Admissibility of Finite Groups over Semi-Global Fields

by

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(Under the Direction of Daniel Krashen)

Abstract

In 2009, Harbater, Hartmann, and Krashen gave a necessary and sufficient condition for a finite group G to be admissible over a semi-global field F so long as the characteristic of the corresponding residue field does not divide the order of G. They used a method known as field patching in order to show the sufficiency of their condition. Here we explore what happens when the characteristic of F (and also that of the residue field) divides the order of the group G. In particular, we show that if G is any p-group, where p is the characteristic of F, then it is admissible over F.

Index words: Admissibility, crossed products, field patching

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Chapter 1

Introduction and Historical Remarks

1.1 Admissibility over $\mathbb{Q}$

In his 1968 thesis work, Murray Schacher asked if one could find a k-central division algebra containing a root of a given irreducible polynomial $f \in k[x]$. If k is a stable[1] field, e.g. a global field, he showed that this is equivalent to asking if a given field extension K/k appears as a maximal subfield of some k-central division algebra. Even more, he was particularly interested in instances when these maximal subfields were Galois over k. Accordingly, he called a finite group G *k-admissible* if one could find a k-central division algebra with a G-Galois maximal subfield over k.

The majority of Schacher's work in [17] focused on the case where k is a number field. Among other things, he showed that if a finite group G is $\mathbb{Q}$-admissible, then its Sylow subgroups must necessarily be metacyclic, i.e. an extension of a cyclic group by another cyclic group. He believed, but was unable to prove, that this condition is also sufficient. At the time of this writing, the full conjecture remains open though several results ([16], [3], [4], [5], [6], [19]) have surfaced since then. Possibly the strongest result at this time is due

[1] A field k is stable if $\mathrm{per}(A) = \mathrm{ind}(A)$ for any central simple k-algebra A.

to Sonn who in [19] showed that Sylow metacyclic solvable groups are admissible over $\mathbb{Q}$.

1.2 Admissibility over Semi-Global Fields

In 2009, Harbater, Hartmann, and Krashen [9] investigated the admissibility problem in a different setting. Instead of working over global fields like $\mathbb{Q}$ and its finite extensions, they decided to approach the problem over particular semi-global fields. A *semi-global field* is a field which arises as the function field of a curve over a complete discretely valued field. An example would be the field $\mathbb{C}((t))(x)$, which is the function field of the $\mathbb{C}((t))$-curve $\mathbb{P}^1_{\mathbb{C}((t))}$. The crucial advantage that we gain by working over fields of the sort is that it allows us to use a technique known as field patching.

In a nutshell, field patching allows us (in certain cases) to take compatible structures over relatively nicer fields and patch them together to get a corresponding structure over the semi-global field. The main theorem proved in [9] utilizes patching in a way that allows them to solve the admissibility problem one Sylow subgroup at a time and patching these solutions together, ultimately arriving at the following:

Theorem 1.2.1. *Let K be a complete discretely valued field with algebraically closed residue field k and suppose F/K is a finitely generated extension with transcendence degree one. If G is a finite group such that $\mathrm{char}(k) \nmid |G|$, then G is F-admissible if and only if every Sylow subgroup of G is abelian of rank[2] at most two.*

This theorem impressively, though not entirely, characterizes the admissible groups over such semi-global fields. In fact, if $\mathrm{char}(k) = 0$, this theorem addresses the admissibility question over these fields completely; however, this naturally leads us down a path that we are compelled to embark on for the sake of completeness: what can be said in the case where $\mathrm{char}(k) \mid |G|$? It is precisely this question which we pursue in this document. In chapter

[2]The rank of a group is the cardinality of the smallest generating subset.

3, we will follow the methods of Harbater, Hartmann, and Krashen in order to prove the following extension of Theorem 1.2.1 in the case where $\mathrm{char}(K) = \mathrm{char}(k) = p$.

Theorem 3.2.3. Let K be a discretely valued field with algebraically closed residue field k and let F/K be a finitely-generated field extension with transcendence degree one. Assume further that $\mathrm{char}(K) = \mathrm{char}(k) = p$. Let G be a finite group whose Sylow-q subgroups are all abelian of rank at most 2 for $q \neq p$ and whose Sylow-p subgroup is cyclic. Then G is admissible over F.

The cyclicity of the Sylow subgroup corresponding to the characteristic of the residue field is likely far from necessary, but there is evidence that it could be a crucial stepping stone: using Saltman's results in [14], admissibility of cyclic p-groups imply admissibility of arbitrary p-groups, as summarized by the following result.

Theorem 3.2.4. Let K be a discretely valued field with residue field k and let F/K be a finitely-generated field extension with transcendence degree one. Assume that $\mathrm{char}(K) = \mathrm{char}(k) = p$. If G is a p-group, then G is admissible over F.

Because of this, it seems reasonable to expect that *any* group G whose Sylow q-subgroups are abelian of rank at most 2 when $q \neq p$ would be admissible over such fields as above.

Chapter 2

Background

Before we can discuss our main results, it would be helpful to introduce some definitions along with some fundamental results. This chapter will be home to these prerequisite topics. A daring reader may wish to skip this chapter entirely.

Throughout this document, a ring will have a multiplicative identity but will not be commutative unless otherwise specified. The set $\mathbb{N}$ of natural numbers will not contain 0. For a field F, F_s will denote a fixed separable closure of F.

2.1 Central Simple Algebras

Let F be a field. We begin by defining the objects and morphisms in the category of (associative) algebras over F. Throughout, we will only consider F-algebras which are finite-dimensional over F.

Definition 2.1.1. The **center** $Z(A)$ of a ring A is the set

$$Z(A) := \{r \in A \mid ra = ar \ \ \forall a \in A\}$$

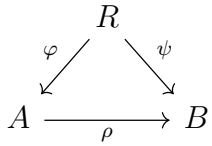
Definition 2.1.2. Let R be a commutative ring. An **algebra** (or R-algebra if we wish to emphasize the ring) is a ring A equipped with a ring homomorphism $\varphi : R \to A$ such that $\varphi(R) \subseteq Z(A)$. The map φ is sometimes referred to as the structure map of the algebra.

Equivalently, an R-algebra is a ring A equipped with the structure of an R-module such that $c \cdot (ab) = (c \cdot a)b = a(c \cdot b)$ for all $c \in R$ and for all $a, b \in A$.

Remark 2.1.3. Observe that the first definition matches the usual one from commutative algebra. The reason we require $\varphi(R) \subseteq Z(A)$ is so that the scalar multiplication is compatible with the ring multiplication, as expressed in the second definition.

Example 2.1.4. Since we require that a ring R must have a multiplicative identity 1_R, and since all ring homomorphisms must map multiplicative identities to multiplicative identities, we see that there always exists a unique ring homomorphism $\varphi\colon \mathbb{Z} \to R$. The image of this homomorphism must necessarily be contained in the center of the ring R, so we see that every ring can be regarded as an algebra over the integers in a unique way.

Definition 2.1.5. Let A and B be R-algebras with structure maps φ and ψ, respectively. An **algebra homomorphism** from A to B is a ring homomorphism $\rho : A \to B$ compatible with the structure maps in the sense that the following diagram commutes:



Although we have defined algebras over arbitrary commutative rings, we will mainly concern ourselves with algebras over fields throughout this document. Not only does this equip us with the tools of linear algebra but also this has the advantage of guaranteeing that our structure maps are injective, thus allowing us to think of our field as a subset of the algebra.

Definition 2.1.6. We define the **dimension** $\dim(A)$ of an F-algebra A to be its dimension as an F-vector space.

Definition 2.1.7. We say that an F-algebra A with structure map $\varphi\colon F \to A$ is **central** if A is finite-dimensional over F and $\varphi(F) = Z(A)$.

We now come to our central (no pun intended) object of study:

Definition 2.1.8. Given a division ring D, one can show that $F = Z(D)$ is indeed a field (see Example 2.1.1 in [7]), and in this case we say that D is an **F-central division algebra**.

Vista 2.1.9. As is often the case in mathematics, our study of central division algebras will greatly benefit from the flexibility afforded by expanding our scope to a larger class of objects. For example, it turns out that division algebras are not conducive for patching; however, this can be fixed by simply expanding our focus to include the more general central simple algebras. Instead of asking whether our division algebras patch together, we will instead show that the resulting central simple algebra is a division algebra.

