Abstract.

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1. 8-20

From **AN06: Del Pezzo and K3 surfaces**:

2.8. Three types of non-symplectic involutions of K3 surfaces

It is natural to divide non-symplectic involutions (X, θ) of K3 and the corresponding DPN surfaces in three types:

Elliptic type: $X^\theta \cong C \cong C_g + E_1 + \cdots + E_k$ where C_g is an irreducible curve of genus $g \geq 2$ (equivalently, $(C_g)^2 > 0$), and $E_1, \ldots, E_k$ are non-singular irreducible rational curves. By Section 2.3, this is equivalent to $r + a \leq 18$ and $(r, a, \delta) \neq (10, 8, 0)$. Then $\mathrm{Aut}(X, \theta)$ is finite because $(C_g)^2 > 0$, see [Nik79] , [Nik83] and Section 3.1 below.

Parabolic type: Either $X^\theta \cong C \cong C_1 + E_1 + \cdots + E_k$ (using the same notation), or $X^\theta \cong C \cong C_1^{(1)} + C_1^{(2)}$ is a union of two elliptic (i. e. of genus 1) curves. By Section 2.3, this is equivalent to either $r + a = 20$ and $(r, a, \delta) \neq (10, 10, 0)$, or $(r, a, \delta) = (10, 8, 0)$. Then $\mathrm{Aut}(X, \theta)$ is Abelian up to finite index and usually non-finite, see [Nik79], [Nik83]. Here $(C_1)^2 = 0$.

2.8. THREE TYPES OF NON-SYMPLECTIC INVOLUTIONS OF K3 SURFACES 49

Hyperbolic type: Either $X^\theta \cong C \cong E_0 + E_1 + \cdots + E_k$ is a union of non-singular irreducible rational curves, or $X^\theta = \emptyset$. By Section 2.3, this is equivalent to either $r + a = 22$, or $(r, a, \delta) = (10, 10, 0)$. Then $\mathrm{Aut}(X, \theta)$ is usually non-Abelian up to finite index, see [Nik79], [Nik83]. Here $C_0 = E_0$ has $C_0^2 = -2$, if $X^\theta \neq \emptyset$.

Thus, pairs (X, θ) of elliptic type are the simplest, and we describe them completely in Chapter 3. On the other hand, for classification of log del Pezzo surfaces of index ≤ 2 we need only these pairs.

The representation-theoretic setup of moduli spaces: see here, which is Sterk section 4, p.14.

2. 8-22

2.1. **Research meeting.** Let $X \xrightarrow{\pi} Y$ be a K3 covering an Enriques surface, so $L := \Lambda_{\mathrm{K3}} := H^2(X; \mathbf{Z}) = U^3 + E_8^2$ and $M := H^2(Y; \mathbf{Z})/\mathrm{tors} = U + E_8$. Let $\eta_Y := e + f$ where e, f form a symplectic basis of U, then the pushforward satisfies $\pi^* \eta_Y = (e + f, e + f, 0)$ where we've written $L = M + M + U$. Moreover the map is

$$\pi^* : M \to M + M + U \quad m \mapsto (m, m, 0).$$

There is an involution

$$I : M \to M \quad (m, m', h) \mapsto (m', m, h),$$

and we let L_+ be the $+1$ eigenspace and L_- the -1 eigenspace. One can show that $L_+^{\perp H_2(X;\mathbf{Z})} = L_-$. Here L_+ is the generic Picard lattice of X.[1] There are identifications

$$L_+ = \left\{ (m, m, 0) \;\middle|\; m \in M \right\} = M(2) = U(2) + E_8(2) = (10, 10, 0)_1$$

and

$$L_- = \left\{ (m, -m, h) \;\middle|\; m \in M, h \in U \right\} = U + U(2) + E_8(2) = (12, 10, 0)_2.$$

