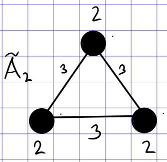
Integral Affine Structures

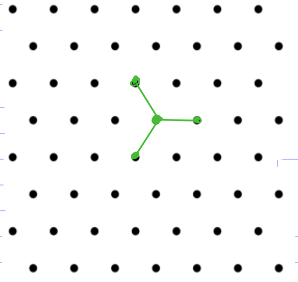
Sterk's Conventions

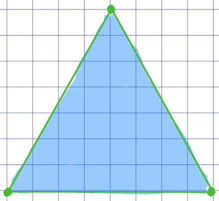
○: -4 Vectors

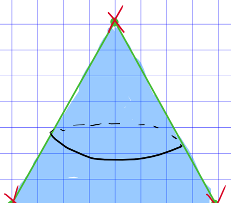
●: -2 Vectors

edges	g_{ij}	m_{ij}
● ●	0	2
●—●	½	3
●═●	$1/\sqrt{2}$	4
●≡●	$\sqrt{3}/2$	6
●—∞—●	1	∞
●—●	1	

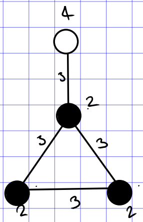

$\tilde{A}_2$
2
3
3
3
2
2

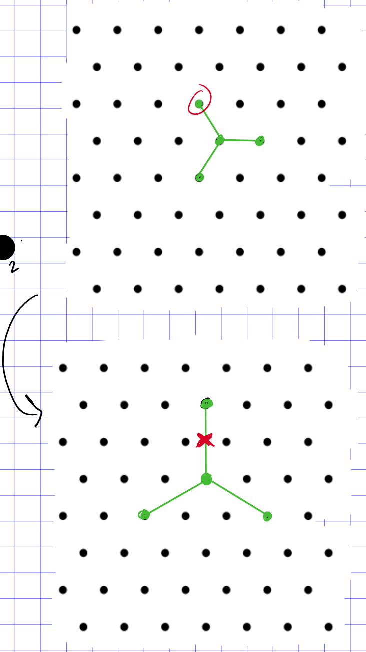


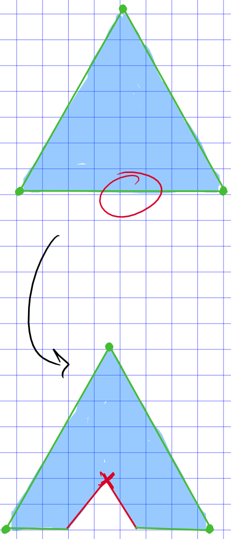


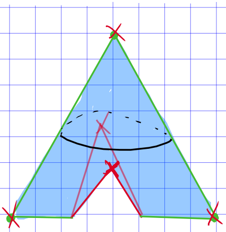


IAS^2, $Q = 3$

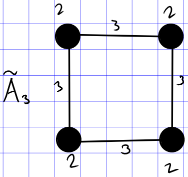


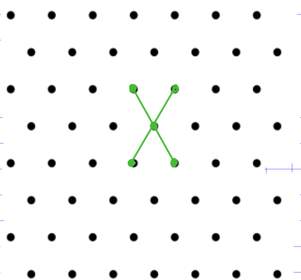


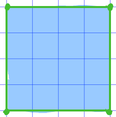


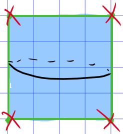


IAS^2, $Q = 3+2 = 5$.

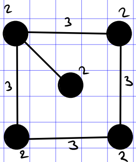

$\tilde{A}_3$
2
3
2
3
3
2
3
2

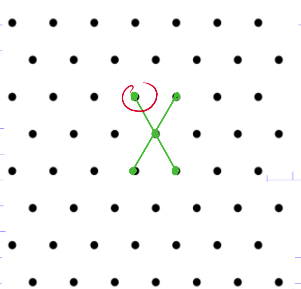


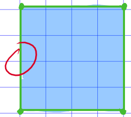


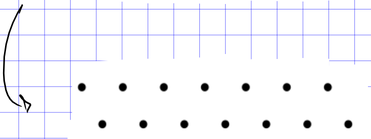


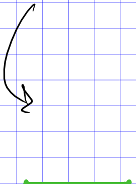
IAS^2, $Q = 4$.

