THE KSBA COMPACTIFICATION OF A 4-DIMENSIONAL FAMILY OF POLARIZED ENRIQUES

SURFACES

by

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(Under the Direction of Valery Alexeev)

ABSTRACT

We describe the moduli compactification by stable pairs (also known as KSBA compactification) of a 4-dimensional family of Enriques surfaces, which arise as specific bidouble covers of the blow up of the projective plane at three general points branched along a configuration of three pairs of lines. The chosen divisor is an appropriate multiple of the ramification locus. We study the degenerations parametrized by the boundary and its stratification. We relate this compactification to the Baily-Borel compactification of the same family of Enriques surfaces. Part of the boundary of this stable pairs compactification has a toroidal behavior, another part is isomorphic to the Baily-Borel compactification, and what remains is a mixture of these two. To conclude, we construct an explicit Looijenga's semitoric compactification whose boundary strata are in bijection with the boundary strata of the KSBA compactification considered.

INDEX WORDS: Moduli spaces, compactifications, stable pairs, Enriques surfaces, stable toric pairs, secondary polytopes, lattices, discrete reflection groups.

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Chapter 1

Introduction

1.1 Motivation and general framework

In the study of moduli spaces, it is a problem to provide compactifications which are functorial and with meaningful geometric and combinatorial properties. A leading example in this sense is the Deligne-Mumford and Knudsen compactification $\overline{M}_{g,n}$ of the moduli space $M_{g,n}$, which parametrizes isomorphism classes of smooth n-pointed curves of genus g (see [DM69, Knu83a, Knu83b]). Another successful example is the toroidal compactification $\overline{A}_g^\tau$, where A_g is the moduli space parametrizing isomorphism classes of g-dimensional principally polarized abelian varieties and τ is the second Voronoi fan. In [Ale02], Alexeev proved that $\overline{A}_g^\tau$ has a modular interpretation, extending previous work of Mumford, Namikawa, and Nakamura (see [Mum72, Nam76, AN99, Ale02]).

After considering curves and abelian varieties, it is natural to look for such "model" compactifications in the case of the moduli space of algebraic surfaces, and there are many techniques that can be used to compactify it. Among these, the natural analogue for surfaces of $\overline{M}_{g,n}$ originated from work of Kollár, Shepherd-Barron, and Alexeev (see [KSB88, Ale94, Ale96a, Ale96b] and [Fuj15, Kol09, Kol13a, KP15, HMX14] for recent developments), and it is usually referred to as the KSBA compactification. This compactification has the advantage of being automatically functorial, and in particular gives a modular interpretation of the points of the boundary. In addition, although the KSBA compactification depends on the choice of a divisor on the surfaces we want to study, it can be defined canonically provided there is a natural choice for this divisor. The disadvantage is that in general the KSBA compactification is hard to study. In addition, it is not

known what to expect in terms of the structure of the boundary. One of the difficulties comes from the substantial use of the Minimal Model Program (MMP), which also makes some results conjectural for moduli of varieties of dimension strictly greater than 2 (most difficulties with applying higher-dimensional MMP have been recently overcome). Therefore, it is important to understand examples of such compactifications and build techniques that allow to get around these difficulties.

In this dissertation, we are interested in studying the KSBA compactification in the case of the moduli space of Enriques surfaces. Enriques surfaces were introduced in 1896 and are very classical objects in algebraic geometry (see [Dol16] and references there). The universal cover of an Enriques surface is a K3 surface, and compactifications of the moduli space of polarized K3 surfaces also received a lot of attention (see [Sca87, Sha80, Sha81, Laz16, AT15]). More specifically, we consider a 4-dimensional family of Enriques surfaces which are called $D_{1,6}$-polarized, and we study its moduli compactifications in the context of Hodge theory, toric geometry, and MMP. The interplay between these different points of view is the object of this dissertation (see Chapter 8).

1.2   Main results

$D_{1,6}$-polarized Enriques surfaces were considered in [Oud10]. $D_{1,6}$ denotes the sublattice of $\mathbb{Z} \oplus \mathbb{Z}^6(-1)$ of vectors of even square. A $D_{1,6}$-polarization on an Enriques surface S is a primitive embedding of $D_{1,6}$ into $\mathrm{Pic}(S)$ with some additional requirements (see Definition 8.2). More geometrically, an appropriate $\mathbb{Z}_2^2$-cover of the blow up of $\mathbb{P}^2$ at three general points branched along a configuration of three pairs of lines is an Enriques surface with two choices of $D_{1,6}$-polarization (see Proposition 8.8). Our choice of divisor consists of the ramification divisor multiplied by a rational number ϵ, $0 < \epsilon \ll 1$. Let $\overline{M}^{\nu}_{D_{1,6}}$ be the normalization of the closure in the coarse moduli space of stable surface pairs of the locus of points parametrizing these Enriques surfaces with our choice of divisor (the precise definition can be found in Section 8.1.5).