We recall that for Λ any 2-elementary lattice, $\Lambda^\vee/\Lambda \cong (\mathbf{Z}/2)^a$ for some a, and these invariants are $(r, a, \delta)_i$ where $r \coloneqq \operatorname{rank}_{\mathbf{Z}} \Lambda$, the integer a is as above, and δ is the so-called coparity: $\delta = 0$ if the associated quadratic form q_L satisfies $q_L(A_L) \subseteq \mathbf{Z}$, so $q_L(x) \equiv 0 \bmod \mathbf{Z}$ (a co-even lattice), and $\delta = 1$ otherwise (a co-odd lattice). There is a trick to computing the coparity.[2]

We construct the usual period domain

$$\Omega_- \coloneqq \Omega^{L_+} = \Omega_{L_+^\perp} = \Omega_{L_-} = \left\{ v \in \mathbf{P}(L_- \otimes_{\mathbf{Z}} \mathbf{C}) \;\middle|\; v^2 = 0,\, v\overline{v} > 0 \right\}.$$

These are periods of X, which are necessarily orthogonal to L_+, the generic Picard lattice. We have $\dim_{\mathbf{C}} \Omega_- = 10$, with an associated 10-dimensional moduli space $E_? = \Gamma_? \backslash \Omega_{L_-}$ for some discrete group $\Gamma_?$. Note that $\mathrm{O}(\Lambda_{\mathrm{K3}}) \supseteq \mathrm{Stab}(L_+) = \Gamma_-$, and $\Gamma_- \curvearrowright \Omega_-$.Note also that $\mathrm{Stab}(L_+) \supseteq \Gamma_h \coloneqq \mathrm{Stab}(h)$, where $h \in M \subseteq \Lambda_{\mathrm{K3}}$ is a numerical polarization on an Enriques surfaces.

It is a fact that there are only finitely many such moduli spaces of Enriques surfaces. Letting $\mathcal{E}_\emptyset$ be the moduli of unpolarized Enriques surfaces and $\mathcal{E}_h$ be the moduli of Enriques surfaces with polarization h, there are finitely many choices up to isomorphism for what $\mathrm{Stab}(h)$ can be. This induces a finite poset of moduli spaces of the form $\mathcal{E}_h$, whose minimal element is $\mathcal{E}_\emptyset$ and whose maximal element in $\mathcal{E}_{\max}$:

Link to Quiver diagram

[1] Like $\mathrm{Pic}(\mathcal{X}_t)$ for for $\mathcal{X}_t \in F_{2d}$ a generic point in a moduli space?

[2] Can't quite remember the trick…for L, it's something like take $L^\dagger(2)$ and check if it is even…? Valery knows how to do this easily. For example, for A_1, you get $\left\langle \frac{-1}{2} \right\rangle (2) = \langle -1 \rangle$ which is odd, and so A_1 is co-even and $\delta = 0$? I don't think I did this correctly. # 8-24