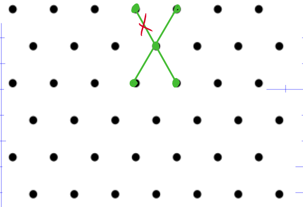


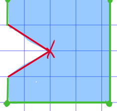


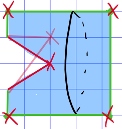








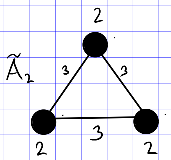




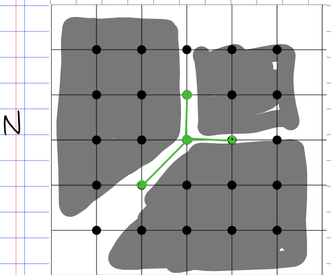
IAS^2, $Q = 4+2=6$.

A full KPP model for $\tilde{A}_2$:

Step 1: Coxeter diagram


2
$\tilde{A}_2$
3
3
3
2
2

Step 2: Toric fan Σ whose T-invariant curves have intersection complex $\tilde{A}_2$



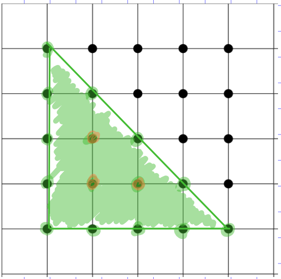
Σ: $e_0, e_1, -e_0-e_1$.

Cones

fan

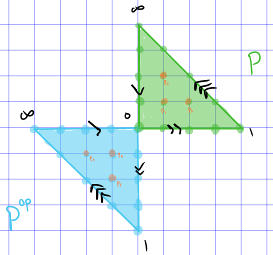
Step 3: Polytope P_Σ (or P_Σ°?)

M



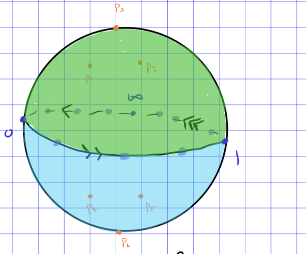
Orange just tracks interior verts.

Step 4: Glue $P \cup P^{op}$



$\cong$

Homeomorphic


P_3
P_2
P_1
∞
0
1
P_4
P_5
P_6

12 Verts on equator

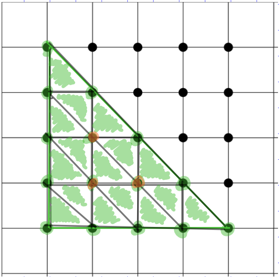
$2 \cdot 3 = 6$ verts in hemispheres.

18 total verts

$= S^2$ "with a lattice"

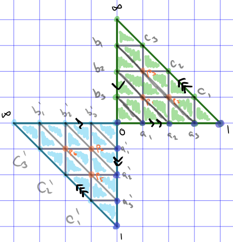
This hides some details: really we need...

Step 3a: Complete to a triangulation by basis triangles.



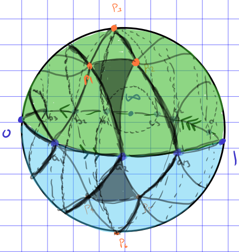
I see...
$2 \cdot 6 + 4 = 16$
triangles of area ½ each.

Step 4, redux:



∞
b_1
b_2
b_3
c_3
c_2
c_1
0
a_1
a_2
a_3
1
∞
b_1'
b_2'
b_3'
c_3'
c_2'
c_1'
a_1'
a_2'
a_3'
1

$\cong$
Homeo



P_3
P_1
∞
0
1
b_3
b_2
a_1
a_2
a_3
P_0

A little complicated to draw precisely

Let's work w/ this instead

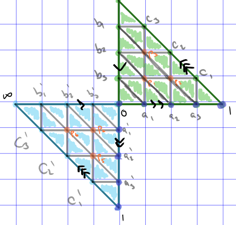
Step 5: The "tropical" surface $= X_0$ for X a KPP model

(= union of (almost) toric sfcs = "broken toric sfc")

$a_i = a_i'$
$b_i = b_i'$
$c_i = c_i'$

Some counts:

After identifications, we


∞
b_1
b_2
b_3
c_3
c_2
c_1
0
a_1
a_2
a_3
1
b_1'
b_2'
b_3'
c_3'
c_2'
c_1'
a_1'
a_2'
a_3'

have a simplicial set with...