Definition 2.1.10. Let A be a ring. We say that A is **simple** if it has no proper nonzero two-sided ideals.

Definition 2.1.11. Let A be an algebra with structure map $\varphi : F \to A$. We say that A is a **central simple F-algebra** (CSA) if it is both central and simple.

Example 2.1.12. We can immediately see from centrality that the only commutative central simple algebra over a given field is the field itself.

Example 2.1.13. By definition, every nonzero element of a division ring D is a unit, and the only ideal of D containing a unit is D itself. Thus, every division ring D is a central simple algebra over its center.

Example 2.1.14. Let D be a central division F-algebra, and let $M_n(D)$ be the collection of n-by-n matrices with entries in D. Then $M_n(D)$ itself is an F-algebra, with structure map

$$\begin{aligned} \varphi\colon F &\to M_n(D) \\ c &\mapsto c \cdot I \end{aligned}$$

where I is the identity matrix. Consider the matrices $E_{ij} \in M_n(D)$ whose entries are all 0 except for the ij-th entry, which is 1. If a matrix A has a nonzero ij-th entry a_{ij} where $i \neq j$, then $E_{ji}A \neq AE_{ji}$ (to see this, observe that the ii-th entry on the left is 0 while the ii-th entry on the right is a_{ij}). Thus, any matrix which is not a diagonal matrix cannot be in the center of $M_n(D)$. Now, if the matrix A is a diagonal matrix with entries a_{ij}, observe that the ij-th entry of $E_{ij}A$ is a_{jj}, while the ij-th entry of AE_{ij} is a_{ii}, which means that any matrix in the center must be a scalar matrix given by an element of D. Since $Z(D) = F$, we find that $Z(M_n(D)) = \varphi(F)$ and so $M_n(D)$ is central.

Now, let $I \subseteq M_n(D)$ be a nonzero two-sided ideal and let $A \in I$ be a nonzero matrix with nonzero entry a_{kl}, which we may assume to be equal to 1. Then, $E_{ij} = E_{ik}AE_{lj} \in I$ for all i and j. Since the E_{ij} generate $M_n(D)$, it follows that $I = M_n(D)$ and so $M_n(D)$ is also simple and thus a CSA over F.

This example illustrates how we may create new central simple algebras starting with a central division algebra D. Surprisingly, it turns out that every finite dimensional central simple algebra can be constructed in this way.

Theorem 2.1.15 (Wedderburn)**.** *Let A be a finite dimensional simple algebra over a field F. Then there exists a division F-algebra D, unique up to isomorphism, along with some $n \in \mathbb{N}$ such that $A \cong M_n(D)$.*

Proof. This is Theorem 2.1.3 in [7]. □

Definition 2.1.16. We say that a finite dimensional central simple algebra A over a field F is **split** if $A \cong M_n(F)$ for some positive integer n.

Proposition 2.1.17. *If k is an algebraically closed field, then every central simple k-algebra is split.*

Proof. This is Corollary 2.1.7 in [7]. □

Proposition 2.1.18. *Let A be a finite dimensional algebra over a field F and let K be some field extension of F. Then A is central simple over F if and only if $A \otimes_F K$ is central simple over K.*

Proof. See Proposition b.(ii) in section 12.4 of [12]. □

Corollary 2.1.19. *If A is a finite dimensional central simple algebra over a field F, then* $\dim(A)$ *is a square.*

Proof. According to the proposition, $A \otimes_F \bar{F}$ is a central simple $\bar{F}$-algebra; however, every central simple algebra over an algebraically closed field is split, so $A \otimes_F \bar{F} \cong M_n(\bar{F})$ for some $n \in \mathbb{N}$. Since $\dim(M_n(\bar{F})) = n^2$ and dimension is preserved by extending scalars, we may conclude that $\dim(A) = n^2$ which completes the proof. □

Definition 2.1.20. We define the **degree** of a central simple F-algebra A to be $\sqrt{\dim(A)}$.

Proposition 2.1.21. *Let A be a finite dimensional CSA over a field F. Then there exists a finite extension K of F such that $A \otimes_F K$ is split as a CSA over K. We may further assume that K is separable, or even Galois, over F.*

Proof. See Corollaries 2.2.11 and 2.2.12 in [7]. □

2.2 Brauer Groups and Cohomology

Proposition 2.2.1. *If A and B are finite dimensional central simple algebras over some field F, then so is $A \otimes_F B$.*

Proof. This is Proposition b.(i) in section 12.4 of [12]. □

Let $\mathrm{CSA}(F)$ be the collection of isomorphism classes of finite dimensional central simple algebras over F. We define an equivalence relation $\sim$ on $\mathrm{CSA}(F)$ by

$$A \sim B \iff \exists n, m \in \mathbb{N} \text{ such that } A \otimes_F M_n(F) \cong B \otimes_F M_m(F)$$

We shall refer to this equivalence relation as "Brauer equivalence." We define $\mathrm{Br}(F)$ to be the quotient $\mathrm{CSA}(F)/\sim$. We will refer to the equivalence class containing the central simple algebra A as $[A]$. One can show (see section 12.5 of [12] for example) that this quotient inherits a binary operation which turns $\mathrm{Br}(F)$ into a monoid via the tensor product using Proposition 2.2.1 with identity $[F]$. In fact, even more is true. We shall require the following definition and proposition.

Definition 2.2.2. Let A be an F-algebra with multiplication given by $\star$. The **opposite algebra** A^{op} is defined as follows. As an abelian group, it is exactly the same as $(A, +)$. As a ring, we define the multiplication $\hat{\star}$ by

$$a \,\hat{\star}\, b := b \star a$$

Notice that if $\varphi : F \to A$ is the structure map for the algebra A, then the same function will satisfy the requirements for a structure map on A^{op}, and so A^{op} is also an F-algebra.

Remark 2.2.3. One can immediately convince oneself that if A is central, then also A^{op} is central. Similarly, if A is simple then so is A^{op}. Thus by Proposition 2.2.1, so is $A \otimes_F A^{\mathrm{op}}$.

Proposition 2.2.4. *If A is an n-dimensional simple F-algebra, then $A \otimes_F A^{\mathrm{op}} \cong M_n(F)$.*

Proof. Since A is simple, it follows that both A^{op} and $A \otimes_F A^{\mathrm{op}}$ are also simple. Now, we define a function $\sigma : A \otimes_F A^{\mathrm{op}} \to \mathrm{End}(A)$ by taking $a \otimes b$ to the map $x \mapsto axb$ and extending by linearity. Observe that in fact, σ is an F-algebra homomorphism (pay particular attention to how the multiplication in the opposite algebra is being used). Since $A \otimes_F A^{\mathrm{op}}$ is simple and $\sigma(1 \otimes 1) = \mathbb{1}_A$, σ must have a trivial kernel and so must be injective. Comparing dimensions and noting $M_n(F) \cong \mathrm{End}(A)$ as F-algebras completes the proof. □

In other words, we find that in the quotient $\mathrm{Br}(F)$, $[A] \cdot [A^{\mathrm{op}}] = [M_n(F)] = [F]$, which means $\mathrm{Br}(F)$ is in fact an abelian group. We refer to $\mathrm{Br}(F)$ as the Brauer group of F. One may also speak of the Brauer group in a more relative manner. Let $\mathrm{Br}(K|F)$ be the Brauer equivalence classes of finite dimensional CSAs over F split by the extension K/F. We call this subgroup of $\mathrm{Br}(F)$ the Brauer group of F relative to K. Since every central simple algebra is split by a Galois extension K/F, we see that

$$\mathrm{Br}(F) = \bigcup \mathrm{Br}(K|F)$$

where the union is over all finite Galois extensions K/F inside a fixed separable closure F_s of F.

As it turns out, it will be fruitful to consider a different characterization of the Brauer group using Galois cohomology. Here, we will skip over the general theory, instead focusing on just the second cohomology group.

Definition 2.2.5. Let G be a finite group. An abelian group A is a **(left) G-module** if it is equipped with a left action by G, i.e. if there is a group homomorphism $\varphi\colon G \to \mathrm{End}_{\mathbb{Z}} A$. As is standard, we will denote $(\varphi(\sigma))(a)$ simply as $\sigma \cdot a$.