Theorem 1.1. *The boundary of $\overline{M}^{\nu}_{D_{1,6}}$ has two divisorial components and one of codimension* 3. *For a full description of the stratification of the boundary and the degenerations parametrized by it, see Theorem 8.43.*

Let us explain the techniques used to prove this result. Denote by Δ the toric boundary of $(\mathbb{P}^1)^3$ and let $B \subset (\mathbb{P}^1)^3$ be a general effective divisor of class $(1, 1, 1)$. Then the $\mathbb{Z}_2^2$-cover of

B branched along $\Delta|_B$ is a $D_{1,6}$-polarized Enriques surface (see Proposition 8.14), and we can reduce our problem to considering the pairs $\left(B, \left(\frac{1+\epsilon}{2}\right)\Delta|_B\right)$ (this is explained in Remark 8.11). The advantage of this setting is that if Q denotes the unit cube, then $((\mathbb{P}^1)^3, B)$ is a stable toric pair of type $\leq Q$ (see Definition 6.4), and therefore we can take advantage of the moduli theory for such pairs, which is developed in [Ale02]. Observe that given the stable toric pair $((\mathbb{P}^1)^3, B)$, then $((\mathbb{P}^1)^3, \Delta + \epsilon B)$ is a stable pair (see [Ale06, Lemma 4.4]), which is not yet what we want. Let $\overline{M}_Q^\nu$ be the normalization of the projective coarse moduli space parametrizing $((\mathbb{P}^1)^3, B)$ and its degenerations. The next result summarizes Theorems 8.40 and 8.41.

Theorem 1.2. *If* $\mathrm{Sym}(Q)$ *denotes the symmetry group of* Q, *then* $\overline{M}_Q^\nu / \mathrm{Sym}(Q) \cong \overline{M}_{D_{1,6}}^\nu$. *On a dense open subset, the isomorphism maps the* $\mathrm{Sym}(Q)$-*class of a stable toric pair* (X, B) *to* $\left(B, \left(\frac{1+\epsilon}{2}\right)\Delta|_B\right)$, *where* Δ *denotes the toric boundary of* X.

One of the main problems in the proof of Theorem 1.2 is to compute the image of (X, B) when $\left(B, \left(\frac{1+\epsilon}{2}\right)\Delta|_B\right)$ is unstable, which happens if and only if the polyhedral subdivision of $(Q, Q \cap \mathbb{Z}^3)$ associated to (X, B) has a *corner cut* (see Definition 8.22). This aspect is analyzed in Section 8.2, where we describe a modification of (X, B), which we denote by $(X^\bullet, B^\bullet)$, such that $\left(B^\bullet, \left(\frac{1+\epsilon}{2}\right)\Delta^\bullet|_{B^\bullet}\right)$ is a stable pair. Moreover, this modification can be performed in a one-parameter family. At this point, given the isomorphism of Theorem 1.2, we can explicitly describe $\overline{M}_{D_{1,6}}^\nu$ because the moduli space $\overline{M}_Q^\nu$ is the projective toric variety associated to the secondary polytope of Q, and it is stratified according to the (regular) polyhedral subdivisions of Q.

Consider the Baily-Borel compactification $\overline{\mathcal{D}/\Gamma}^{BB}$, where $\mathcal{D}$ is the period domain parametrizing $D_{1,6}$-polarized Enriques surfaces. This compactification was studied in [Oud10], and its boundary has two rational 1-cusps and three 0-cusps. The following theorem is proved in Section 8.5.

Theorem 1.3. *There exists a birational morphism* $\overline{M}_{D_{1,6}}^\nu \to \overline{\mathcal{D}/\Gamma}^{BB}$ *extending the period map to the boundary of* $\overline{M}_{D_{1,6}}^\nu$ *(see Theorem 8.52).*

In our case, proving that the period map extends is nontrivial because the boundary of $\overline{M}_{D_{1,6}}^\nu$ is not divisorial, and therefore we cannot make use of the standard extension criterion [BJ06, Corollary III.22.19]. Also in this case, we obtain the desired extension using one-parameter families. To compute the limit of a one-parameter family in the Baily-Borel compactification, it is crucial

to understand the Kulikov type of a KSBA degeneration of K3 surface pairs which are the double covers of our Enriques surface pairs. This is studied in Section 8.5.1.

Label the 0-cusps of $\overline{\mathcal{D}/\Gamma}^{BB}$ even, odd of type 1, and odd of type 2 (see Figure 8.7). We observe that the boundary of $\overline{M}^{\nu}_{D_{1,6}}$ has a toroidal behavior in a neighborhood of the preimage of the even 0-cusp, and is isomorphic to the Baily-Borel compactification in a neighborhood of the preimage of the odd 0-cusp of type 2. Above the odd 0-cusp of type 1 the behavior of $\overline{M}^{\nu}_{D_{1,6}}$ is neither toroidal nor Baily-Borel. These considerations make us consider Looijenga's semitoric compactifications (see [Loo03]), which generalize the Baily-Borel and toroidal compactifications in the case of type IV Hermitian symmetric domains. A semitoric compactification $\overline{\mathcal{D}/\Gamma}^{\Sigma}$ depends on a choice of combinatorial data Σ called an admissible decomposition of the conical locus of $\mathcal{D}$ (see Definition 8.58). To construct Σ in our case, for each 0-cusp we consider the associated hyperbolic lattice, and we compute a fundamental domain for the discrete reflection group generated by the reflections with respect to the (-1)-vectors. The computation at the odd 0-cusp of type 1 in Section 8.6.2 is noteworthy: Vinberg's algorithm produces in one step an infinite Coxeter diagram (this also happens in [Con83]). We obtain the following result.

Theorem 1.4. *The admissible decomposition Σ described in Definition 8.59 produces a semitoric compactification $\overline{\mathcal{D}/\Gamma}^{\Sigma}$ birational to $\overline{M}^{\nu}_{D_{1,6}}$ and whose boundary strata are in bijection with the boundary strata of $\overline{M}^{\nu}_{D_{1,6}}$. This bijection preserves the dimensions of the strata and the intersections between them (see Theorem 8.67).*

Motivated by these observations, we make the following conjecture, which will be object of further investigation.