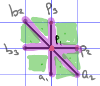
- $3+3+12=18$ vertices
- 3 edges
- 2 faces

Star @ p_1 = a fan.

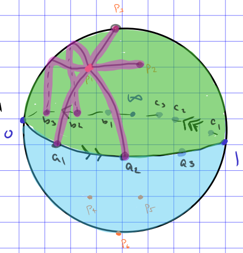
Stars at vertices!

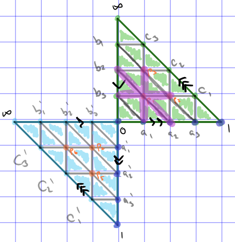
Some interior:

p_1:


b_2
p_3
p_1
b_3
p_2
a_1
a_2

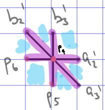
$= Bl_2(\mathbb{P}^1 \times \mathbb{P}^1) \rightsquigarrow$


p_3
p_2
p_1
∞
c_3
c_2
c_1
b_3
b_2
b_1
0
a_1
a_2
a_3
1
p_4
p_5
p_6

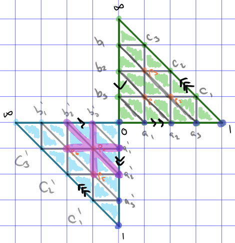

∞
b_1
b_2
b_3
c_3
c_2
c_1
0
a_1
a_2
a_3
1
b_1'
b_2'
b_3'
c_3'
c_2'
c_1'
a_1'
a_2'
a_3'

---

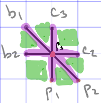
p_4:


b_2'
b_3'
p_4
a_2'
p_6
p_5
a_3'

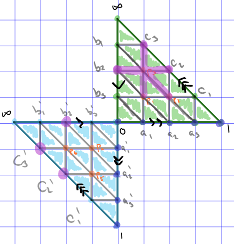
$= Bl_2(\mathbb{P}^1 \times \mathbb{P}^1)$


∞
b_1
b_2
b_3
c_3
c_2
c_1
0
a_1
a_2
a_3
1
b_1'
b_2'
b_3'
c_3'
c_2'
c_1'
a_1'
a_2'
a_3'

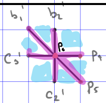
P_3 :


b_1
c_3
b_2
p_3
c_2
p_1
p_2

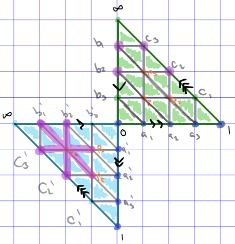
$= Bl_2(\mathbb{P}^1 \times \mathbb{P}^1)$


∞
b_1
c_3
b_2
c_2
p_3
b_3
c_1
p_1
p_2
∞
b_1'
b_2'
b_3'
0
a_1
a_2
a_3
1
a_1'
p_6
p_4
c_3'
a_2'
p_5
c_2'
a_3'
c_1'
1

P_6 :

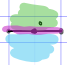

b_1'
b_2'
c_3'
p_6
p_4
p_5
c_2'

$= Bl_2(\mathbb{P}^1 \times \mathbb{P}^1)$


∞
b_1
c_3
b_2
c_2
p_3
b_3
c_1
p_1
p_2
∞
b_1'
b_2'
b_3'
0
a_1
a_2
a_3
1
a_1'
p_6
p_4
c_3'
a_2'
p_5
c_2'
a_3'
c_1'
1

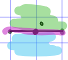
Some corners:

0 : $a_1 = a_1'$



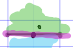
$b_3 = b_3'$ $= \mathbb{P}^1$

∞ : $b_1 = b_1'$



$c_3 = c_3'$ $= \mathbb{P}^1$

1 : $c_1 = c_1'$



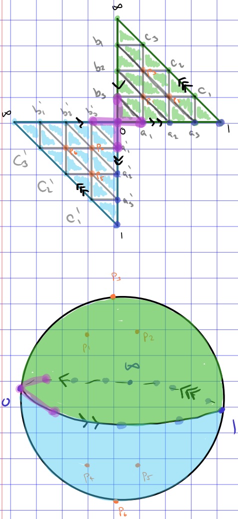
$a_3 = a_3'$ $= \mathbb{P}^1$

Not surfaces, what happened??.