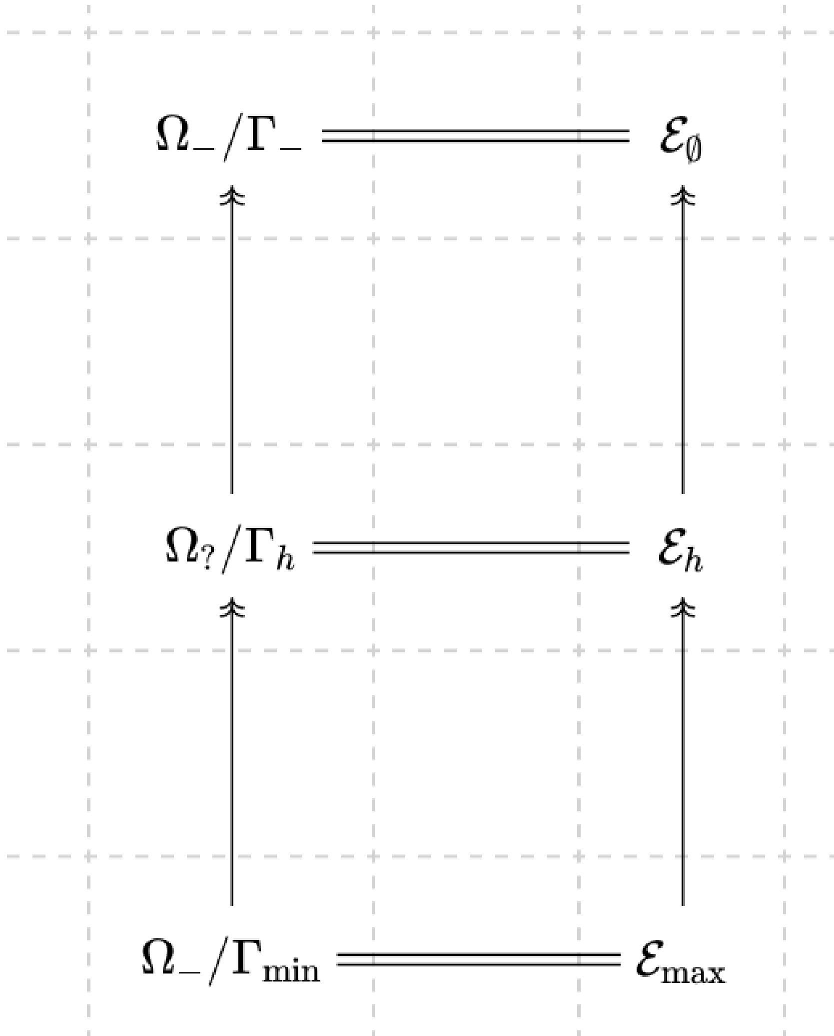


FIGURE 1. Pasted image 20230824135448.png

We can form a moduli space[3] $M_{\overline{h}}$ with $\overline{h} \in \mathrm{Pic}(Y)/\mathrm{tors}$, which will either be of the form M_L or $M_L \amalg M_{L'}$. Note that we can also consider the more standard moduli of polarized Enriques surfaces (Y, L) with $L \in \mathrm{Pic}(Y)^{\mathrm{amp}}$.[4]

Sterk takes Γ to be the image of $\left\{ g \in \mathrm{O}(L) \;\middle|\; g \circ I = I \circ g \, g(h) = h \right\}$ in $\mathrm{O}(L_-)$, which seems to precisely be something like $\mathrm{Stab}_{\mathrm{O}(L_-)}(h)$? [5] here $h = (e + f, e + f, 0)$. These isometries fix a $U(2)$ summand. If we take $U(2) = \langle E, F \rangle$, we either have $E \rightleftharpoons E, F \rightleftharpoons F$, or $E - F \rightleftharpoons F - E$ and $E \rightleftharpoons F$.

Note that we can build $F_{4,\mathrm{h.e.}}$ as $F_{U(2)}$, and we have the following diagram.

Link to Quiver diagram

[3] Missed what this is a moduli space *of*, can't quite remember what $\overline{h}$ and L, L' were.

[4] Why do we not use this moduli space? Seems pretty natural. Maybe it coincides with something we already use? Valery might have said something along these lines that I've forgotten.

[5] Would help to know this explicitly, since it's a much simpler description than this centralizer description