Definition 2.2.6. Let G be a finite group. A **1-cocycle** a with values in some G-module A is a function

$$a\colon G \to A$$

such that for any $\sigma, \tau \in G$,

$$a(\sigma\tau) = a(\sigma) + \sigma \cdot a(\tau)$$

Definition 2.2.7. Let G be a finite group. A **2-cocycle** c with values in some G-module A is a function

$$c : G \times G \to A$$

such that for any $\sigma, \tau, \rho \in G$,

$$\sigma \cdot c(\tau, \rho) - c(\sigma\tau, \rho) + c(\sigma, \tau\rho) - c(\sigma, \tau) = 0$$

We say that the 2-cocycle c is a **2-coboundary** if there exists a 1-cocycle a such that

$$c(\sigma, \tau) = \sigma \cdot a(\tau) - a(\sigma\tau) + a(\sigma)$$

Example 2.2.8. Let E/F be a degree n cyclic field extension with Galois group G generated by σ and let $b \in E^\times$. For each $\tau \in G$, we may associate a unique integer i with $0 \leq i < n$ such that $\tau = \sigma^i$. Using this, we define a function $c\colon G \times G \to E^\times$ as follows:

$$c(\sigma^i, \sigma^j) = \begin{cases} 1 & i + j < n \\ b & \text{else} \end{cases}$$

We can check to see that c is indeed a 2-cocycle.

One can show using the machinery of Galois cohomology that the Brauer group of a field F is in fact isomorphic to the second Galois cohomology group $H^2(F, F_s^\times)$ whose elements are (continuous) 2-cocycles modulo 2-coboundaries (Theorem 4.4.3 in [7]). We can describe this isomorphism more explicitly using crossed-product algebras.

Construction 2.2.9. Suppose E/F is a G-Galois extension for some finite group G and let $c\colon G \times G \to E$ be a 2-cocycle with values in E (which is naturally a G-module via the Galois action). Let $\Delta(E, G, c)$ be the (left) E-vector space with basis $\{u_\sigma \mid \sigma \in G\}$. We endow $\Delta(E, G, c)$ a ring structure by defining

$$u_\sigma u_\tau := c(\sigma, \tau) u_{\sigma\tau}$$

$$u_\sigma e := \sigma(e) u_\sigma$$

for all $\sigma, \tau \in G$ and for all $e \in E$. Of course, one should wonder if this multiplication is associative, and this is precisely where we would use the 2-cocycle condition. In fact, we can show that $\Delta(E, G, c)$ is indeed a central simple F-algebra.

Definition 2.2.10. We say that a central simple F-algebra A is a **crossed-product algebra** if it is isomorphic to $\Delta(E, G, c)$ for some choice of E, G, and c as above.

If the 2-cocycle we choose is 2-coboundary, then the corresponding algebra $\Delta(E, G, c)$ is Brauer trivial. Using this construction gives us a map $H^2(F, F_s^\times) \to \mathrm{Br}(F)$ which is indeed an isomorphism (Corollary 7.8 in [15]). Thus, every central simple algebra is Brauer equivalent to a crossed-product algebra.

2.3 Witt Vectors and Cyclic Extensions

Throughout this section, let p be some prime number. Let F be a field and suppose that $\mathrm{char}(F) = p$. Artin-Schreier theory tells us that any degree p cyclic extension of F is gotten

by adjoining to F a root α of some irreducible polynomial of the form $x^p - x - a \in F[x]$. In fact, the existence of degree p cyclic extensions is equivalent to the existence of some $a \in F$ such that $x^p - x - a$ is irreducible over F. Even more, Artin and Schreier showed that the same hypothesis guarantees the existence of degree p^2 cyclic extensions; however, for our purposes here, we will need to be able to construct degree p^m cyclic extensions over such fields F for arbitrarily large m. This was done by Witt using Witt vectors.

For explicit constructions, refer to [18] or [10]. To motivate the construction of Witt vectors, let us first recall some examples of ring structures one might impose on the set R^m given some ring R.

Example 2.3.1. The most natural way to do this is by applying the operations in R component-wise, which would give our standard ring structure on R^m. More explicitly, if $(x_0, ..., x_{m-1})$ and $(y_0, ..., y_{m-1})$ are two elements of R^m, we have

$$(x_0, ..., x_{m-1}) + (y_0, ..., y_{m-1}) = (x_0 + y_0, ..., x_{m-1} + y_{m-1})$$
$$(x_0, ..., x_{m-1}) \cdot (y_0, ..., y_{m-1}) = (x_0 y_0, ..., x_{m-1} y_{m-1})$$

Example 2.3.2. Let us now consider a slightly more mysterious ring structure on R^4. Given elements $(x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3) \in R^4$, define addition component-wise as before but this time, define multiplication as:

$$(x_0, x_1, x_2, x_3) \cdot (y_0, y_1, y_2, y_3) = (x_0 y_0 + x_1 y_2, x_0 y_1 + x_1 y_3, x_2 y_0 + x_3 y_2, x_2 y_1 + x_3 y_3)$$

It may seem bizarre at first until you realize that this multiplication endows R^4 with the familiar structure of $M_2(R)$.

An interesting and important observation from Example 2.3.2 is that we are not restricted to using only the corresponding components from the two inputs we are given. So long as

the appropriate ring axioms are satisfied, we are welcome to use more intricate polynomials if we wish. The only thing to be careful about is where we take our coefficients from. Notice that the component polynomials defining these binary operations in both examples make sense regardless of the ring R. This is because every ring R is an algebra over $\mathbb{Z}$ and all coefficients of the component polynomials are integers.

We will now define new binary operations on R^m using polynomials $S_0, ..., S_{m-1}$ for addition and $P_0, ..., P_{m-1}$ for multiplication. Special care must be taken to make sure these polynomials would make sense for any ring R, which is to say that we will want these polynomials to have integer coefficients as discussed above.

Let $X_* = (X_n)_{n=0}^{\infty}$ be a sequence of indeterminates and consider the polynomials

$$\begin{aligned} W_0(X_*) &= X_0 \\ W_1(X_*) &= X_0^p + pX_1 \\ W_2(X_*) &= X_0^{p^2} + pX_1^p + p^2X_2 \\ &\vdots \\ W_n(X_*) &= \sum_{i=0}^{n} p^i X_i^{p^{n-i}} \end{aligned}$$

The polynomials $W_n(X_*)$ are known as the ghost components of X_*. Let $Y_* = (Y_n)_{n=0}^{\infty}$ be another sequence of indeterminates. Then of course we also have ghost components for Y_*. The idea then is to define the sum (respectively, product) of X_* and Y_* so that the corresponding ghost components of the sum (respectively, product) will be the component-wise sum (respectively, product) of the ghost components for X_* and Y_*. It may not be immediately clear that we can define this sum (respectively, product) in terms of polynomials in X_* and Y_* with integer coefficients, but we certainly can, as the next theorem (which is Theorem 6 in section 2.6 of [18] and whose proof we will omit here) guarantees.

Theorem 2.3.3. *Let $\Phi \in \mathbb{Z}[X, Y]$. Then there is a unique sequence of elements $\phi_* = (\phi_n)_{n=0}^{\infty}$ in $\mathbb{Z}[X_*, Y_*]$ such that*

$$W_n(\phi_*) = \Phi(W_n(X_*), W_n(Y_*))$$

In particular, the polynomial $\Phi = X + Y$ gives rise to a sequence of polynomials $S_* = (S_n)_{n=0}^{\infty}$ and the polynomial $\Phi = XY$ gives rise to a sequence of polynomials $P_* = (P_n)_{n=0}^{\infty}$. It will be helpful to collect some useful facts concerning these polynomials.

Facts 2.3.4.

1. The polynomials S_n and P_n are in $\mathbb{Z}[X_0, \ldots, X_n, Y_0, \ldots, Y_n]$.
2. In particular, $S_n = X_n + Y_n + \gamma$ where γ_n is a polynomial in $X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}$.
3. The polynomials S_n and P_n have 0 constant term.