Gluing introduced an I_1 singularity, giving us an extra "monodromy" ray in the fan! We have a different

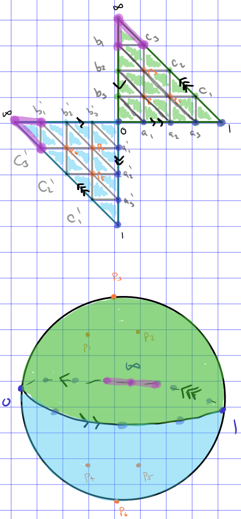
procedure for sfcs at singular pts.

0



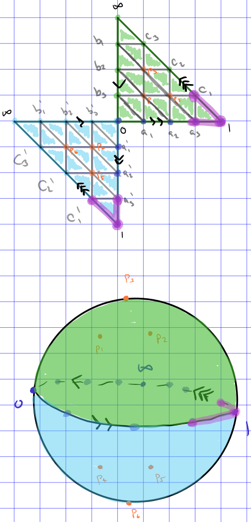
∞
b1
b2
b3
c3
c2
c1
b1'
b2'
b3'
0
a1
a2
a3
1
a1'
a2'
a3'
C3'
C2'
c1'
1
p3
p1
p2
∞
0
1
p4
p5
p6

∞



∞
b1
b2
b3
c3
c2
c1
b1'
b2'
b3'
0
a1
a2
a3
1
a1'
a2'
a3'
C3'
C2'
c1'
1
p3
p1
p2
∞
0
1
p4
p5
p6

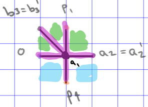
1



∞
b1
b2
b3
c3
c2
c1
b1'
b2'
b3'
0
a1
a2
a3
1
a1'
a2'
a3'
C3'
C2'
c1'
1
p3
p1
p2
∞
0
1
p4
p5
p6

Same on
the edges:

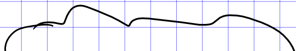
a_1:

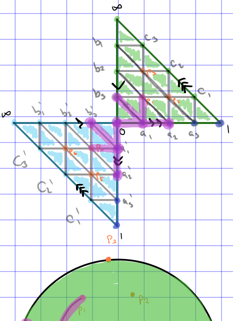


b3=b3'
p1
0
a2=a2'
a1
p4

$= Bl_1(\mathbb{P}^1 \times \mathbb{P}^1)$

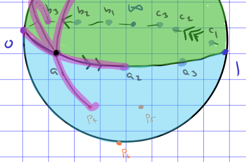
$= Bl_1(\mathbb{F}_n)$



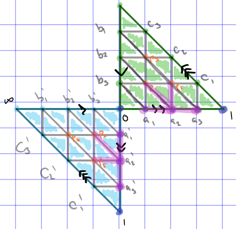


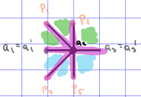
∞
b1
b2
b3
c3
c2
c1
b1'
b2'
b3'
0
a1
a2
a3
1
a1'
a2'
a3'
C3'
C2'
c1'
p3
p1
p2

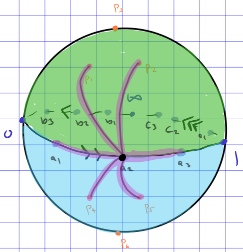
Note: no singularities on these vertices! So toric!


b3
b2
b1
∞
c3
c2
c1
0
1
a1
a2
a3
p4
p5
p6

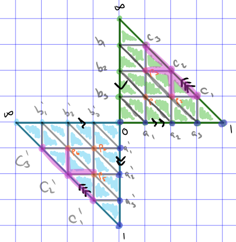
a_2:

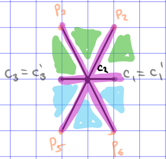

b1
b2
b3
c3
c2
c1
∞
b1'
b2'
b3'
0
a1
a2
a3
1
c3'
c2'
c1'
a1'
a2'
a3'
1
p3
p1
p6
p4
p5


p1
p2
a1=a1'
a2
a3=a3'
p4
p5


p3
p1
p2
∞
0
b3
b2
b1
c3
c2
c1
1
a1
a2
a3
p4
p5
p6

c_2:


∞
b1
b2
b3
c3
c2
c1
∞
b1'
b2'
b3'
0
a1
a2
a3
1
c3'
c2'
c1'
a1'
a2'
a3'
1
p6
p4
p5

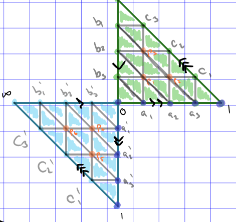

p3
p2
c2
c3=c3'
c1=c1'
p5
p6

... etc!