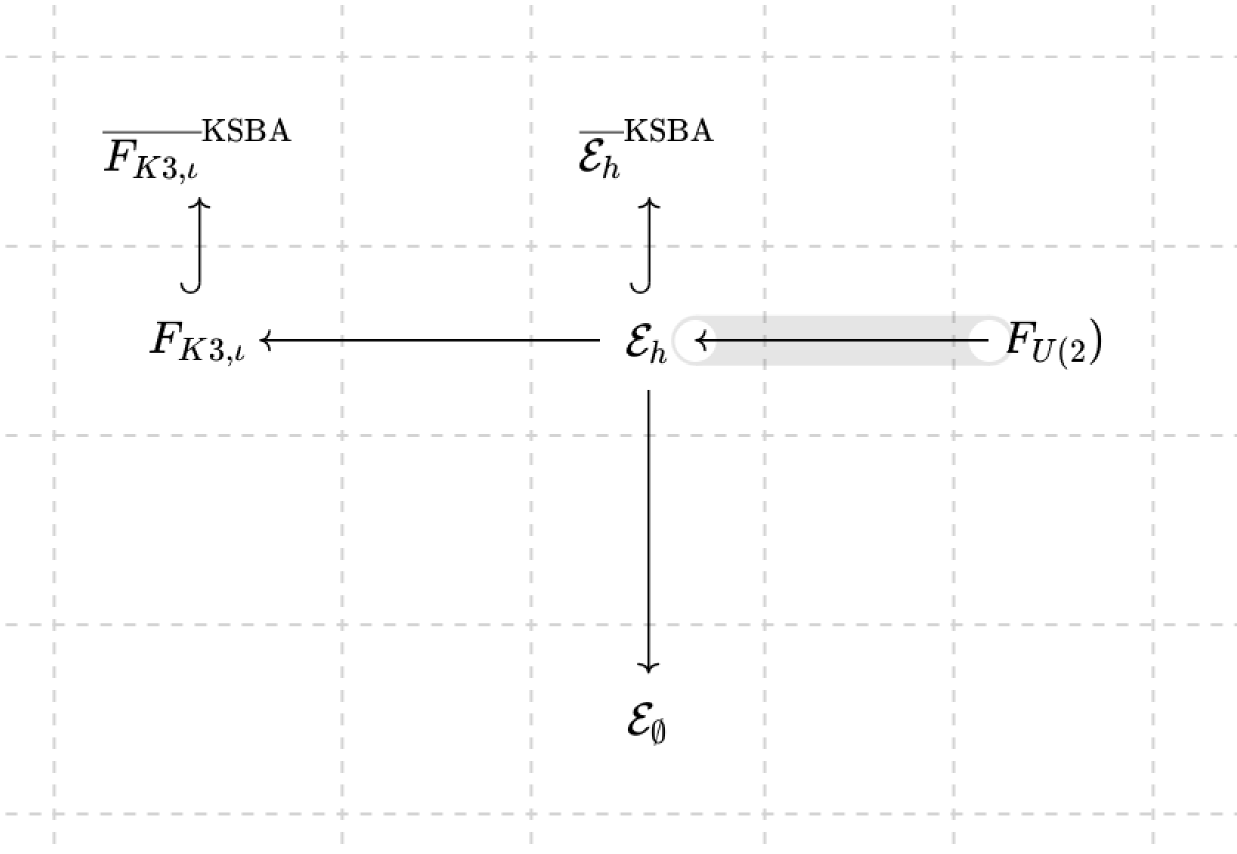


Figure 2. Pasted image 20230824143638.png

Here $F_{K3,\iota}$ is a moduli of K3 surfaces with involution.[6]

Let $T = (12, 10, 0)_2 \ni e$ with $e^2 = 0$; we can identify this with $U + U(2) + E_8(2) = (2, 0, 0)_1 + (2, 2, 0)_1 + (8, 0, 0)_0$. Note that given $(r, a, \delta)_i$, one can generally construct this decomposition into pieces of the following forms:

- $\langle 2 \rangle = (1, 1, 1)_1$
- $U = (2, 0, 0)_1$
- $U(2) = (2, 2, 0)_1$
- $E_8 = (8, 0, 0)_0$
- $E_8(2) = (8, 8, 0)_0$
- $A_1 = \langle -2 \rangle = (1, 1, 1)_0$
- $E_7 = (7, 1, 1)_0$
- $D_{4n} = (4n, 2, 0)_0$
- $D_{4n+2} = (4n + 2, 2, 1)_0$

