposition $V = V_- \oplus V_+$, and every subspace of $V_\pm$ clearly is a $\mathbb{Z}/2$-submodule: up to isomorphism there exist therefore exactly two irreducible representations of $\mathbb{Z}$: the one-dimensional space $\mathbb{C}$ with g acting as -1 $+1$.

Taking averages has another important application. If V is a G-module then the linear mapping

$$V \ni v \mapsto \frac{1}{|G|} \sum_{g \in G} gv$$

is a projection — the orthogonal projection with respect to a G-invariant metric — to the fixed space

$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}.$$

If W is another G-module then the space of linear mappings $\mathrm{Hom}(V,W)$ inherits a G-module structure where $g \in G$ acts as

$$\mathrm{Hom}(V,W) \ni f \longmapsto g \circ f \circ g^{-1} \in \mathrm{Hom}(V,W),$$

so that $\mathrm{Hom}(V,W)^G$ consists just of the G-equivariant linear maps.

Lemma Let V and W be irreducible G-modules. Then

$$\mathrm{Hom}(V,W)^G \simeq \begin{cases} \mathbb{C} & \text{if } V \simeq W, \\ \{0\} & \text{else.} \end{cases}$$

Proof If V and W are isomorphic G-modules we may assume $V = W$. Thus let $f: V \to V$ be an equivariant endomorphisms. For any $\lambda \in \mathbb{C}$ the kernel of $\lambda - f$ is a G-submodules, and by irreducibility it is either zero or all V. The latter occurs for exactly one λ, and this proves that $f = \lambda$ is scalar.

We now show that V must be isomorphic if there exists a non-zero equivariant homomorphism $f: V \to W$. Indeed the kernel of f is a proper submodule of V, and as V is irreducible the kernel is zero, and f injective. Likewise the image of f is a non-trivial submodule of W, hence all W since W also is irreducible. Therefore $f: V \simeq W$ is an isomorphism.

Note The group G is abelian if and only if all irreducible representations of G have dimension one.

Proof Let $\rho: G \to GL(V)$ be a representation of an abelian group G. Since there exists a G-invariant metric on V all elements of the image $\rho(G) \subset GL(V)$ are diagonalisable; since they commute among each other they can be diagonalised even simultaneously. Thus there exists a common base of eigenvectors, and the line spanned by each base vector is a G-stable submodule.

Conversely, given a finite group G we assume that all irreducible representations of G are one-dimensional. There certainly exists an injective representation $\rho: G \to GL(V)$: for instance put $V = \bigoplus_{h \in G} \mathbb{C}$ and let $g \in G$ cyclically permute the canonical base vectors of V:

$$g \cdot (x_h)_{h \in G} = (x_{gh})_{h \in G}.$$

In view of the assumption V decomposes into G-stable lines, so that ρ factors as

$$\begin{array}{ccc} G & \xrightarrow{\quad\rho\quad} & GL(V) \\ & \searrow \qquad \nearrow & \\ & \prod GL(1,\mathbb{C}) & \end{array}$$

where the product is indexed by those lines. Since $GL(1,\mathbb{C}) = \mathbb{C}^*$ is abelian we have thus embedded G into an abelian group.

Theorem Given the finite group G there are, up to isomorphism, but finitely many irreducible representations of G (as many as there are conjugacy classes in G).

Remark For abelian groups the theorem directly follows from the note preceding it.

Corollary Let $I_1, \ldots, I_k$ a complete set of pairwise non-isomorphic irreducible G-modules. Then for any G-module W the evaluation map

$$\bigoplus_{j=1}^{k} \mathrm{Hom}(I_j, W)^G \otimes I_j \longrightarrow W, \quad f \otimes v \mapsto f(v)$$

is an isomorphism.

Proof Three observations suffice:

- If W is irreducible then the statement follows at once from the lemma.
- W is a direct sum of irreducible representations.
- The expressions on either side of the evaluation map are additive with respect to direct sums.