Conjecture 1.5. *The compactifications $\overline{M}^{\nu}_{D_{1,6}}$ and $\overline{\mathcal{D}/\Gamma}^{\Sigma}$ are isomorphic.*

The double nature, namely toroidal and Baily-Borel, of the boundary of the moduli space $\overline{M}^{\nu}_{D_{1,6}}$ produced by the theory of stable pairs is remarkable: it illustrates the behavior one should expect in general when considering compactifications by stable pairs (see also [AT15, Remark 1.5]).

After reducing the problem to considering lines on the blow up of $\mathbb{P}^2$ at three general points, it is natural to ask whether the techniques in [Ale08, Ale15, AP09, Hu14] apply in our case. The answer to this is no, and the reason is we do not allow any (-1)-curve to be part of our chosen divisor, making our stable pairs not compatible with only considering lines in $\mathbb{P}^2$ (see Remark 8.12).

More connections and differences with the cited works are discussed in Remarks 8.5 and 8.72. A compact moduli space for del Pezzo surfaces using stable pairs was constructed in [HKT09], but in our case we have a different choice of the divisor. Our work is also related to [Hac04] in the case of sextic plane curves, but here we consider the blow up of $\mathbb{P}^2$. Some of the techniques we use are closely related to [AT15] (for instance the use of stable toric pairs and polyhedral subdivisions).

The reason why we focused on this particular type of Enriques surface is the following. $D_{1,6}$-polarized Enriques surfaces come with a degree six polarization given by half the ramification locus of the $\mathbb{Z}_2^2$-cover (this is the polarization which gives the classical construction of Enriques). The techniques used in this dissertation can be generalized to describe a compactification by stable pairs of the entire 10-dimensional family of polarized Enriques surfaces of degree six. This will be object of future work.

Chapter 8 is organized as follows. In Section 8.1 we define the $D_{1,6}$-polarized Enriques surfaces and the surface pairs we want to study. Here we also define our moduli spaces of interest: $\overline{M}_Q$ and $\overline{M}_{D_{1,6}}$. Section 8.2 is devoted to the definition of $(X^\bullet, B^\bullet)$ for a stable toric pair (X, B) of type $\leq Q$. In Section 8.3 we prove that $\left(B^\bullet, \left(\frac{1+\epsilon}{2}\right)\Delta^\bullet|_{B^\bullet}\right)$ is stable for every stable toric pair (X, B) of type $\leq Q$. In Section 8.4 we first construct a morphism $\overline{M}_Q^\nu \to \overline{M}_{D_{1,6}}$, and then we prove the isomorphism $\overline{M}_Q^\nu / \operatorname{Sym}(Q) \cong \overline{M}_{D_{1,6}}^\nu$, which allows us to describe the stable pairs parametrized by the boundary of $\overline{M}_{D_{1,6}}^\nu$ and its stratification. In Section 8.5 we relate $\overline{M}_{D_{1,6}}^\nu$ with the Baily-Borel compactification $\overline{\mathcal{D}/\Gamma}^{BB}$. Finally, the connection with Looijenga's semitoric compactifications is studied in Section 8.6.

1.3 Background

The background part of this dissertation is articulated in 6 chapters. Chapter 2 is a brief introduction to lattice theory, and it includes the main definitions and notations. Particular attention is devoted to the classification of indefinite unimodular lattices. In Chapter 3 we consider the specific case of hyperbolic lattices, and we review Vinberg's method to determine a fundamental domain for a given discrete reflection group. In Chapter 4 we define K3 surfaces and Enriques surfaces. We study their basic numerical invariants and geometric properties. In particular, we classify the lattice $H^2(S; \mathbb{Z})$ (resp. $H^2(S; \mathbb{Z})$ modulo torsion), where S is a K3 surface (resp. an Enriques

surface). In Chapter 5 we define the notion of secondary polytope, and we illustrate it with two examples: the Stasheff polytope and the secondary polytope of the unit cube. In Chapter 6 we survey the theory of stable toric pairs and their moduli. Chapter 7 is devoted to the study of stable pairs. We start with the definition of weighted stable curve (which is the very first example of stable pair). Then we review the definition of log canonical singularities and its extension to the non-normal case, i.e. semi-log canonical singularities. After this, we define stable pairs and their corresponding Viehweg's moduli functor.

We work over an arbitrary algebraically closed field $\Bbbk$ of characteristic zero, exception made for Sections 8.5 and 8.6 where we work over $\mathbb{C}$.

Chapter 2

Lattice Theory

This chapter is a brief introduction to lattice theory, and it includes the main definitions and notations. Particular attention is devoted to the classification of indefinite unimodular lattices. Our main references are [BHPV04, Chapter I, Section 2] and [Ser73, Chapter V].

2.1 Basic definitions and facts

Definition 2.1. A *lattice* is a pair (L, b_L), where L is a free finitely generated $\mathbb{Z}$-module and $b_L \colon L \times L \to \mathbb{Z}$ is a symmetric bilinear form. A lattice is usually denoted by its underlying $\mathbb{Z}$-module L. Another lattice $L' \subseteq L$ is a *sublattice* of L if $b_{L'} = b_L|_{L' \times L'}$. Two lattices L_1, L_2 are *isometric* if there exists a $\mathbb{Z}$-modules isomorphism $\varphi \colon L_1 \to L_2$ such that $b_{L_2}(\varphi(a), \varphi(b)) = b_{L_1}(a, b)$ for all $a, b \in L_1$. Such isomorphism φ is called an *isometry*.