We can now define the ring $W_m(R)$. As a set, it is simply R^m. Given two elements $a = (a_0, \ldots, a_{m-1})$ and $b = (b_0, \ldots, b_{m-1})$, define

$$a + b := (S_0(a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1}), \ldots, S_{m-1}(a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1}))$$
$$ab := (P_0(a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1}), \ldots, P_{m-1}(a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1}))$$

These operations make $W_m(R)$ into a ring. Furthermore, given another characteristic p ring S and a ring homomorphism $\varphi\colon R \to S$, there is an induced homomorphism $\phi_*\colon W_m(R) \to W_m(S)$ where we apply ϕ component-wise. Since the binary operations in these rings of Witt vectors are given by polynomials which are necessarily preserved by φ, this makes sense. One can check that this defines a functor from the category of characteristic p rings to characteristic p^m rings.

Equipped with these definitions, we now proceed to the theory of degree p^n cyclic extensions of a field F with characteristic p. To begin, let K/F be a Galois extension with Galois group G. Then we can equip $W_m(K)$ with a G action by declaring that $\sigma \cdot \rho := \sigma_*(\rho)$ (that is, we let σ act component-wise on ρ, which makes sense as discussed above). This gives $W_m(K)$ the structure of a G-module.

Proposition 2.3.5. *Let F be a field with $\mathrm{char}(F) = p$. Then $H^1(F, W_m(F_s)) = 0 \ \forall m \in \mathbb{N}$.*

Proof. The proof is by induction on m. The case $m = 1$ is the additive version of Hilbert's Theorem 90. Suppose the statement holds for any $k < m$ for some $m \in \mathbb{N}$ and let G be the absolute Galois group of F. We have a homomorphism $T\colon W_m(F_s) \to W_{m-1}(F_s)$ defined by $(\beta_0, \ldots, \beta_{m-2}, \beta_{m-1}) \mapsto (\beta_0, \ldots, \beta_{m-2})$ whose kernel can be identified with $W_1(F_s)$. Thus we have a short exact sequence of G-modules

$$1 \to W_1(F_s) \to W_m(F_s) \to W_{m-1}(F_s) \to 1$$

which gives rise to the following exact sequence

$$H^1(F, W_1(F_s)) \to H^1(F, W_m(F_s)) \to H^1(F, W_{m-1}(F_s))$$

Since the outer groups are trivial by hypothesis, so is the middle group. □

In order to generalize Artin-Schreier theory, following Witt, we will need to extend the Artin-Schreier map on F to one on $W_m(F)$. Let $P\colon W_m(F) \to W_m(F)$ be the homomorphism induced by the Frobenius endomorphism on F and define $\wp\colon W_m(F) \to W_m(F)$ by

$$\wp(\rho) = P(\rho) - \rho$$

It should be noted that just as the usual Artin-Schreier map on F is not a ring homomor-

phism, neither is this one on $W_m(F)$. It is, however, an endomorphism of the additive group of $W_m(F)$.

Proposition 2.3.6. *Let F be a field with $\mathrm{char}(F) = p$ and let $\beta = (\beta_0, \ldots, \beta_{m-1}) \in W_m(F)$. Then there exists $\rho = (\rho_0, \ldots, \rho_{m-1}) \in W_m(F_s)$ such that $\wp(\rho) = \beta$ and ρ_i is a root of*

$$f_i(x) = x^p - x - \gamma_i(\rho_0, \ldots, \rho_{i-1}, \beta_0, \ldots, \beta_{i-1}) - \beta_i$$

where γ_i is the polynomial $S_i - X_i - Y_i$ described in Facts 2.3.4. Furthermore, if $\rho' = (\rho'_0, \ldots, \rho'_{m-1}) \in W_m(F_s)$ also satisfies $\wp(\rho') = \beta$, then ρ'_i must be a root of

$$f'_i(x) = x^p - x - \gamma_i(\rho'_0, \ldots, \rho'_{i-1}, \beta_0, \ldots, \beta_{i-1}) - \beta_i$$

Proof. The proof is by induction on m. For $m = 1$, we can let ρ_0 be a root of the separable polynomial $f_1(x) = x^p - x - \beta_0$. Now suppose the proposition holds for some $m \in \mathbb{N}$ and consider the Witt vector $(\rho_0, \ldots, \rho_{m-1}, x)$. If

$$(\rho_0^p, \ldots, \rho_{m-1}^p, x^p) - (\rho_0, \ldots, \rho_{m-1}, x) = (\beta_0, \ldots, \beta_m)$$

then by Facts 2.3.4 we know that $x^p = x + \beta_m + \gamma_m(\rho_0, \ldots, \rho_{i-1}, \beta_0, \ldots, \beta_{m-1})$ where γ_m is as described in Facts 2.3.4. Then if we let ρ_m be a root of

$$f_m(x) = x^p - x - \gamma_m(\rho_0, \ldots, \rho_{i-1}, \beta_0, \ldots, \beta_{m-1}) - \beta_m$$

we find that $\wp(\rho_0, \ldots, \rho_m) = (\beta_0, \ldots, \beta_m)$. Finally, if $\rho' = (\rho'_0, \ldots, \rho'_{m-1}) \in W_m(F_s)$ also satisfies $\wp(\rho') = \beta$, then the same process applied above shows that ρ'_i is a root of the polynomial $f'_i(x) = x^p - x - \gamma_i(\rho'_0, \ldots, \rho'_{i-1}, \beta_0, \ldots, \beta_{i-1}) - \beta_i$. □

Proposition 2.3.7. *Let F be a field with $\mathrm{char}(F) = p$, let F' be the prime subfield of F, and let G be the absolute Galois group of F. Then there is an isomorphism*

$$\delta\colon W_m(F)/\wp(W_m(F)) \to \mathrm{Hom}(G, W_m(F')).$$

Proof. By Proposition 2.3.6, the map $\wp\colon W_m(F_s) \to W_m(F_s)$, whose kernel is $W_m(F')$, is surjective. Equipping $W_m(F_s)$ with its usual Galois action (by letting $\sigma \in G$ act componentwise), we get an exact sequence of G-modules

$$0 \longrightarrow W_m(F') \longrightarrow W_m(F_s) \xrightarrow{\wp} W_m(F_s) \longrightarrow 0$$

Since G acts on $W_m(F')$ trivially, we find that $H^1(F, W_m(F')) \cong \mathrm{Hom}(G, W_m(F'))$. Using this along with the short exact sequence above gives us an exact sequence

$$W_m(F) \xrightarrow{\wp} W_m(F) \xrightarrow{\tilde{\delta}} \mathrm{Hom}(G, W_m(F')) \longrightarrow H^1(F, W_m(F_s))$$

where the last group is trivial by Proposition 2.3.5. Descending $\tilde{\delta}$ to the quotient gives us the required isomorphism δ. □

It will be immensely helpful for us to examine the map δ above. First, let us recall how the boundary map $\tilde{\delta}$ is defined. Given $\beta \in W_m(F)$, choose $\rho \in W_m(F_s)$ such that $\wp(\rho) = \beta$. Now we may define a homomorphism $\chi_\rho\colon G \to W_m(F')$ by the assignment $\sigma \mapsto \sigma(\rho) - \rho$. This homomorphism, as it turns out, does not depend on our choice of ρ, and the assignment $\beta \mapsto \chi_\rho$ defines $\tilde{\delta}$.

The significance of this proposition is that it allows us to construct cyclic extensions of F of degree p^k. Let K_ρ be the kernel of χ_ρ. This then corresponds to a Galois extension L/F with Galois group isomorphic to G/K_ρ, which in turn is isomorphic to a subgroup of $W_m(F')$. But since $W_m(F') \cong \mathbb{Z}/p^m\mathbb{Z}$, every subgroup will be cyclic and so the extension L/F is a cyclic extension.

But can we construct the field L more concretely using β? Well, recall that L is the fixed field of K_ρ, the kernel of χ_ρ. By definition, K_ρ consists of $\sigma \in G$ such that $\sigma(\rho) = \rho$. Thus if $\rho = (\rho_0, \ldots, \rho_{m-1})$, $L = F(\rho_0, \ldots, \rho_{m-1})$. As discussed, χ_ρ (and so also K_ρ and L) does not depend on the choice of ρ. For this reason, we will denote $F(\rho_0, \ldots, \rho_{m-1})$ by $F(\wp^{-1}(\beta))$.

All that remains is to find the degree of this extension; luckily, the degree of the extension is equal to the size of the corresponding subgroup of $W_m(F')$, which itself is equal to the order of $\chi_\rho \in \mathrm{Hom}(G, W_m(F'))$ and by the proposition, this is equal to the order of $\overline{\beta} \in W_m(F)/\wp(W_m(F))$. Finally, the following result (which is essentially Lemma 3 in section 8.11 of [10]) tells us precisely when $\overline{\beta}$ has order p^m in the quotient $W_m(F)/\wp(W_m(F))$.