So what is our broken sfc explicitly?

We interpret this picture

Some counts:


b₁
b₂
b₃
c₃
c₂
c₁
p₃
p₁
p₂
∞
b₁'
b₂'
b₃'
0
a₁
a₂
a₃
1
a₁'
a₂'
a₃'
p₄
p₆
p₅
c₃'
c₂'
c₁'
1

- $3+3+12=18$ vertices
- 3 edges
- 2 faces

as...

$$X_0 = \left(\bigcup_{i=0}^{6} V_{p_i}\right) \cup \left(\bigcup_{i=1}^{3} V_{a_i}\right) \cup \left(\bigcup_{i=1}^{3} V_{b_i}\right) \cup \left(\bigcup_{i=1}^{3} V_{c_i}\right)$$

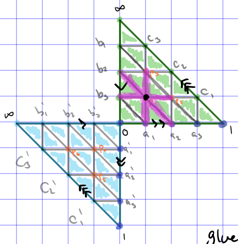
$$\cup\, V_0 \cup V_1 \cup V_\infty$$

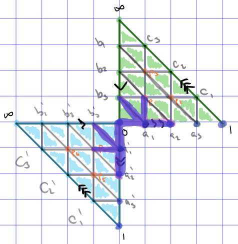
$= 6+3+3+3+3 = 18$ sfcs glued together

where V_0, V_1, V_∞ are semitoric sfcs with one I_1 singularity (equiv: charge $=1$, one nontoric blowup of some toric sfc.)

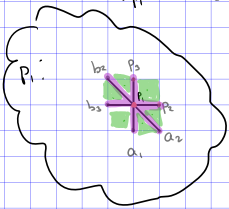
and the rest are honest toric sfcs.

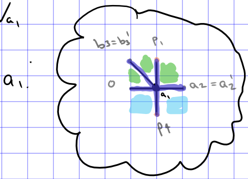
How are they glued? E.g.

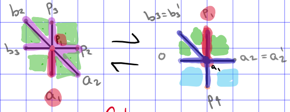

∞
b1
b2
b3
c3
c2
c1
p3
p2
0
a1
a2
a3
1
∞
b1'
b2'
b3'
C3'
C2'
C1'
p6
p5
p4
a1'
a2'
a3'
1


∞
b1
b2
b3
c3
c2
c1
p3
p2
0
a1
a2
a3
1
∞
b1'
b2'
b3'
C3'
C2'
C1'
p6
p5
p4
a1'
a2'
a3'
1

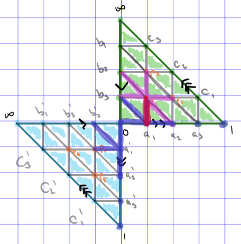
$V_{p_1} \overset{\text{glue}}{\rightleftharpoons} V_{a_1}$


P1:
b2
P3
b3
P1
P2
a2
a1

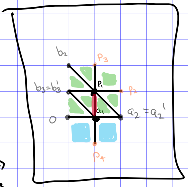

a1:
b3=b3'
P1
0
a2=a2'
a1
P4


b2
P3
P1
b3
P2
a2
a1
b3=b3'
P1
0
a2=a2'
a1
P4

Glue red


∞
b1
b2
b3
c3
c2
c1
0
a1
a2
a3
1
∞
b1'
b2'
b3'
C3'
C2'
C1'
a1'
a2'
a3'
1

yields


b2
P3
P1
P2
b3=b3'
a1
a2=a2'
0
P4

This describes

V_{p_1} glued to V_{a_1} along a

common curve; the edge

$p_1 \to a_1$ in red

So $\mathcal{X}_0 = (15 \text{ toric sfcs}) \cup \left(\begin{array}{c} 3 \text{ almost toric sfcs} \\ \text{with charge} = 1 \end{array}\right)$

Q: Can this be smoothed to a family $\mathcal{X}$?