Definition 2.2. Let L be a lattice and let $e_1, \ldots, e_n$ be a $\mathbb{Z}$-basis for L. Then the *Gram* matrix of L associated to the chosen $\mathbb{Z}$-basis is the matrix $(b_L(e_i, e_j))_{1 \leq i,j \leq n}$.

Definition 2.3. Let L be a lattice and let M be a Gram matrix for L. Then we define the *rank* (resp. *discriminant*, *signature*) of L as the rank (resp. determinant, signature) of M. These definitions do not depend on the choice of M. If N is another Gram matrix for L, then there exist an integral matrix B with integral inverse such that $N = B^t M B$. Therefore, N and M have the same rank and signature. In addition, $\det(N) = \det(B^t) \det(M) \det(B) = \det(B)^2 \det(M) = 1 \cdot \det(M) = \det(M)$. We denote the rank (resp. discriminant) of L as $\mathrm{rk}(L)$ (resp. $\mathrm{discr}(L)$). The signature of M is denoted by (t_+, t_-), where t_+ (resp. t_-) is the number of positive (resp. negative) eigenvalues. L is *indefinite* if $t_+, t_- \geq 1$.

Definition 2.4. Let L be a lattice. We say that L is *unimodular* (resp. *nondegenerate*) if $\operatorname{discr}(L) = \pm 1$ (resp. $\operatorname{discr}(L) \neq 0$). Observe that if L is nondegenerate, then its rank equals the rank of L as a free $\mathbb{Z}$-module.

Assumption. In what follows we are concerned with nondegenerate lattices. Therefore, when we consider a lattice, it is assumed to be nondegenerate.

Observation 2.5. Let L be a (nondegenerate) lattice. Then denote by L^* its dual, i.e. $\operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. L^* is naturally identified with

$$\{v \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid b_L(v, w) \in \mathbb{Z} \text{ for all } w \in L\}.$$

L^* has a natural bilinear form with values in $\mathbb{Q}$ induced by L, and it is called the *dual lattice* of L. (Observe that, strictly speaking, L^* is not a lattice. To be more rigorous, one can define a lattice to have the bilinear form defined over $\mathbb{Q}$, and then call a lattice *integral* if its bilinear form is defined over $\mathbb{Z}$.) Let $n = \operatorname{rk}(L)$ and let M be a Gram matrix for L. Then L is isometric to $(\mathbb{Z}^n, M)$ and L^* is generated over $\mathbb{Z}$ by the rows (or the columns) of M^{-1}. In particular, $\operatorname{rk}(L) = \operatorname{rk}(L^*)$ and $\operatorname{discr}(L^*) = 1/\operatorname{discr}(L)$.

Definition 2.6. Let L be a lattice. We define the *discriminant group* of L as the abelian group $A_L = L^*/L$.

Lemma 2.7 ([BHPV04, Chapter I, (2.1) Lemma]). *Let L be a lattice. Then the following hold:*

(i) $|\operatorname{discr}(L)| = |A_L|$ *(in particular, A_L is a finite group);*

(ii) *If $L' \subset L$ is a sublattice of the same rank, then*

$$|L/L'|^2 = \operatorname{discr}(L')/\operatorname{discr}(L).$$

Definition 2.8. A lattice L is said to be *even* if $b_L(a, a)$ is an even integer for all $a \in L$. Otherwise, we call L an *odd* lattice.

2.2 Unimodular lattices

Examples 2.9. Denote by I_n the $n \times n$ identity matrix. In what follows we give some fundamental examples of unimodular lattices.

(a) Let $p, q \in \mathbb{Z}_{\geq 0}$. Then denote by $\mathbb{Z}^{p,q}$ the lattice $\mathbb{Z}^{p+q}$ with the symmetric bilinear form given by the matrix

$$\left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array} \right),$$

where the "0" symbols denote zero matrices of appropriate dimensions. Then $\mathbb{Z}^{p,q}$ is odd, unimodular, and of signature (p, q).

(b) The lattice $U = (\mathbb{Z}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right))$ is obviously even, unimodular, and of signature $(1, 1)$.

(c) The lattice E_8 is given by $(\mathbb{Z}^8, B)$, where

$$B = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}.$$

Observe that, if A denotes the adjacency matrix of the E_8 Dynkin diagram (see Figure 2.1[1]), then $B = A - 2I_8$. One can check that the lattice E_8 is even, unimodular, and of signature $(0, 8)$.

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[1] All the figures in this dissertation were realized using the software GeoGebra, Copyright ©International GeoGebra Institute, 2013.
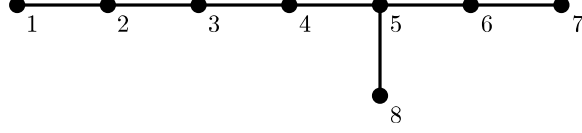
Figure 2.1: The E_8 Dynkin diagram

The next theorem completely classifies indefinite unimodular lattices.

Theorem 2.10 ([Mil58, Theorem 1] or [Ser73, Chapter V, Theorem 6]). *Indefinite even (resp. odd) unimodular lattices of the same signature are isometric. More explicitly, up to isometry, the only indefinite unimodular lattices of signature (p, q), are given by*

- $\mathbb{Z}^{p,q}$ *(odd case);*

- $U^{\oplus p} \oplus E_8^{\oplus(q-p)/8}$ *if $p \leq q$, or $U^{\oplus q} \oplus E_8(-1)^{\oplus(p-q)/8}$ otherwise (even case). (Given a lattice L and an integer n, $L(n)$ denotes the lattice (L, nb_L).)*

Example 2.11. Let L be an even unimodular lattice of signature $(2, 10)$. Then by Theorem 2.10 L has to be isometric to $U^{\oplus 2} \oplus E_8$.