Lemma 2.3.8. *Let $\beta = (\beta_0, \ldots, \beta_{m-1}) \in W_m(F)$. Then $\overline{\beta} \in W_m(F)/\wp(W_m(F))$ has order p^m if and only if the polynomial $x^p - x - \beta_0$ has no roots in F.*

We will end this section with a discussion on what we gain by considering these constructions over a field which is complete with respect to a non-Archimedean absolute value. We begin by collecting some results which we will use in this discussion. The first result makes precise the fact that the roots of a polynomial vary continuously in terms of its coefficients.

Lemma 2.3.9. *Let F be a normed field, $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in F[x]$, and fix an extension of the norm to the algebraic closure $\overline{F}$. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $g(x) = b_n x^n + \cdots + b_1 x + b_0 \in F[x]$ satisfies $|a_i - b_i| < \delta$ for each i, then there exists orderings of the roots $\alpha_1, \ldots, \alpha_n$ of f and the roots $\beta_1, \ldots, \beta_n$ of g such that $|\alpha_i - \beta_i| < \varepsilon$ for all i.*

Proof. This follows from Theorem 2 in [2]. □

The following result is an analogue of the previous one in the realm of Witt vectors. It says that an Artin-Schreier root $\wp^{-1}(\beta)$ also depends continuously on β.

Lemma 2.3.10. *Suppose F is field with $\mathrm{char}(F) = p$ such that F_s is a normed field. Let $\beta = (\beta_0, \ldots, \beta_{m-1}) \in W_m(F)$ and $\rho = (\rho_0, \ldots, \rho_{m-1}) \in W_m(F_s)$ be such that $\wp(\rho) = \beta$. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\beta' = (\beta'_0, \ldots, \beta'_{m-1}) \in W_m(F)$ satisfies $|\beta_i - \beta'_i| < \delta$ for each i, then we may find $\rho' = (\rho'_0, \ldots, \rho'_{m-1}) \in W_m(F_s)$ satisfying $\wp(\rho') = \beta'$ and $|\rho_i - \rho'_i| < \varepsilon$ for each i.*

Proof. The proof is by induction on m. For the case $m = 1$, the result is a direct consequence of Lemma 2.3.9. Now suppose the statement holds for some $m \in \mathbb{N}$; let $\varepsilon > 0$, and let $\beta = (\beta_0, \ldots, \beta_m) \in W_{m+1}(F)$ and $\rho = (\rho_0, \ldots, \rho_m) \in W_{m+1}(F_s)$ be such that $\wp(\rho) = \beta$. We have a polynomial $c_m = \gamma_m + Y_m \in \mathbb{Z}[X_0, \ldots, X_{m-1}, Y_0, \ldots, Y_m]$ where $\gamma_m \in \mathbb{Z}[X_0, \ldots, X_{m-1}, Y_0, \ldots, Y_{m-1}]$ is the one given by Facts 2.3.4. By Proposition 2.3.6, we know that ρ_m is a root of the polynomial $f(x) = x^p - x - c_m(\rho_0, \ldots, \rho_{m-1}, \beta_0, \ldots, \beta_m)$. Again by Lemma 2.3.9, there exists $\overline{\varepsilon} > 0$ such that $|c_m(\rho_0, \ldots, \rho_{m-1}, \beta_0, \ldots, \beta_m) - c| < \overline{\varepsilon}$ implies that the polynomial $x^p - x - c$ has a root ρ'_m such that $|\rho_m - \rho'_m| < \varepsilon$. Now, if we endow F^{2m-1} the product topology induced by the norm on F, evaluation of c_m gives a continuous map $F^{2m-1} \to F$, so there exists $\delta_1 > 0$ such that whenever $|\rho_i - \rho'_i| < \delta_1$ for $0 \leq i \leq m-1$ and $|\beta_j - \beta'_j| < \delta_1$ for $0 \leq j \leq m$, $|c_m(\rho_0, \ldots, \rho_{m-1}, \beta_0, \ldots, \beta_m) - c_m(\rho'_0, \ldots, \rho'_{m-1}, \beta'_0, \ldots, \beta'_m)| < \overline{\varepsilon}$. Without loss of generality, we may assume that $\delta_1 \leq \varepsilon$. Letting $\overline{\beta} = (\beta_0, \ldots, \beta_{m-1})$ and $\overline{\rho} = (\rho_0, \ldots, \rho_{m-1})$, we know that $\wp(\overline{\rho}) = \overline{\beta}$, and so we may apply the inductive hypothesis to find $\delta_2 > 0$ such that whenever $\overline{\beta}' = (\beta'_0, \ldots, \beta'_{m-1}) \in W_m(F)$ satisfies $|\beta_i - \beta'_i| < \delta_2$ for $0 \leq i \leq m-1$, then we may find $\overline{\rho}' = (\rho'_0, \ldots, \rho'_{m-1}) \in W_m(F_s)$ satisfying $\wp(\overline{\rho}') = \overline{\beta}'$ and $|\rho_i - \rho'_i| < \delta_1$ for $0 \leq i \leq m-1$. Thus if we let $\delta = \min(\delta_1, \delta_2)$, then whenever we have $\beta' = (\beta'_0, \ldots, \beta'_m) \in W_m(F)$ satisfying $|\beta_j - \beta'_j| < \delta$ for each i, we can find $\rho'_0, \ldots, \rho'_{m-1} \in F_s$ such that $|\rho_i - \rho'_i| < \delta_1$ for $0 \leq i \leq m-1$. Since also $|\beta_j - \beta'_j| < \delta_1$ for $0 \leq j \leq m$, it follows that $|c_m(\rho_0, \ldots, \rho_{m-1}, \beta_0, \ldots, \beta_m) - c_m(\rho'_0, \ldots, \rho'_{m-1}, \beta'_0, \ldots, \beta'_m)| < \overline{\varepsilon}$. Finally, letting ρ'_m be a root of $g(x) = x^p - x - c_m(\rho'_0, \ldots, \rho'_{m-1}, \beta'_0, \ldots, \beta_m)$ such that $|\rho_m - \rho'_m| < \varepsilon$ completes the proof. $\square$

A striking result in algebraic number theory states that when working over a field F which is complete with respect to some non-Archimedean norm, one can, in some sense, approximate separable extensions. Suppose L/F is a finite separable extension. Then by the primitive element theorem, $L = F(\alpha)$ where α is a root of some polynomial $m(x) \in F[x]$. The assertion, then, is even if we perturb the coefficients of $m(x)$ up to some threshold, adjoining a root of the new polynomial yields the same extension. If the new coefficients belong to a subfield of F, this process allows us to then pull back the extension L/F to a separable extension over the subfield.

Our goal for the remainder of this section is to prove an analogous statement in terms of Witt vectors. Mirroring the proof of the previous assertion, ours will crucially rely on Krasner's lemma which we now state.

Lemma 2.3.11 (Krasner's Lemma)**.** *Let F be a complete non-Archimedean normed field and let $\alpha \in F_s$ with Galois conjugates $\alpha_0, \alpha_1, \ldots, \alpha_n \in F$ where $\alpha_0 = \alpha$. If $\beta \in F_s$ such that*

$$|\alpha - \beta| \leq |\alpha - \alpha_i|$$

for $1 \leq i \leq n$, then $\alpha \in F(\beta)$.