A: When $Q = 24$, yes, we use a theorem of Freedman

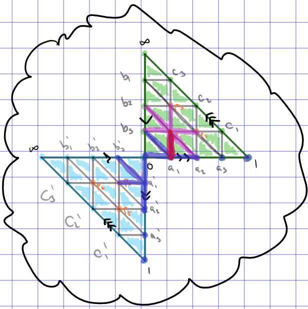
(Note: $\mathcal{X}$ is a highly non-unique KPP model).

One last lie to clear up: what is the actual semitoric sfc at a node?

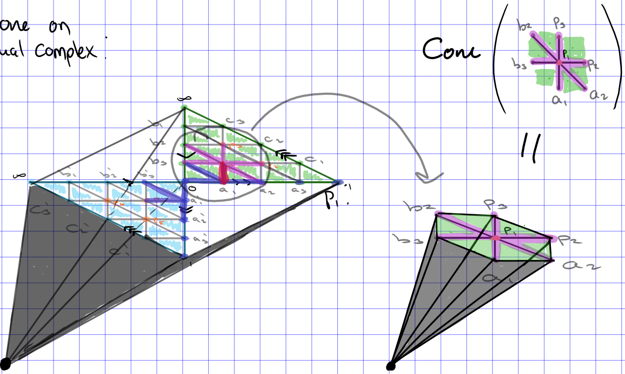
Each node p_j represents an ACP (V_j, D_j) where $D_j = \sum_i (D_j)_i$.

Set $\mathcal{F}(V_i, D_i) := \text{Cone}\left(\underbrace{\Gamma(D_i)}_{\text{dual complex}}\right)$, a triangulated IA sfc.

Dual complex =

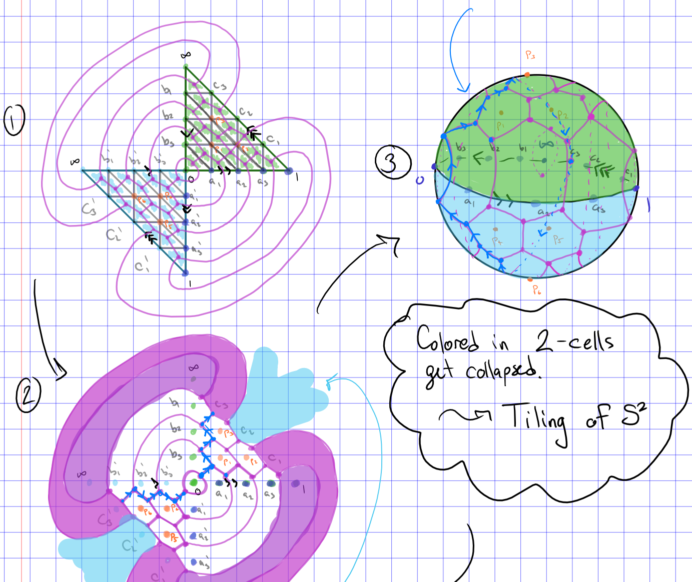

∞
c_3
b_1
b_2
c_2
b_3
c_1
∞
b_1'
b_2'
b_3'
0
a_1
a_2
a_3
1
c_3'
a_1'
a_2'
c_2'
a_3'
c_1'
1

Cone on dual complex:


∞
b_1
b_2
b_3
c_3
c_2
c_1
0
a_1
a_2
a_3
P_1
1
b_1'
b_2'
b_3'
c_3'
c_2'
c_1'
a_1'
a_2'
a_3'
Cone
b_2
P_3
b_3
P_1
P_2
a_1
a_2
//

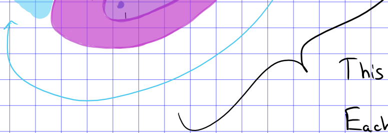
The actual degeneration:

Blue = a cartoon "tropical divisor"
(a weighted balanced subgraph).


①
②
③
∞
b_1
b_2
b_3
c_3
c_2
c_1
0
a_1
a_2
a_3
1
P_1
P_2
P_3
P_4
P_5
P_6
b_1'
b_2'
b_3'
c_3'
c_2'
c_1'
a_1'
a_2'
a_3'

Colored in 2-cells
get collapsed.