Note The corollary goes beyond the statement that every representation admits a decomposition into irreducible ones: unlike the latter the decomposition of W into the images of the $\mathrm{Hom}(I_j, W)^G \otimes I_j$ is unique. These images are called the *isotypical summands* of W as each of them may be (non-uniquely) written as a sum of copies of one and the same irreducible representation.

All this carries over to equivariant vector bundles $E \to X$ over a space X without G-action, considered as a G-space whose action is trivial: $X = X^G$. The averaging homomorphism now defines a bundle endomorphism $E \to E$ which still is a projection, so that its image is a subbundle of $\mathrm{End}\, E$. In particular the equivariant bundle homomorphisms from E to another G-bundle $F \to X$ form the subbundle

$$\mathrm{Hom}(E, F)^G \subset \mathrm{Hom}(E, F)\,.$$

Proposition Let X be a trivial G-space, let $I_1, \ldots, I_k$ be a complete set of pairwise non-isomorphic irreducible G-modules, and let $F \to X$ be an arbitrary G-bundle over X. Then the evaluation homomorphism

$$\bigoplus_{j=1}^{k} \mathrm{Hom}(X \times I_j, F)^G \otimes (X \times I_j) \longrightarrow F$$

is an isomorphism of G-bundles.

Note The point is that the first factor of each tensor product may be a non-trivial bundle on which G acts trivially, while the second is trivial as a bundle but with non-trivial G-action.

Example Up to equivariant isomorphism the $\mathbb{Z}/2$-bundles on a trivial $\mathbb{Z}/2$-space X are the Whitney sums of any bundle with trivial action, and any other on which the non-trivial group element acts as the scalar -1.

We now consider vector bundles over a free G-space.

Lemma If X is a Hausdorff space with free G-action then the quotient mapping $X \xrightarrow{q} X/G$ is a covering projection.

Proof Let $x \in X$ be a point. Using the Hausdorff property we choose pairwise disjoint neighbourhoods $U_g \subset X$ of gx for all $g \in G$. Then

$$U := \bigcap_{g \in G} g^{-1} U_g$$

is a neighbourhood of x with $gU \subset U_g$ for each $g \in G$. Therefore the sets gU are pairwise disjoint, and the group action restricts to a homeomorphism $G \times U \approx GU$. The resulting commutative diagram

$$\begin{array}{ccccc}
G \times U & \xrightarrow{\approx} & GU & \hookrightarrow & X \\
\Big\downarrow{\scriptstyle \{*\} \times \mathrm{id}_U} & & \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle q} \\
(G/G) \times U & \xrightarrow{\approx} & (GU)/G & \hookrightarrow & X/G
\end{array}$$

shows that q restricts to a trivial projection over the neighbourhood $(GU)/G$ of $Gx \in X/G$.

Proposition Let X be a free G-space, $X \xrightarrow{q} X/G$ the quotient map. Then the pull-back operation

$$\mathrm{Vect}(X/G) \xrightarrow{q^*} \mathrm{Vect}_G\, X$$

defines a bijection between isomorphism classes of vector bundles on X and equivariant isomorphism classes of G-vector bundles on X/G.

Proof If $F \to X/G$ is a bundle then $q^*F = \big\{(x,v) \in X \times F \,\big|\, v \in F_{Gx}\big\}$ is a G-bundle under the action

$$g\cdot(x,v) = (gx, v)\,.$$

Let now $E \xrightarrow{\pi} X$ be a G-vector bundle. Since π is equivariant we can form its quotient $E/G \xrightarrow{\overline{\pi}} X/G$. It is a family of vector bundles as for any point $x \in X$ the element $g \in G$ sends the fibre E_x linearly and bijectively to E_{gx}. From the commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E/G \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\overline{\pi}} \\ X & \xrightarrow{q} & X/G \end{array}$$

we at once obtain an isomorphism $q^*(E/G) \simeq E$ of families over X. On the other hand $X \xrightarrow{q} X/G$ is a covering projection, in particular a local homeomorphism: this proves that the family $\overline{\pi}$ is locally trivial.

The assignment of $E/G \xrightarrow{\overline{\pi}} X/G$ to $E \xrightarrow{\pi} X$ gives the desired inverse of the pull-back operation.