Applications of this theorem to algebraic geometry can be found in Chapter 4.

2.3 The $D_{p,q}$ lattice

Definition 2.12. Let $p, q \in \mathbb{Z}_{\geq 0}$. Then define the lattice $D_{p,q}$ to be the sublattice of $\mathbb{Z}^{p,q}$ (see Examples 2.9 (a)) of vectors of even square. Equivalently, $D_{p,q}$ is the sublattice of $\mathbb{Z}^{p,q}$ of vectors of even sum of coordinates with respect to the canonical basis of $\mathbb{Z}^{p+q}$.

Observation 2.13. $D_{p,q}$ is by definition even and it has index 2 in $\mathbb{Z}^{p,q}$ (if $x \in \mathbb{Z}^{p,q}$, then $2x \in D_{p,q}$). Therefore, $D_{p,q}$ and $\mathbb{Z}^{p,q}$ have the same signature. Moreover, using Lemma 2.7 (ii), we have that

$$\operatorname{discr}(D_{p,q}) = \operatorname{discr}(\mathbb{Z}^{p,q}) \cdot |\mathbb{Z}^{p,q}/D_{p,q}|^2 = (-1)^q \cdot 2^2 = (-1)^q \cdot 4.$$

In particular, the discriminant group $A_{D_{p,q}}$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ (from what follows, we can see that both can occur as discriminant groups).

Definition 2.14. For $n \geq 4$, denote by D_n the usual root lattice given by $(\mathbb{Z}^n, A - 2I_n)$, where A is the adjacency matrix of the D_n Dynkin diagram (see Figure 2.2). One can check that D_n is even, has signature $(0, n)$, and $\mathrm{discr}(D_n) = (-1)^n \cdot 4$. In particular, it is well known that

$$A_{D_n} \cong \begin{cases} \mathbb{Z}_4 & \text{if } n \text{ is odd}, \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } n \text{ is even}. \end{cases}$$

(See for instance [Ebe13, Section 1.4].) Equivalently, D_n is the sublattice of vectors of $\mathbb{Z}^{0,n}$ of even square, i.e. $D_n = D_{0,n}$ (this is the standard definition of D_n). It follows that the lattice $D_{p,q}$ is the natural generalization of D_n to arbitrary signatures.
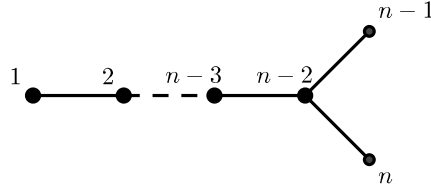
Figure 2.2: The D_n Dynkin diagram

In Chapter 8 we are concerned with the lattice $D_{1,6}$.

Proposition 2.15. *The lattice $D_{1,6}$ has discriminant group $\mathbb{Z}_4$.*

Proof. Let $e_0, \ldots, e_6$ be the canonical basis for $\mathbb{Z}^7$. Then it is easy to check that $2e_0, e_0 - e_1, \ldots, e_0 - e_6$ is a basis for $D_{1,6}$. The vector $v = \frac{1}{2}(e_0 + \ldots + e_6)$ is an element of $D_{1,6}^*$ by Observation 2.5. Since $2v \notin D_{1,6}$, we can conclude that $v + D_{1,6}$ has order 4 in $A_{D_{1,6}}$. Therefore, $A_{D_{1,6}} \cong \mathbb{Z}_4$. $\qquad\square$

Observe that Proposition 2.15 trivially follows from the following more general result.

Proposition 2.16. *Let $p, q \in \mathbb{Z}_{>0}$ with $p < q$. Then $D_{p,q}$ is isometric to $U^p \oplus D_{q-p}$.*

Proof. The lattice $U^p \oplus D_{q-p}$ is the sublattice of $U^p \oplus \mathbb{Z}^{0,q-p} \cong \mathbb{Z}^{p,q}$ of vectors of even square, and therefore it can be identified with $D_{p,q}$. $\qquad\square$

Chapter 3

Vinberg's Algorithm

In this chapter we study discrete reflection groups for hyperbolic lattices. In particular, we review Vinberg's method to determine a fundamental domain for a given discrete reflection group. The main reference for what follows is [Vin75].

3.1 Roots and reflections

Notation 3.1. In what follows, given a lattice L and vectors $v, w \in L$, we denote $b_L(v, w)$ (resp. $b_L(v, v)$) simply by $v \cdot w$ (resp. v^2).

Definition 3.2. Let L be a lattice. A *root* of L is a vector $\alpha \in L$ such that $\alpha^2 \neq 0$ and the fraction $2(\alpha \cdot v)/\alpha^2$ is an integer for all $v \in L$.

Example 3.3. Vectors $\alpha \in L$ such that $\alpha^2 = \pm 1, \pm 2$ are examples of roots. We call (-1)-*vector* (resp. (-2)-*vector*) a vector α such that $\alpha^2 = -1$ (resp. $\alpha^2 = -2$).

The roots of a lattice L can be used to define special isometries of L called *reflections*.