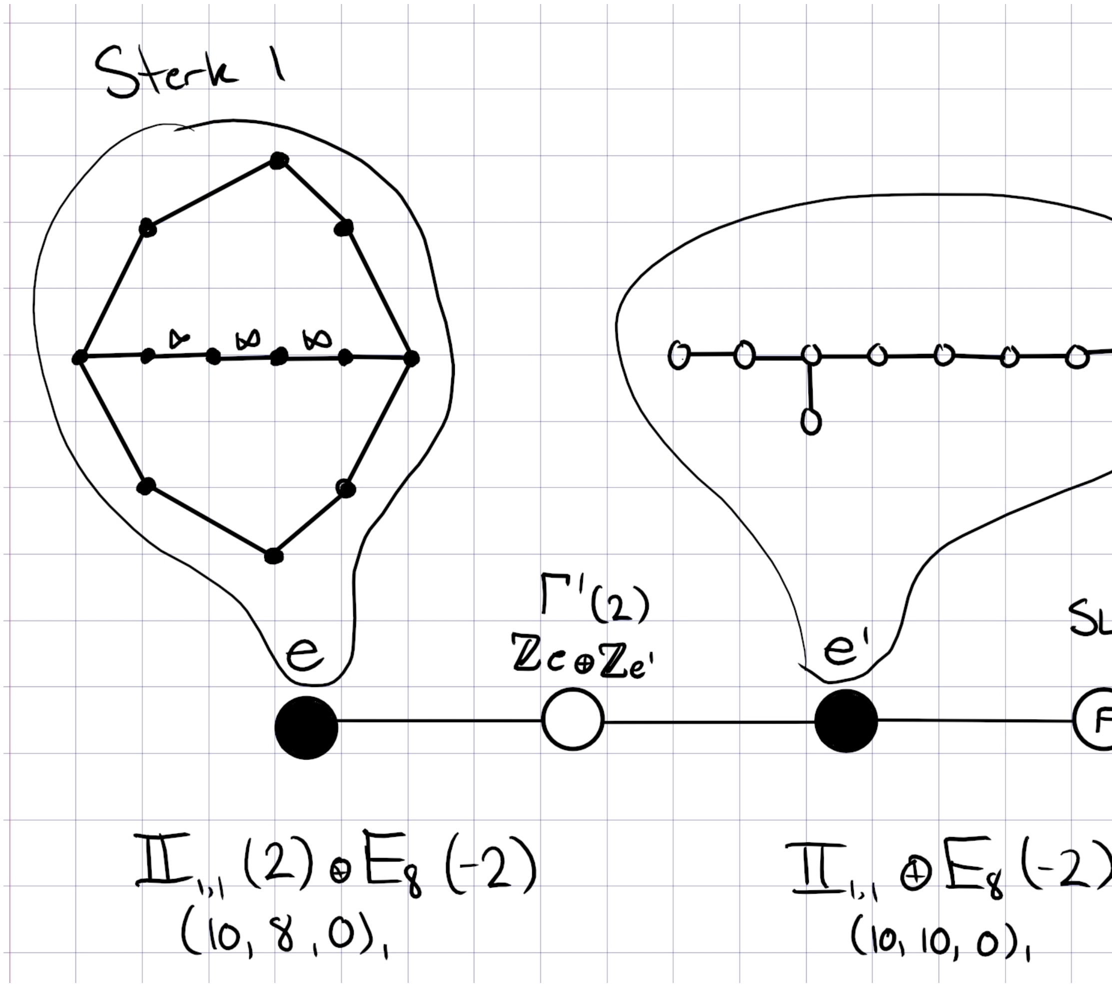
We have $\overline{T} = e^\perp/e \ni f$ and $f^\perp/f = \overline{\overline{T}} = J^\perp/J$.

As a quick aside:

- A_n has $\Lambda^\vee/\Lambda = \mathbf{Z}_{n+1}$ and $\Lambda^\vee(2) =?$
- D_n has $\Lambda^\vee/\Lambda = (\mathbf{Z}/2)^2$ or $\mathbf{Z}/4$ for n odd and $\Lambda^\vee(2) =?$

[6]Not entirely sure which moduli space this is yet. Like moduli of K3s with a nonsymplectic involution, lattice-polarized by a particular S? Or are we doing something like the existence of $F_{2d,\iota}$, a moduli of degree $2d$ K3 surfaces equipped with a (possibly symplectic, possibly not) involution?