Let us briefly discuss a few aspects of bundles over a general (but always compact) G-space X, which generalise results from Section 4. While we can no longer average endomorphisms of a bundle $E \to X$ we can still average sections: $g \in G$ acts on the space of sections ΓE by $(gs)(x) = g\big(s(g^{-1}\cdot x)\big)$, and the averaging operator projects ΓE onto the subspace $(\Gamma E)^G$ of invariant sections.

Proposition Let X be a G-space, $S \subset X$ a closed G-stable subspace, and $E \to X$ a G-vector bundle. Then every invariant section of $E|S$ admits an extension to an invariant section of E over X.

Proof From Section 4 we know that there exists an extension at all, to which we just apply the averaging operator.

The equivariant analogues of the basic results from Section 4 now are but formal consequences: bundles that are equivariantly isomorphic over S are still so over some neighbourhood of S in X, and G-homotopic mappings induce isomorphic G-bundles from a given bundle. Likewise every G-bundle can be embedded as a Whitney summand in a trivial bundle (whose G-action need not be trivial of course).

As in the non-equivariant case the set $\mathrm{Vect}_G\, X$ of isomorphism classes of equivariant vector bundles over the G-space X is a commutative semi-group under the direct sum and tensor product operations. Applying the K-functor we obtain the *equivariant K-theory* $K_G X$ and thus a ring-valued functor on the category of compact G-spaces.

On the one-point space $X = \{*\}$ this functor reduces to a classical object of representation theory, the *(complex) representation ring*

$$R(G) = K_G\{*\}$$

of the group G: thus $K_G X$ is a common generalisation of KX and $R(G)$. By the result quoted before every representation of G is isomorphic to a unique sum of irreducible representations: this means that the semi-group formed by the isomorphism classes of representations is always isomorphic to a direct product of copies of the semi-group $\mathbb{N}$. It is thus embedded in $R(G)$, which is a free abelian group of finite rank (equal to the number of conjugacy classes in G). By contrast the multiplicative structure of the ring $R(G)$ contains much more detailed information about the group G.

Examples (1) Up to isomorphim the irreducible representations of $\mathbb{Z}/2$ are the trivial representation 1 and the non-trivial one χ of dimension 1. Since clearly $\chi^2 = 1$ the representation ring is $\mathbb{Z}[\chi]/(\chi^2-1)$.

(2) The symmetric group Sym_3 has three non-isomorphic irreducible representations: the trivial one, another one-dimensional one given by the sign character $\chi\colon \mathrm{Sym}_3 \to \{\pm 1\} \subset GL(1,\mathbb{C})$, and the representation ρ on $\big\{(x,y,z)\in\mathbb{C}^3 \,\big|\, x+y+z=0\big\}$ that permutes the coordinates. The representation ring turns out to be

$$R(\mathrm{Sym}_3) = \mathbb{Z}[\chi,\rho]\big/\big((\chi^2-1, (\chi-1)\rho, \rho^2-\rho-\chi-1\big)\,.$$

(3) If X is given the trivial G-action then what we have learnt above gives us a canonical isomorphism $K_GX \simeq KX \otimes_{\mathbb{Z}} R(G)$.

(4) In particular the map $S^1 \to \{*\}$ induces an isomorphism

$$R(G) = K_G\{*\} \simeq K_GS^1$$

where the circle carries the trivial action: $K_G\{*\} \simeq K\{*\}\otimes R(G) \simeq KS^1 \otimes R(G) \simeq K_GS^1$.

(5) For a free G-space X the pull-back operation induces an isomorphism $K(X/G) \xrightarrow{q^*} K_GX$.