Proof. See p. 141 in [13]. □

Proposition 2.3.12. *Let F be a field which is complete with respect to a non-Archimedean norm with $\mathrm{char}(F) = p$ and suppose we have $\beta = (\beta_0, \ldots, \beta_{m-1}) \in W_m(F)$ such that $\overline{\beta} \in W_m(F)/\wp(W_m(F))$ has order p^m. Then there exists some $\varepsilon > 0$ such that for any $\beta' = (\beta'_0, \ldots, \beta'_{m-1}) \in W_m(F)$ satisfying $|\beta_i - \beta'_i| < \varepsilon$ for all i, $F(\wp^{-1}(\beta)) = F(\wp^{-1}(\beta'))$.*

Proof. The proof is by induction on m. We will denote by $|\cdot|$ the extension of the norm of F to the separable closure. For the case $m = 1$, let ρ_0 be a root of the polynomial $f_0(x) = x^p - x - \beta_0$ (which must be irreducible by the assumption on $\overline{\beta}$) and let δ be the

minimum distance between any two roots of $f_0(x)$. By Lemma 2.3.10, there exists $\varepsilon > 0$ such that for any $\beta_0' \in F$ with $|\beta_0 - \beta_0'| < \varepsilon$, $g_0(x) = x^p - x - \beta_0'$ has a root ρ_0' satisfying $|\rho_0 - \rho_0'| < \delta$. Then by Krasner's lemma, $F(\wp^{-1}(\beta_0)) \subseteq F(\wp^{-1}(\beta_0'))$. But $[F(\wp^{-1}(\beta_0')) : F] \leq [F(\wp^{-1}(\beta_0)) : F]$, so in fact $F(\wp^{-1}(\beta_0)) = F(\wp^{-1}(\beta_0'))$. Now suppose the statement holds for some $m \in \mathbb{N}$. Let $\beta = (\beta_0, \ldots, \beta_m) \in W_{m+1}(F)$ and choose $\rho = (\rho_0, \ldots, \rho_m) \in W_{m+1}(F_s)$ such that $\wp(\rho) = \beta$. Let $L = F(\wp^{-1}(\overline{\beta}))$ where $\overline{\beta} = (\beta_0, \ldots, \beta_{m-1})$. As in the proof of Lemma 2.3.10, let $c_m = \gamma_m + Y_m$. By Proposition 2.3.6, we know that $F(\wp^{-1}(\beta)) = L(\rho_m)$ where ρ_m is a root of the polynomial $f(x) = x^p - x - c_m(\rho_0, \ldots, \rho_{m-1}, \beta_0, \ldots, \beta_m)$. We should first note that $f(x)$ is irreducible over L, for otherwise the degree of $F(\wp^{-1}(\beta))/F$ would be strictly less than p^{m+1}, contradicting the assumption on $\overline{\beta}$. As before, let δ be the minimum distance between any two roots of $f(x)$. Again by Lemma 2.3.10, there exists $\varepsilon_1 > 0$ such that for any $\beta' = (\beta_0', \ldots, \beta_m') \in W_{m+1}(F)$ satisfying $|\beta_i - \beta_i'| < \varepsilon_1$ for each i, then there exists $\rho' = (\rho_0', \ldots, \rho_m') \in W_m(F_s)$ such that $\wp(\rho') = \beta'$ and $|\rho_i - \rho_i'| < \delta$ for each i. By the inductive hypothesis, there exists $\varepsilon_2 > 0$ such that whenever $\overline{\beta}' = (\beta_0', \ldots, \beta_{m-1}') \in W_m(F)$ satisfies $|\beta_i - \beta_i'| < \varepsilon_2$ for $0 \leq i \leq m-1$, $L = F(\wp^{-1}(\overline{\beta})) = F(\wp^{-1}(\overline{\beta}'))$. Taking $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, we then see that given any $\beta' = (\beta_0', \ldots, \beta_m') \in W_{m+1}(F)$ satisfying $|\beta_i - \beta_i'| < \varepsilon$ for each i, we have $\rho' = (\rho_0', \ldots, \rho_m') \in W_m(F_s)$ such that $\wp(\rho') = \beta'$ and $|\rho_i - \rho_i'| < \delta$ for each i. Thus,

$$F(\wp^{-1}(\beta')) = L(\rho_m') = L(\rho_m) = F(\wp^{-1}(\beta))$$

where the second equality again follows from Krasner's lemma as it did in the base case. $\square$

Chapter 3

Admissibility over Semi-Global Fields

The main result of this document will be discussed in section 2. Before then, we will formalize our understanding of field patching, as well as state some fundamental results from [9] which will be used later.

3.1 Field Patching

The method of patching over fields was developed by Harbater and Hartmann back in 2007 and takes inspiration from "cut-and-paste" constructions in topology and analysis. Other forms of patching exist, e.g. formal and rigid patching, but we shall not discuss those here. Most of what follows come from [8] and [9].

Definition 3.1.1. Let I be a finite partially ordered set and let $\mathcal{F} = \{F_i\}_{i \in I}$ be an inverse system of fields indexed by I with inclusion maps $\iota_{ij} : F_i \to F_j$ whenever $i \geq j$. A **patching problem** $\mathcal{V}$ for the inverse system $\mathcal{F}$ is given by the following data:

1. A finite dimensional F_i-vector space V_i for each $i \in I$.

2. F_i-linear maps $\nu_{ij} \colon V_i \to V_j$ whenever $i \geq j$ such that the induced maps $V_i \otimes_{F_i} F_j \to V_j$ are isomorphisms.

Definition 3.1.2. Given patching problems $\mathcal{V} = \{V_i\}_{i\in I}$ and $\mathcal{V}' = \{V_i'\}_{i\in I}$ on the inverse system $\mathcal{F} = \{F_i\}_{i\in I}$, a **morphism of patching problems** $\mathcal{V} \to \mathcal{V}'$ will be given by an F_i-linear map $\eta_i \colon V_i \to V_i'$ for each $i \in I$ such that whenever $i \geq j$, the following square commutes:

$$\begin{array}{ccc} V_i & \xrightarrow{\nu_{ij}} & V_j \\ {\scriptstyle \eta_i}\downarrow & & \downarrow{\scriptstyle \eta_j} \\ V_i' & \xrightarrow[\nu_{ij}']{} & V_j' \end{array}$$

Thus we have a category $\mathrm{PP}(\mathcal{F})$ of patching problems over the inverse system $\mathcal{F}$. Let $\mathrm{Vect}(k)$ be the category of finite dimensional vector spaces over the field k and suppose that the inverse limit F of the system $\mathcal{F}$ is a field. Given a finite dimensional F-vector space V, let $V_i = V \otimes_F F_i$ and let $\nu_{ij} \colon V_i \to V_j$ be the maps $\mathbb{1} \otimes \iota_{ij}$. This data gives us a patching problem $\beta(V)$. We may similarly induce a morphism of patching problems from an F-linear map, and one can check that this indeed results in a functor

$$\beta \colon \mathrm{Vect}(F) \to \mathrm{PP}(\mathcal{F})$$

The functor β is what we shall refer to as the base change functor. Of course, what we are much more interested in is the reverse direction. As such, there are two questions one might naturally ask concerning our set up:

1. For which inverse systems $\mathcal{F}$ is the base change functor an equivalence of categories?

2. Can we impose more structure on our patching problems instead of only requiring vector spaces over the given fields?

Definition 3.1.3. Let $\mathcal{F}$ be a finite inverse system of fields with inverse limit F and let $\mathcal{V}$ be a patching problem on $\mathcal{F}$. We say that an F-vector space V is a **solution** to the patching problem $\mathcal{V}$ if $\beta(V) \cong \mathcal{V}$.

In [8], Harbater and Hartmann provide some answers to these questions. The inverse systems they consider all come from geometric settings, and this is where semi-global fields come into play. Here, we will discuss only the relevant constructions. Throughout, let T be a complete discrete valuation ring with uniformizer t, residue field k, and fraction field K. Finally, let F/K be a finitely generated field extension with transcendence degree one.

Definition 3.1.4. Let $\widehat{X}$ be a regular connected projective T-curve with function field F such that the reduced irreducible components of the closed fiber X are regular (given F, such an $\widehat{X}$ always exists; see [1] or [11]). Let $f : \widehat{X} \to \mathbb{P}^1_T$ be a finite morphism such that the inverse image S of ∞ contains all the points where distinct irreducible components of X meet. We will call $(\widehat{X}, S)$ a **regular T-model** of F, following the terminology used in [9].

Construction 3.1.5. Given such a regular T-model, we extract an inverse system of fields as follows. For each point $Q \in S$, we let R_Q be the local ring of $\widehat{X}$ at Q and let $\widehat{R}_Q$ be its completion with field of fractions F_Q. For each connected component U of $X \setminus S$, we let R_U be the subring of F consisting of functions which are regular at each point of U and let $\widehat{R}_U$ be its t-adic completion. Again, we let F_U be the field of fractions of $\widehat{R}_U$. Now, if a point Q is a limit point of a connected component U, then there is a unique branch $\mathfrak{p}$ of X at Q lying on $\bar{U}$ (since $\bar{U}$ is regular). In this case, $\mathfrak{p}$ is a height one prime in the ring $\widehat{R}_Q$ which contains t and we let $\widehat{R}_\mathfrak{p}$ be the completion of the discrete valuation ring obtained by localizing $\widehat{R}_Q$ along the ideal $\mathfrak{p}$. As before, we denote by $F_\mathfrak{p}$ the field of fractions of $\widehat{R}_\mathfrak{p}$.