- $E_{6,7,8}$ has $\Lambda^\vee/\Lambda = \mathbf{Z}/3$ or $\mathbf{Z}/2$? And $\Lambda^\vee(2) = ?$



We get a cusp diagram:

Note that $U + E_8(2) \cong U(2) + D_8$.

Something about $(\mathrm{ADE})^\perp \subseteq \Lambda_{(24,0,0)_1}$.

Let us quickly review our setup: consider $U(2) \subseteq U(2) + E_8(2)$, and take its perp to obtain $L_- \subseteq \Lambda_{18} := U^{\oplus^2} + D_{16}$. Let's call the associated period domain Ω_2, we then want a morphism $\Omega_2 \hookrightarrow \Omega_{4,h.e.}$, where we have group actions $\Lambda_2 \curvearrowright \Omega_2$ and $\Lambda_{4,h.e.} := \mathrm{O}(\Lambda_{18}) \curvearrowright \Omega_{4,h.e.}$. Note that $\Gamma_2 \neq \mathrm{O}(L_-)$!

Is there an induced morphism on the quotients $\Omega_2/\Gamma_2 \to \Omega_{4,h.e.}/\Gamma_{4,h.e.}$? The answer is yes, and injectivity follows from applying the Torelli theorem and using a geometric argument. As a result, we get an embedding of quasiprojective varieties $\varphi : E_2 \hookrightarrow F_{4,h.e.}$. Applying an extension result that Luca found and added to the paper, in this case we do get an extension of this morphism to the BB compactifications,

$$\overline{\varphi} : \overline{E_2}^{BB} \to \overline{F_{4,h.e.}}^{BB}$$

and Luca proceeded to study the cusp correspondence.

Question: is there a morphism $\Gamma_2 \to \Gamma_{4,h.e.}$?

A neat trick: for any lattice L, there is a well-defined diagonal map

$$\varphi : L(2) \to L \oplus L \quad x \mapsto (x, x)$$

Why this is true:

$$(\varphi(x), \varphi(x))_{L\oplus L} = ((x,x),(x,x))_{L\oplus L} = (x,x)_L + (x,x)_L = 2(x,x)_L = (x,x)_{L(2)}$$

In what follows, we take $e \in L_- \subseteq \Lambda_{18}$ and consider $e^{\perp L_-}/e \subseteq e^{\perp \Lambda_{18}}/e$ and make identifications $U(2) + E_8(2) \subseteq U(2) + E_8^2$ and $U + E_8(2) \subseteq U + E_8^2$. ## Question 1a

In the Laza-O'Grady paper, where to they compute $e^\perp/e$? We believe $e^\perp/e \cong U(2) + E_8^2$ or $U + E_8^2$; we should look into theorem 2.8 for the definitions of the $\mathrm{III_a}$ and $\mathrm{III_b}$ conventions. Is this computed more explicitly in one of Valery's papers? Or Scattone?

We do know that in Sterk's paper, he shows $e^\perp/e \cong U(2) + E_8(2)$ or $U + E_8(2)$.

2.2. **Question 1b.** We have the morphism

$$\overline{\varphi} : \overline{E_2}^{BB} \to \overline{F_{4,h.e.}}^{BB}$$

But we are not so sure it is injective. What Luca can say for sure is that the restriction of this map onto its image is in fact the normalization.

2.3. **Question 2.** Luca has found an abstract extension result that describes when a morphism symmetric spaces lifts to a morphism on their BB compactifications. It is somewhat abstract and not very symmetric.

I mentioned here that there is some interest from people in the UK who study a reverse problem: given a period domain cooked up from a lattice, does it "come from geometry"? They are usually classifying spaces of Hodge structures, but what spaces realize those?

2.4. **Question 3.** We would like to know the actual presentations of the lattices of the form $\pi^\perp/\pi$ where π is an isotropic plane for E_2; Sterk describes 9 of these.

There is more we talked about, but I haven't recorded it here yet.