Let X be a G-space and $E \to X$ a G-vector bundle. There are obvious induced actions of G on the projective bundle $P(E) \to X$ and on the tautological line bundle $T \to P(E)$, making the latter and its dual $H \to P(E)$ again G-bundles. Note that in the particular case of $E = \mathbf{1} \oplus L$ with a G-line bundle $L \to X$ the canonical sections X_0 and X_∞ of $P(\mathbf{1}\oplus L) \to X$ are G-stable. On inspection it turns out that everything in the proof of the Periodicity Theorem respects, or can easily be made to respect the G-actions. This results in the

Equivariant Periodicity Theorem Let $L \to X$ be a G-line bundle on the G-space X. Then

$$K\big(P(\mathbf{1}\oplus L)\big) = KX\,[h]\,\big/\big((h-1)(l\,h-1)\big)$$

as K_GX-modules, where as usual h stands for the class of the hyperplane bundle $H \to P(\mathbf{1}\oplus L)$ on the left, and for an indeterminate on the right hand side.

The Periodicity Theorem gives K_G the cohomological properties discussed in the non-equivariant case, in particular use of the suspension isomorphism $\tilde{K}_GX \simeq \tilde{K}_G(\Sigma^2X)$ gives the exact sequence for space pairs and suggests the definition of $K_G^*X = K_G^0X \oplus K_G^1X$ to turn it into the exact triangle of graded K-theory. Nevertheless the picture is not quite complete unless G is abelian: in a true equivariant cohomology theory *every* representation space V of G should give rise to a suspension isomorphism $\tilde{K}_GX \simeq \tilde{K}_G(V^c\wedge X)$ involving the G-sphere V^c, the one-point compactification V. While this does hold for equivariant K-theory it is not a conseqeunce of the Periodicity Theorem unless V is a sum of one-dimensional representations.

Theorem Assume that the G-vector bundle $E \to X$ is equivariantly isomorphic to a Whitney sum $\bigoplus_i L_i$ of d line bundles. Then

$$K_G^*P(E) = K_G^*X\,[h]\,\big/\prod_{i=1}^{d}\big(1-[L_i]\,h\big) = K_G^*X\,[h]\,\big/\big(\lambda_{-1}[E](h)\big)$$

where the last version involves the well-known expression

$$\lambda_{-1}[E](h) = \sum_{i=0}^{d}(-1)^i[\Lambda^iE]\,h^i \in K_GX\,[h]$$

that does not depend on a particular decomposition of E as a Whitney sum.

Note As in the non-equivariant set-up this theorem remains true even without the decomposability condition, but it then would require a new proof. The problem is that the local triviality of E cannot be immediately exploited since it no longer makes good sense to restrict E to arbitrary small open sets: a useful open set must be G-stable and thus contain at least one full G-orbit in X.

The *Thom space* of a G-vector bundle $E \to X$ is the quotient space

$$\Theta(E) = P(\mathbf{1} \oplus E)/P(E)$$

which compactifies the fibres of E by a single point at infinity common to them all — in view of the fact that the base X is compact it might alternatively be defined as the one-point compactification of E. Assuming that E decomposes into line bundles the theorem computes the absolute terms in the K-theory exact sequence of the pair $\big(P(\mathbf{1} \oplus E), P(E)\big)$

$$K_G^*\big(P(\mathbf{1} \oplus E), P(E)\big) \longleftarrow K_G^* P(E) \longleftarrow K_G^* P(\mathbf{1} \oplus E) \longleftarrow K_G^*\big(P(\mathbf{1} \oplus E), P(E)\big) \longleftarrow K_G^* P(E)$$

as

$$\cdots \longleftarrow K_G^* X[h]/\big(\lambda_1[E](h)\big) \longleftarrow K_G^* X[h]/\big((1-h)\cdot\lambda_1[E](h)\big) \longleftarrow K_G^*\big(P(\mathbf{1} \oplus E), P(E)\big) \longleftarrow \cdots,$$

with $h \in K_G^* P(\mathbf{1} \oplus E)$ mapping to $h \in K_G^* P(E)$ since the hyperplane bundle on $P(\mathbf{1} \oplus E)$ restricts to the hyperplane bundle on $P(E)$. In particular the corresponding arrow is surjective, and we may put zeros in the dotted positions and still have an exact sequence. We see that the element $\lambda_{-1}[E](h)$ — the so-called *Thom class* —lives on the Thom space:

$$\theta_E := \lambda_{-1}[E](h) \in K_G\big(P(\mathbf{1} \oplus E), P(E)\big) = \tilde{K}_G \Theta(E),$$

and we further read off the

Thom Isomorphism Theorem Under the assumptions of the previous theorem multiplication by θ_E yields an isomorphism

$$K_G^* X \xrightarrow{\simeq} \tilde{K}_G^* \Theta(E)$$

of $K_G X$-modules.