From the definition of $\widehat{R}_\mathfrak{p}$, we see that $\widehat{R}_Q$ naturally includes into $\widehat{R}_\mathfrak{p}$. Moreover, we can observe that R_U and R_Q have the same localization at the generic point of $\bar{U}$. One can further show that this localization naturally includes into the t-adically complete ring $\widehat{R}_\mathfrak{p}$, which implies that $\widehat{R}_U$ also includes into $\widehat{R}_\mathfrak{p}$. From these inclusions, we get corresponding inclusions of the fields F_Q and F_U into $F_\mathfrak{p}$. Taking all of these fields together along with these inclusions gives us an inverse system $\mathcal{F}$ whose inverse limit is the semi-global field F.

Letting $\mathcal{F}$ be the inverse system from Construction 3.1.5, Harbater and Hartmann [8] showed that the base change functor $\beta\colon \mathrm{Vect}(F) \to \mathrm{PP}(\mathcal{F})$ is indeed an equivalence of categories. In fact, they showed even more. Suppose we have a category $\mathcal{A}(L)$ of algebraic structures over a field L (i.e. L-vector spaces, possibly with some additional structure, e.g. L-algebras), along with base change functors $\mathcal{A}(L) \to \mathcal{A}(L')$ whenever $L \subseteq L'$. Then we can similarly define $\mathcal{A}$-patching problems as we did when $\mathcal{A}(L) = \mathrm{Vect}(L)$. We would then have a similar base change functor $\beta\colon \mathcal{A}(F) \to \mathrm{PP}_{\mathcal{A}}(\mathcal{F})$ where $\mathrm{PP}_{\mathcal{A}}(\mathcal{F})$ is the category of $\mathcal{A}$-patching problems over the inverse system $\mathcal{F}$.

Theorem 3.1.6 (Harbater, Hartmann; 2007)**.** *Let $\mathcal{F}$ be the finite inverse system resulting from Construction 3.1.5. If $\mathcal{A}(L)$ is one of the following categories of algebraic structures:*

1. *finite dimensional L-algebras,*
2. *G-Galois L-algebras for some fixed finite group G, with G-equivariant morphisms, or*
3. *central simple L-algebras,*

then the base change functor $\beta\colon \mathcal{A}(F) \to \mathrm{PP}_{\mathcal{A}}(\mathcal{F})$ is an equivalence of categories.

We will now apply Theorem 3.1.6 towards solving the admissibility problem.

Definition 3.1.7. Let F be a field. We say that a field extension L/F is **adequate** if there exists a central division F-algebra D such that L is isomorphic to a maximal subfield of D as an F-algebra.

Definition 3.1.8. Let F be a field and let G be a finite group. We say that G is **admissible** over F if there exists a division algebra D which contains a maximal subfield L such that L/F is Galois with group G.

Lemma 3.1.9. *Let G be a finite group and let F, $\widehat{X}$, S, and $\mathcal{F}$ be as in Construction 3.1.5. Suppose that for each $Q \in S$ we are given subgroups $H_Q \subseteq G$ and an H_Q-Galois adequate*

extension L_Q/F_Q such that $L_Q \otimes_{F_Q} F_{\mathfrak{p}}$ is a split extension $F_{\mathfrak{p}}^{|H_Q|}$ over $F_{\mathfrak{p}}$ for each branch $\mathfrak{p}$. Further suppose that the greatest common divisor of the indices $[G : H_Q]$ is 1. Then there exists a G-Galois adequate extension E/F.

Proof. For a detailed proof, see Lemma 4.2 in [9]. We will provide a sketch of the proof here to see how patching over fields is utilized. Let $n = |G|$. We are given H_Q-Galois adequate extensions L_Q/F_Q for each $Q \in S$, which means we also have F_Q division algebras D_Q for each $Q \in S$. We will extend this to all other fields in our inverse system by letting $D_\xi = L_\xi = F_\xi$ for all other ξ in our indexing set. Each of these new extensions are thus H_ξ-Galois over their respective base fields, where H_ξ is trivial.

Now that we have a Galois extension L_ξ/F_ξ for each field F_ξ in our inverse system, let $E_\xi = \mathrm{Ind}_{H_\xi}^G(L_\xi)$ (see page 31 of [8] for the definition) be the induced G-Galois algebra. Also, if $n_\xi = [G : H_\xi]$, let $A_\xi = M_{n_\xi}(D_\xi)$. As shown in [9], E_ξ embeds into A_ξ as a maximal commutative separable subalgebra for all ξ. At this point, one checks using the splitness assumption that the G-Galois algebras E_ξ, the central simple algebras A_ξ, and the inclusions $E_\xi \to A_\xi$ can all be patched together to get a G-Galois algebra E which embeds as a maximal commutative separable subalgebra into a central simple algebra A.

All that remains is to show that A is in fact a division algebra, from which it follows that E must be a G-Galois maximal subfield, thus proving the claim. We do this by showing that the index of A is equal to the degree of A. Note that for all ξ,

$$n/n_\xi = |H_\xi| = \deg(D_\xi) \mid \mathrm{ind}(A) \mid \deg(A)$$

which means that the least common multiple of the n/n_ξ must also divide the index of A. But by the hypothesis on the indices $[G : H_Q]$, we know that this least common multiple

must be n, which incidentally is equal to the degree of A. Thus,

$$\deg(A) = n \mid \mathrm{ind}(A) \mid \deg(A)$$

so in fact $\deg(A) = \mathrm{ind}(A)$ as claimed $\square$

3.2 Results

In [14], Saltman showed that if K is a field with characteristic p and D is a K-central cyclic division algebra of degree p^n, then in fact D contains a maximal subfield which is G-Galois over K where G is *any* group of order p^n. This rather surprising result tells us that in order to show that arbitrary p-groups are admissible over a semi-global field with characteristic p, it is enough to construct cyclic division algebras of degree p^n over such semi-global fields for any $n \in \mathbb{N}$. Of course, we shall require more in order to patch these along with the results of [9], but Saltman's results tell us that the cyclic case will play a central role in pursuing admissibility when the characteristic of the semi-global field divides the order of the group in question.

Throughout this section, we assume that T is a complete discretely valued ring with uniformizer t, residue field k, and fraction field K, both with characteristic p . We also assume that F/K is a finitely generated field extension with transcendence degree one. Let $(\widehat{X}, S)$ be a regular T-model of F. Finally, let $\widehat{R}_Q$, $\widehat{R}_U$, $\widehat{R}_{\mathfrak{p}}$, F_Q, F_U, and $F_{\mathfrak{p}}$ be defined as in Construction 3.1.5 for all Q, U, and $\mathfrak{p}$.

Lemma 3.2.1. *Let R be a domain with fraction field F and suppose there exists a height one prime ideal $\mathfrak{p} \subset R$ such that the completion $\widehat{R}_{\mathfrak{p}}$ is a complete discrete valuation ring with uniformizer b, valuation v, and fraction field $\widehat{F}$. Furthermore, suppose there exists a cyclic*

extension L/F with cyclic Galois group of order n, generated by an element σ, such that $\widehat{L} := L \otimes_F \widehat{F}$ is an unramified field extension of $\widehat{F}$. Then L/F is adequate.