⤳ Tiling of S^2

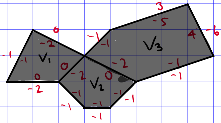


This is the intersection complex, not the dual complex!!

This is $\mathcal{X}_0$ for a KPP degn.

Each tile corresponds to an irreducible component V_i of $\mathcal{X}_0 = \bigcup_i V_i$

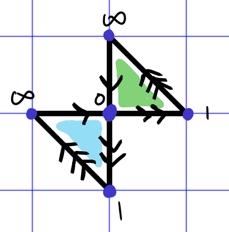
I'm omitting necessary self-intersection numbers along edges, e.g.

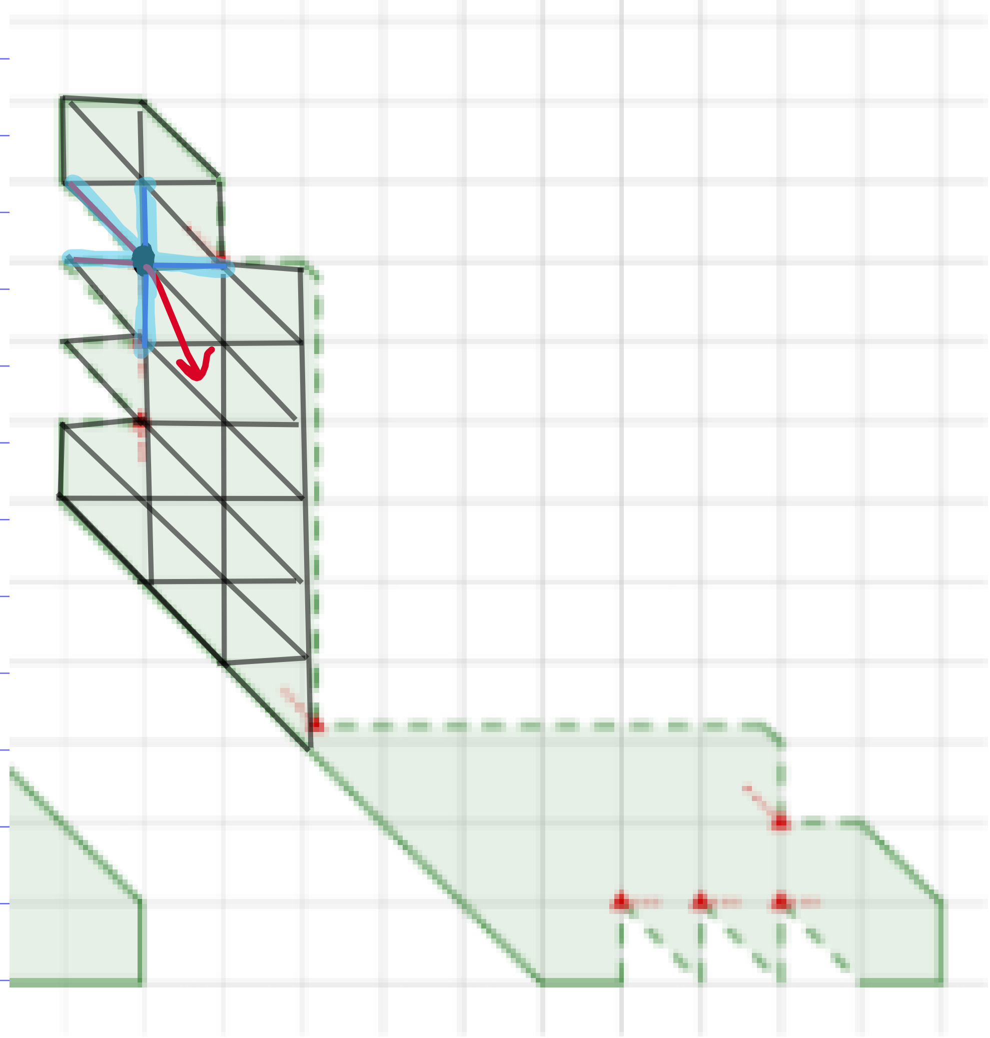


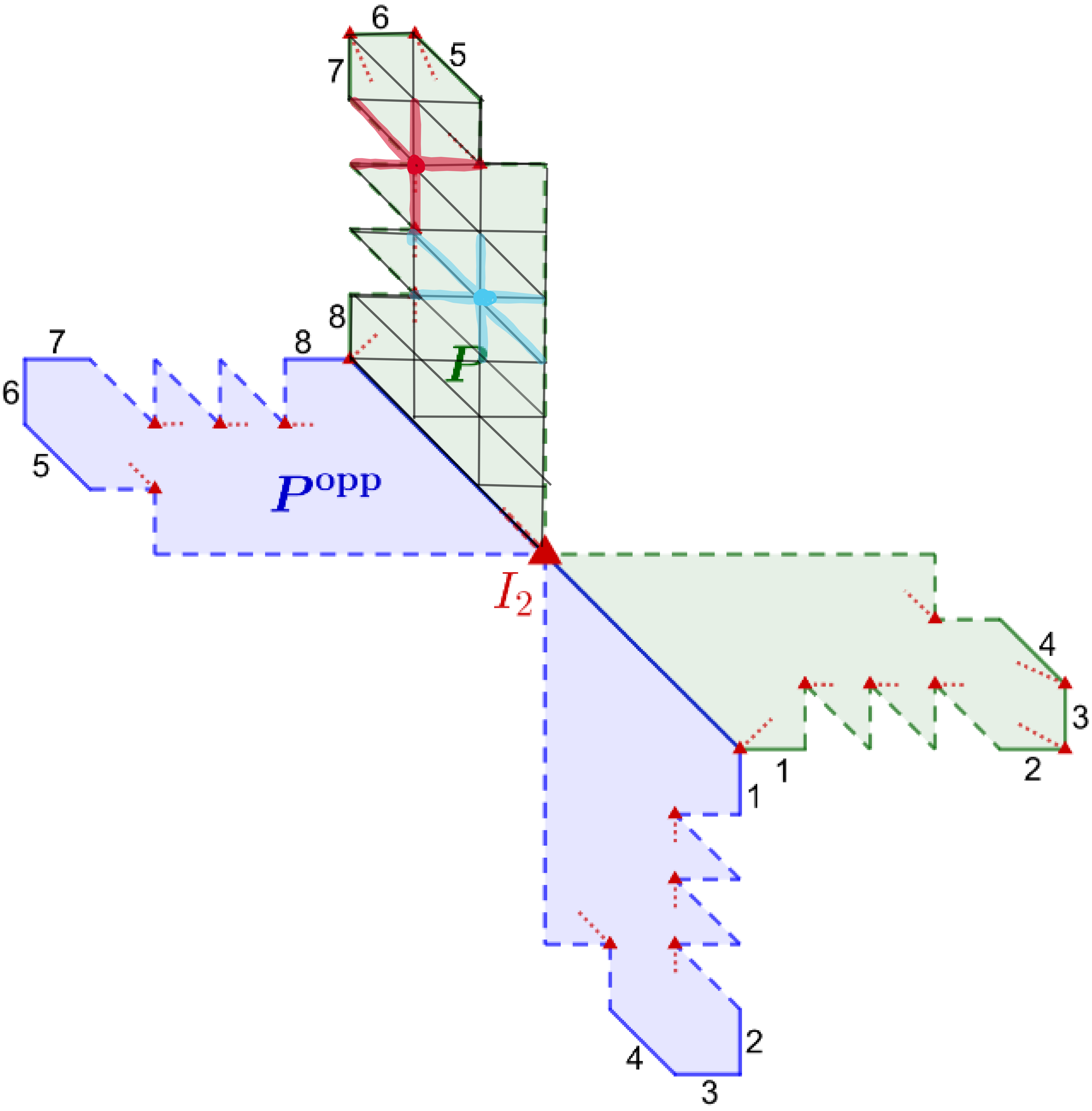
V_1
V_2
V_3
0
-2
-1
-1
0
0
-2
-1
-1
-2
-2
0
-1
-1
-1
-1
-1
3
-5
4
-6
-1
-1

Not accurate #'s; they must satisfy a triple pt. formula!

Adjacent V_i are glued along an anticanonical cycle of curves.


∞
∞




P
P^opp
I_2