Note Again this is still true even if E does not decompose, but it requires an essentially different proof.

An equivalent, apparently less canonical description of the Thom space requires the choice of an invariant metric on the G-bundle $E \to X$. Then the equivariant disk and sphere bundles

$$D(E) = \big\{v \in E \,\big|\, |v| \le 1\big\} \to X \quad \text{and} \quad S(E) = \big\{v \in E \,\big|\, |v| = 1\big\} \to X$$

are defined and — see Problem 33 — we may identify $\Theta(E) = D(E)/S(E)$.

Proposition Let X be a G-space with $K_G^1 X = \{0\}$ and let $E \to X$ be a decomposable vector bundle with an invariant metric. There is a natural exact sequence

$$0 \longleftarrow K_G^0 S(E) \longleftarrow K_G^0 X \longleftarrow K_G^0 X \longleftarrow K_G^1 S(E) \longleftarrow 0$$

where the middle arrow is multiplication by $\lambda_{-1}[E] = \sum_i (-1)^i \lambda^i[E]$.

Proof This is what remains of the exact sequence of the pair $\big(D(E), S(E)\big)$

$$K_G^1\big(D(E), S(E)\big) \longleftarrow K_G^0 S(E) \longleftarrow K_G^0 D(E) \longleftarrow K_G^0\big(D(E), S(E)\big) \longleftarrow K_G^1 S(E) \longleftarrow K_G^1 D(E),$$

taking into account that the projection $D(E) \to X$ is an equivariant homotopy equivalence and substituting the relative term $K_G^*\big(D(E), S(E)\big) = \tilde{K}_G^* \Theta(E)$ by the Thom isomorphism: note that the restriction of H to the zero section of $E \to X$ is a trivial line bundle.

Corollary Let G be abelian, and V a G-module. Then, denoting by $S \subset V$ the unit sphere with respect to an invariant metric, there is a natural exact sequence

$$0 \longleftarrow K^0_G S \longleftarrow R(G) \longleftarrow R(G) \longleftarrow K^1_G S \longleftarrow 0$$

where the middle arrow is multiplication by the virtual representation $\lambda_{-1}[V] = \sum_i (-1)^i [\Lambda^i V]$.

Proof Put $X = \{*\}$.

Corollary Assume furthermore that the action of G on $V\backslash\{0\}$ is free — this quite restrictive assumption forces G to be a cyclic group. Then the exact sequence becomes

$$0 \longleftarrow K^0(S/G) \longleftarrow R(G) \longleftarrow R(G) \longleftarrow K^1(S/G) \longleftarrow 0\,.$$

Examples (1) For an integer $n > 0$ we choose $V = \mathbb{C}^n$ and let the non-trivial element of $G = \mathbb{Z}/2$ act as -1 on V: thus the class of V in the representation ring is the sum of n copies the non-trivial irreducible representation $\chi \in R(\mathbb{Z}/2) = \mathbb{Z}[\chi]/(\chi^2 - 1)$, and the middle arrow of the exact sequence

$$0 \longleftarrow K^0(\mathbb{R}P^{2n-1}) \longleftarrow R(\mathbb{Z}/2) \longleftarrow R(\mathbb{Z}/2) \longleftarrow K^1(\mathbb{R}P^{2n-1}) \longleftarrow 0$$

is multiplication by $\big(\lambda_{-1}[nV]\big)^n = \big(1 - \lambda^1(\chi)\big)^n = (1-\chi)^n$. Putting $x = 1 - \chi$, from the relation $(1-x)^2 = \chi^2 = 1$ we obtain $x^2 = 2x$ and more generally

$$x^{j+1} = 2^j x \quad \text{for all } j > 0\,,$$

so that in terms of the base $(1, x)$ for the free $\mathbb{Z}$-module $R(\mathbb{Z}/2)$ the middle arrow has the matrix