Definition 3.4. If $\alpha \in L$ is a root, then the map $r_\alpha \colon L \to L$ given by

$$v \mapsto v - 2\frac{\alpha \cdot v}{\alpha^2}\alpha,$$

is an isometry of L called *reflection with respect to α*. Observe that $r_\alpha \circ r_\alpha = \mathrm{id}_L$ and $r_\alpha(\alpha) = -\alpha$. The hyperplane $\alpha^\perp \otimes_\mathbb{Z} \mathbb{R}$ in $L_\mathbb{R} = L \otimes_\mathbb{Z} \mathbb{R}$ is called the *mirror* of the reflection r_α, and we denote it by H_α.

Definition 3.5. Let $O(L)$ denote the group of isometries of L. Then a subgroup of $O(L)$ generated by reflections is called a *reflection group* for L.

3.2 Hyperbolic lattices and hyperbolic spaces

Definition 3.6. Let L be a lattice. We say that L is *hyperbolic* if it has signature $(1, n)$ for some positive integer n.

Definition 3.7. Let L be a hyperbolic lattice and endow $L_\mathbb{R}$ with the euclidean topology. Then the subset $\{x \in L_\mathbb{R} \mid x \cdot x > 0\} \subset L_\mathbb{R}$ has two connected components. Let C be one of these two components, and denote by $\overline{C}$ its closure in $L_\mathbb{R}$. We call $\overline{C}$ a *light cone*.

Definition 3.8. Let L be a hyperbolic lattice. Then the quotient $\Lambda(L) = C/\mathbb{R}_{>0}$ is called *hyperbolic space*. $\Lambda(L)$ is compactified by $\overline{\Lambda(L)} = (\overline{C} \setminus \{0\})/\mathbb{R}_{>0}$, and the points we add are called *points at infinity*.

Definition 3.9. Let L be a hyperbolic lattice and let $\alpha \in L$ be a root such that $\alpha^2 < 0$. Then H_α has nonempty intersection with C, hence it defines a hyperplane in the hyperbolic space $\Lambda(L)$ given by $h_\alpha = (H_\alpha \cap C)/\mathbb{R}_{>0}$. If we have another such hyperplane $h_\beta \subset \Lambda(L)$, we say that h_α and h_β are

- *adjacent* if $h_\alpha \cap h_\beta \neq \emptyset$;

- *parallel* if $\overline{h}_\alpha, \overline{h}_\beta \subset \overline{\Lambda(L)}$ intersect at a point at infinity;

- *divergent* if $\overline{h}_\alpha \cap \overline{h}_\beta = \emptyset$.

If the hyperplanes are adjacent, then the angle between them is equal to π/m, where

$$\cos\left(\frac{\pi}{m}\right) = \frac{\alpha \cdot \beta}{\sqrt{\alpha^2 \beta^2}}.$$

3.3 Fundamental domains and Coxeter diagrams

Definition 3.10. Let Γ be a discrete reflection group for a hyperbolic lattice L. Then the mirrors of the reflections in Γ decompose the hyperbolic space $\Lambda(L)$ into convex polyhedra which are in the same Γ-orbit. A *fundamental domain* for Γ is the choice of one of these convex polyhedra.

Definition 3.11. Let Γ be a discrete reflection group for a hyperbolic lattice L. Then we can construct a graph $G(\Gamma)$ as follows. Choose a fundamental domain for Γ and let $\mathcal{X}$ be the set of hyperplanes in $\Lambda(L)$ which bound it.

- Let $\{v_\alpha \mid h_\alpha \in \mathcal{X}\}$ be the set of vertices of $G(\Gamma)$;

- An edge between v_α, v_β is determined as follows. Consider $h_\alpha, h_\beta \in \mathcal{X}$.

    - If h_α, h_β are adjacent and the angle between them is π/m with $m \neq 2$, then draw an edge between v_α and v_β labeled by $m - 2$;

    - If h_α, h_β are parallel, then draw an edge between v_α and v_β labeled by ∞;

    - If h_α, h_β are divergent, then draw a dotted edge between v_α and v_β.

The graph $G(\Gamma)$ obtained this way is called the *Coxeter diagram* of Γ.

Definition 3.12. Let $G(\Gamma)$ as in Definition 3.11 and let G be a connected subdiagram of $G(\Gamma)$. Then G is called *elliptic* if the corresponding Gram matrix is definite. If G has t vertices, then G is called *parabolic* if the corresponding Gram matrix is semidefinite of rank $t - 1$. A subdiagram of $G(\Gamma)$ is called *elliptic* (resp. *parabolic*) if all its connected components are such.

3.4 Algorithm to determine a fundamental domain

Fix a hyperbolic lattice L of signature $(1, n)$ and let $\Gamma = \langle r_\alpha \mid \alpha \in \mathcal{R} \rangle$ be a discrete reflection group, where $\mathcal{R} \subset L$ consists of roots of negative square. In [Vin75], Vinberg gives an algorithm to determine a fundamental domain P for Γ. (We are interested in running Vinberg's algorithm to produce interesting decompositions of light cones, and we use these decompositions in Chapter 8 to construct compactifications of certain moduli spaces.)

Description of the algorithm.

- **Preparation:** Fix a vector v_0 in the light cone $\overline{C}$. From the set $\{\alpha \in \mathcal{R} \mid v_0 \cdot \alpha = 0\}$ choose a subset $\mathcal{S}_0$ of fundamental roots. Let $n = 1$.

- **Step:** Define

$$\mathcal{S}_n = \mathcal{S}_{n-1} \cup \{\alpha \in \mathcal{R} \mid v_0 \cdot \alpha = n \text{ and } \alpha \cdot \beta \geq 0 \text{ for all } \beta \in \mathcal{S}_{n-1}\}.$$

Then let $n = n + 1$ and repeat **Step**.