Proof. Let A be the cyclic F-algebra (L, σ, b) and consider $B = A \otimes_F \widehat{F} = (\widehat{L}, \sigma, 1 \otimes b)$, which is also a cyclic algebra, this time over $\widehat{F}$. Corollary 4.7.5 from [7] states that the period of the cyclic algebra B is equal to the order of $1 \otimes b$ in $\widehat{F}^\times / \operatorname{N}_{\widehat{L}|\widehat{F}}(\widehat{L}^\times)$. Corollary 2.4 in [18] says that for every $x \in \widehat{L}$, $v\left(\operatorname{N}_{\widehat{L}|\widehat{F}}(x)\right) = n \cdot w(x)$ where w is the valuation on $\widehat{L}$. Thus we get a map

$$\widehat{F}^\times / \operatorname{N}_{\widehat{L}|\widehat{F}}(\widehat{L}^\times) \to \mathbb{Z}/n\mathbb{Z}$$

sending $1 \otimes b$ to $\overline{v(b)} = 1$, which has order n in $\mathbb{Z}/n\mathbb{Z}$. Tracing back through these homomorphisms, we find that

$$n \mid \operatorname{per}(B) \mid \operatorname{per}(A) \mid \operatorname{ind}(A) \mid \deg(A) = n$$

which forces $\operatorname{ind}(A) = \deg(A)$, implying that A had to be a division algebra to begin with. □

Lemma 3.2.2. *Let $Q \in X$ be a point at which X is regular and let $\widehat{R}_Q$ be the completion of the local ring of $\widehat{X}$ at Q with fraction field F_Q. Then, for every $n \in \mathbb{N}$, there exists an adequate cyclic extension L_Q/F_Q of degree p^n which is split along $\mathfrak{p}$ where $\mathfrak{p}$ is the unique branch of X at Q.*

Proof. Let $\mathfrak{m} \subseteq \widehat{R}_Q$ be the maximal ideal. Since X is regular at Q, there is a unique height one prime ideal $\mathfrak{p} \subseteq \widehat{R}_Q$ containing t, which must necessarily be a principal ideal, i.e. $\mathfrak{p} = (t_Q)$ for some $t_Q \in \widehat{R}_Q$. Since t_Q is irreducible, $t_Q \notin \mathfrak{m}^2$ and so we can find some $x \in \widehat{R}_Q$ such that $m = (t_q, x)$. In fact, the equicharacteristic hypothesis tells us that $\widehat{R}_Q \cong k[[t_q, x]]$. Let $\widehat{R}_x$ be the x-adic completion of $\widehat{R}_Q$ and let F_x be its fraction field and k_x the residue field. In particular, this means that $\widehat{R}_x \cong k((t))[[x]]$ and $k_x \cong k((t))$. In order to use Lemma 3.2.1, we begin by carefully constructing a degree p^n cyclic unramified extension of F_x.

Consider the polynomial $\overline{f}(z) = z^p - z - \frac{1}{\overline{t_Q}} \in k_x[z]$. Since $\mathrm{char}(k_x) = p$, this polynomial is irreducible if and only if it has no roots. So suppose $\alpha \in \overline{k_x}$ is a root of $\overline{f}$ and let v be the $\overline{t_Q}$-adic discrete valuation on k_x. Then,

$$-1 = v\left(\frac{1}{\overline{t_Q}}\right) = v(\alpha^p - \alpha) \geq \min(p \cdot v(\alpha), v(\alpha))$$

which means that $v(\alpha) \neq p \cdot v(\alpha)$. Thus we actually have that

$$-1 = \min(p \cdot v(\alpha), v(\alpha)) = p \cdot v(\alpha)$$

which is impossible since $v(\alpha) \in \mathbb{Z}$. Thus $\overline{f}$ is irreducible, and so is the polynomial $f(z) = z^p - z - \frac{1}{t_Q} \in F_x[z]$. Let $\beta = \left(\frac{1}{t_Q}, 0, \ldots, 0\right) \in W_n(F_x)$ and denote by $\overline{\beta}$ its image in the quotient $W_n(F_x)/\wp(W_n(F_x))$. By Lemma 2.3.8, $\overline{\beta}$ has order p^n, and thus if we let $L_x = F_x(\wp^{-1}(\beta))$, we find that L_x/F_x is cyclic of degree p^n. Also, since the residue field of L_x is $l_x = k_x\left(\wp^{-1}\left(\frac{1}{\overline{t_Q}}, 0, \ldots, 0\right)\right)$ (which is Galois over k_x) and $[l_x : k_x] = p^n$, L_x is unramified over F_x.

Next, let $|\cdot|_x$ be the x-adic norm on F_x and let $|\cdot|_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic norm on $F_{\mathfrak{p}}$. Proposition 2.3.12 says that for some $\varepsilon_1 > 0$, any $\alpha_0 \in F_x$ such that $\left|\alpha_0 - \frac{1}{t_Q}\right|_x < \varepsilon_1$ will guarantee that $F_x(\wp^{-1}(\alpha)) = L_x$ where $\alpha = (\alpha_0, 0, \ldots, 0) \in W_n(F_x)$. Likewise, there exists $\varepsilon_2 > 0$ such that for any $\alpha_0 \in F_{\mathfrak{p}}$ such that $|\alpha_0|_{\mathfrak{p}} < \varepsilon_2$, the polynomial $z^p - z - \alpha_0 \in F_{\mathfrak{p}}[z]$ has a root by Hensel's lemma. Using weak approximation along with the fact that F_Q is dense in both $F_{\mathfrak{p}}$ and F_x, we can find $\alpha_0 \in F_Q$ such that $\left|\alpha_0 - \frac{1}{t_Q}\right|_x < \varepsilon_1$ and $|\alpha_0|_{\mathfrak{p}} < \varepsilon_2$. Let $\alpha = (\alpha_0, 0, \ldots, 0) \in W_n(F_Q)$ and let $L_Q = F_Q(\wp^{-1}(\alpha))$. Since L_Q/F_Q is necessarily cyclic of degree p^n and $L_Q \otimes_{F_Q} F_x \cong L_x$ is unramified, Lemma 3.2.1 assures us that L_Q/F_Q is adequate. Lastly, we also find that $L_Q \otimes_{F_Q} F_{\mathfrak{p}} \cong F_{\mathfrak{p}}^{p^n}$, which incidentally completes the proof. $\square$

Theorem 3.2.3. *Let K be a discretely valued field with algebraically closed residue field k and let F/K be a finitely-generated field extension with transcendence degree one. Assume further that* $\mathrm{char}(K) = \mathrm{char}(k) = p$. *Let G be a finite group whose Sylow-q subgroups are all abelian of rank at most 2 for $q \neq p$ and whose Sylow-p subgroup is cyclic. Then G is admissible over F.*

Proof. If $p \nmid |G|$, this follows directly from Proposition 4.4 in [9]. Now assume $p \mid |G|$. Following the proof of Proposition 4.4 in [9], let $Q_0, ..., Q_{r-1} \in X$ be distinct points at which X is regular, where r is the number of distinct primes dividing $|G|$. Thus, X has a unique branch along each of these points. By Proposition 6.6 of [8], there is a finite morphism $f\colon \widehat{X} \to \mathbb{P}^1_T$ such that $S = f^{-1}(\infty) \subset X$ contains each Q_i along with all points where distinct irreducible components of X meet. As was shown in the proof of Proposition 4.4 in [9], the assumption on Sylow-q subgroups where $q \neq p$ allows us to find adequate field extensions L_{Q_i}/F_{Q_i} for $i > 0$ whose Galois groups are the Sylow-q subgroups and such that they are split along their respective branches. As for the Sylow-p subgroup, which is cyclic of order p^n for some $n \in \mathbb{N}$, Lemma 3.2.2 tells us that we may also find an adequate extension L_{Q_0}/F_{Q_0} whose Galois group is cyclic of order p^n and such that it is split along the branch at Q_0. Thus the result follows from Lemma 3.1.9 above. □

Theorem 3.2.4. *Let K be a discretely valued field with residue field k and let F/K be a finitely-generated field extension with transcendence degree one. Assume that* $\mathrm{char}(K) = \mathrm{char}(k) = p$. *If G is a p-group, then G is admissible over F.*

Proof. Consider the cyclic group C_{p^n} of order p^n. By Lemma 3.2.2, we know that there exists a C_{p^n}-Galois adequate extension L_Q/F_Q where $Q \in X$ is a point at which X is regular. Then by Lemma 3.1.9, there exists a C_{p^n}-Galois adequate extension L/F. Thus there exists an F-central division algebra D containing a copy of L as a maximal subfield, which means $\deg(D) = [L : F] = p^n$. In particular, we know that D is a nontrivial p-algebra; Theorem 3

in [14] then tells us that $N_F = \dim_{F'} F/\wp(F)$ is infinite, where F' is the prime subfield of F. But then by Theorem 1' in [14], there exists a G-Galois field extension E/F such that E is a maximal subfield of D. Thus G is admissible over F. $\square$

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