$$\begin{pmatrix} 0 & 0 \\ 2^{n-1} & 2^n \end{pmatrix}.$$

We thus obtain

$$K^0(\mathbb{R}P^{2n-1}) = K^0\{*\} \oplus \tilde{K}^0(\mathbb{R}P^{2n-1}) \simeq \mathbb{Z} \oplus \mathbb{Z}/2^{n-1} \text{ for all } n > 0\,,$$

where consideration of the rank homomorphism makes it clear that the direct sum decompositions correspond as shown. On the other side we read off the kernel

$$K^1(\mathbb{R}P^{2n-1}) = (2-x) = (1+\chi) \simeq \mathbb{Z} \text{ for all } n > 0\,.$$

(2) Our computation may be extended to real projective spaces of even dimension $2n$. To this end we assemble the commutative diagram

$$\begin{array}{ccccccc}
R(\mathbb{Z}/2) & \xleftarrow{\cdot x^{n+1}} & R(\mathbb{Z}/2) & \longleftarrow & K^1(\mathbb{R}P^{2n+1}) & \longleftarrow & 0 \\
\Vert & & \downarrow{\scriptstyle \cdot x} & & \downarrow & & \\
R(\mathbb{Z}/2) & \xleftarrow{\cdot x^{n}} & R(\mathbb{Z}/2) & \longleftarrow & K^1(\mathbb{R}P^{2n-1}) & \longleftarrow & 0
\end{array}$$

from the exact sequences for $\mathbb{R}P^{2n\pm1}$. Multiplication by x^{n+1} has the same kernel as multiplication by x; therefore the downward arrow on the right, that is, the restriction homomorphism is zero. Since the quotient $\mathbb{R}P^m/\mathbb{R}P^{m-1}$ is an m-sphere the exact sequences

$$K^0(\mathbb{R}P^{2n+1}, \mathbb{R}P^{2n}) \longleftarrow K^1(\mathbb{R}P^{2n}) \longleftarrow K^1(\mathbb{R}P^{2n+1}) \longleftarrow K^1(\mathbb{R}P^{2n+1}, \mathbb{R}P^{2n})$$
$$K^0(\mathbb{R}P^{2n}, \mathbb{R}P^{2n-1}) \longleftarrow K^1(\mathbb{R}P^{2n-1}) \longleftarrow K^1(\mathbb{R}P^{2n}) \longleftarrow K^1(\mathbb{R}P^{2n}, \mathbb{R}P^{2n-1})$$

show that the homomorphism $K^1(\mathbb{R}P^{2n+1}) \to K^1(\mathbb{R}P^{2n-1})$ is the product of the surjective factor $K^1(\mathbb{R}P^{2n+1}) \to K^1(\mathbb{R}P^{2n})$ with $K^1(\mathbb{R}P^{2n}) \to K^1(\mathbb{R}P^{2n-1})$ which is injective: thus

$$K^1(\mathbb{R}P^{2n}) = \{0\} \text{ for all } n \in \mathbb{N}\,.$$

Note finally that both $K^1(\mathbb{R}P^{2n+1})$ and $K^1(\mathbb{R}P^{2n+1}, \mathbb{R}P^{2n})$ are infinity cyclic groups. Thus we have an exact sequence

$$0 \longleftarrow \mathbb{Z} \longleftarrow \mathbb{Z} \longleftarrow K^0(\mathbb{R}P^{2n}) \longleftarrow K^0(\mathbb{R}P^{2n+1}) \longleftarrow K^0(\mathbb{R}P^{2n+1}, \mathbb{R}P^{2n})$$

and conclude that the inclusion induces an isomorphism $K^0(\mathbb{R}P^{2n+1}) \to K^0(\mathbb{R}P^{2n})$, so that

$$K^0(\mathbb{R}P^{2n}) = K^0\{*\} \oplus \tilde{K}^0(\mathbb{R}P^{2n}) \simeq \mathbb{Z} \oplus \mathbb{Z}/2^n \quad \text{for all } n \in \mathbb{N}\,.$$