Then the hyperplanes h_α for $\alpha \in \cup_{n \geq 0} \mathcal{S}_n$ bound a fundamental domain P for the Γ-action.

Definition 3.13. We say that Vinberg's algorithm *finishes* if there exists $\overline{n} \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{S}_n = \mathcal{S}_{\overline{n}}$ for all $n > \overline{n}$. It is possible that the algorithm does not finish.

3.5   Stopping criterion

Vinberg gives a criterion to recognize if the algorithm is finished at a given step n.

Observation 3.14. First observe that, if the hyperplanes h_α for $\alpha \in \mathcal{S}_n$ bound a polyhedron P_n of finite volume, then P_n equals the fundamental domain P (see [Vin75, Section 2.4]).

Theorem 3.15 ([Vin75, Theorem 2.6 bis]). *Assume $G(\Gamma)$ contains no dotted edges and no Lanner's subdiagrams (these are listed in [Vin75, Table 3]). Then P has finite volume if and only if every connected parabolic subdiagram of $G(\Gamma)$ is a connected component of some parabolic subdiagram of rank $n - 1$.*

Let us explain how to use Theorem 3.15 as a stopping criterion. At step n of Vinberg's algorithm, draw the Coxeter diagram for the discrete reflection group generated by the reflections r_α for $\alpha \in \mathcal{S}_n$. Then Theorem 3.15 tells us whether P_n has finite volume. If it does, then $P = P_n$ by Observation 3.14, and the algorithm finishes. If P_n does not have finite volume, then we continue with the algorithm.

Examples 3.16. Let L be a hyperbolic lattice and let Γ be the discrete reflection group generated by the reflections with respect to the (-1)-vectors. In Section 8.6.2 we run Vinberg's algorithm and we compute the Coxeter diagram of Γ for $L = \mathbb{Z}^{1,3}$ and $L = \mathbb{Z}^{1,1} \oplus \mathbb{Z}^2(-2)$ (recall that $\mathbb{Z}^{p,q}$ denotes the unique odd unimodular lattice of signature (p, q)).

Chapter 4

K3 Surfaces and Enriques Surfaces

In this chapter we define K3 surfaces and Enriques surfaces. We study their basic numerical invariants and geometric properties. In particular, we classify the lattice $H^2(S; \mathbb{Z})$ (resp. $H^2(S; \mathbb{Z})$ modulo torsion), where S is a K3 surface (resp. an Enriques surface). Our main references are [BHPV04, Bea96, GH94].

4.1 Basic definitions and notations

A *variety* is a connected and reduced scheme of finite type over our fixed base field $\Bbbk$. A *surface* is a 2-dimensional smooth projective variety. Given a surface S, we briefly recall the following notation and definitions:

- $p_g(S) = h^0(S, \omega_S)$ is the *geometric genus*;

- $q(S) = h^1(S, \mathcal{O}_S)$ is the *irregularity*;

- $\kappa(S)$ is the *Kodaira dimension*;

- $b_i(S) = \mathrm{rk}(H^i(S; \mathbb{Z}))$ is the *i-th Betti number*;

- $\chi_{\mathrm{top}}(S) = \sum_{i=0}^{4}(-1)^i b_i(S)$ is the *topological Euler characteristic*. This is equal to $2 - 2b_1(S) + b_2(S)$ by Poincaré duality (see [GH94, Chapter 0, Section 4]);

- $\mathrm{NS}(S) = \mathrm{Pic}(S)/\mathrm{Pic}^0(S)$ is the *Néron-Severi group*;

- $\mathrm{Num}(S) = \mathrm{NS}(S)/_{\equiv}$, where "$\equiv$" denotes numerical equivalence.

Observation 4.1. In this chapter we are concerned with surfaces S with $K_S \equiv 0$. This condition automatically implies that S is *minimal*, which means that S does not contain a (-1)-*curve* (i.e. a curve $E \subset S$ isomorphic to $\mathbb{P}^1$ such that $E^2 = -1$). To show this, assume $E \subset S$ is a (-1)-curve. Then [Bea96, I.15 The genus formula] implies that

$$p_g(E) = 0 = 1 + \frac{-1 + E \cdot K_S}{2} \implies E \cdot K_S = -1,$$

which is a contradiction.

4.2 K3 surfaces

Definition 4.2. A *K3 surface* is a surface S such that $K_S \sim 0$ (so that $p_g(S) = 1$) and $q(S) = 0$.

Example 4.3. The simplest example of a K3 surface is the following. Let S be a smooth quartic hypersurface in $\mathbb{P}^3$. Then $K_S \sim 0$ by adjunction formula. To show that $q(S) = 0$, consider the short exact sequence

$$0 \to \mathcal{O}(-4) \to \mathcal{O} \to \mathcal{O}_S \to 0,$$

where we set $\mathcal{O} = \mathcal{O}_{\mathbb{P}^3}$. Consider the associated long exact sequence in cohomology

$$\ldots \to H^1(\mathbb{P}^3, \mathcal{O}) \to H^1(S, \mathcal{O}_S) \to H^2(\mathbb{P}^3, \mathcal{O}(-4)) \to \ldots$$

Since $H^1(\mathbb{P}^3, \mathcal{O})$ and $H^2(\mathbb{P}^3, \mathcal{O}(-4))$ are both isomorphic to $\{0\}$, we conclude that $H^1(S, \mathcal{O}_S) \cong \{0\}$. Therefore S is a K3 surface.

Example 4.4. Let $S \subset (\mathbb{P}^1)^3$ be a multidegree $(2, 2, 2)$ smooth hypersurface. We have that $K_S \sim 0$ by adjunction formula. Consider the short exact sequence

$$0 \to \mathcal{O}(-2, -2, -2) \to \mathcal{O} \to \mathcal{O}_S \to 0,$$

where we set $\mathcal{O} = \mathcal{O}_{(\mathbb{P}^1)^3}$. Consider the associated long exact sequence in cohomology

$$\ldots \to H^1((\mathbb{P}^1)^3, \mathcal{O}) \to H^1(S, \mathcal{O}_S) \to H^2((\mathbb{P}^1)^3, \mathcal{O}(-2, -2, -2)) \to \ldots$$

By Kodaira vanishing theorem (see [Har77, Chapter III, Remark 7.15]), we obtain that

$$H^1((\mathbb{P}^1)^3, \mathcal{O}) \cong \{0\}, \ H^2((\mathbb{P}^1)^3, \mathcal{O}(-2, -2, -2)) \cong \{0\},$$

hence $q(S) = 0$ and S is a K3 surface.

Proposition 4.5. *Let S be a K3 surface. Then S is minimal, $\kappa(S) = 0$, $b_1(S) = 0$, $b_2(S) = 22$, $h^{1,1} = 20$, and $\pi_1(S) = \{1\}$.*

Proof. S is minimal by Observation 4.1. $\kappa(S) = 0$ follows by the definition of Kodaira dimension together with $K_S \sim 0$. Using [Bea96, Fact III.19], we have that $b_1(S) = 2q(S) = 0$. By Noether's formula (see [Bea96, I.14]), we have that

$$\chi_{\text{top}}(S) = 12\chi(\mathcal{O}_S) = 24 = 2b_0(S) - 2b_1(S) + b_2(S) \implies b_2(S) = 22.$$

The Hodge decomposition (see [GH94, Chapter 0, Section 7]) tells us that

$$H^2(S; \mathbb{C}) \cong H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S).$$

Therefore, $H^{1,1}(S)$ is 20-dimensional because $H^{2,0}(S), H^{0,2}(S)$ are both 1-dimensional and we just proved that $b_2(S) = 22$. To conclude, all K3 surfaces are diffeomorphic to each other (see [BHPV04, Chapter VIII, (8.6) Corollary]). In particular, S is diffeomerphic to a quartic hypersurface in $\mathbb{P}^3$ (see Example 4.3), which is simply-connected by the Lefschetz hyperplane theorem (see [GH94, Chapter 1, Section 2]). $\qquad \square$

Proposition 4.6 ([Huy16, Chapter 1, Proposition 2.4])**.** *Let S be a K3 surface. Then the natural surjections $\mathrm{Pic}(S) \to \mathrm{NS}(S) \to \mathrm{Num}(S)$ are isomorphisms.*

Proof. First, consider the exponential short exact sequence

$$0 \to \underline{\mathbb{Z}}_S \to \mathcal{O}_S \to \mathcal{O}_S^* \to 0,$$

where $\underline{\mathbb{Z}}_S$ denotes the constant sheaf of stalk $\mathbb{Z}$ on S. Recall that $H^1(S, \mathcal{O}_S^*) \cong \mathrm{Pic}(S)$ (see [Har77, Chapter III, Exercise 4.5]). Looking at the associated long exact sequence in cohomology

$$\ldots \to H^1(S, \mathcal{O}_S) \to \mathrm{Pic}(S) \to H^2(S; \mathbb{Z}) \to H^2(S, \mathcal{O}_S) \to \ldots,$$

we obtain that $\mathrm{Pic}(S)$ is isomorphic to the image of c_1 in $H^2(S; \mathbb{Z})$, which is $\mathrm{NS}(S)$.

To see that $\mathrm{Pic}(S) \to \mathrm{Num}(S)$ is an isomorphism, assume by contradiction there exists a nontrivial line bundle $\mathscr{L} \in \mathrm{Pic}(S)$ which is numerically trivial. Let $\mathscr{M}$ be an ample line bundle on S. Then we have that $\mathscr{L} \cdot \mathscr{M} = 0$. Consider the linear system $|\mathscr{L}|$ and assume it is not empty. Since $\mathscr{L} \not\cong \mathcal{O}_S$, we can find a nonzero effective divisor $E \in |\mathscr{L}|$. This implies that $\mathscr{L} \cdot \mathscr{M} = E \cdot \mathscr{M} > 0$, which cannot be. Hence $|\mathscr{L}| = \emptyset$ and $h^0(\mathscr{L}) = 0$. Similarly, we can argue that $h^2(\mathscr{L}) = h^0(\mathscr{L}^{-1}) = 0$. By Riemann-Roch formula (see [Bea96, Theorem I.12]), we have that

$$2 + \frac{1}{2}\mathscr{L} \cdot \mathscr{L} = \chi(\mathscr{L}) = -h^1(\mathscr{L}) \leq 0.$$

This implies that $\mathscr{L} \cdot \mathscr{L} < 0$, which is a contradiction. Therefore, the map $\mathrm{Pic}(S) \to \mathrm{Num}(S)$ is an isomorphism.

To conclude, we showed that $\mathrm{Pic}(S) \to \mathrm{NS}(S)$ and $\mathrm{Pic}(S) \to \mathrm{Num}(S)$ are isomorphism, hence also $\mathrm{NS}(S) \to \mathrm{Num}(S)$ is. $\qquad\square$

Proposition 4.7. *Let S be a K3 surface. Then $(H^2(S; \mathbb{Z}), \smile)$ is a lattice isometric to $U^{\oplus 3} \oplus E_8^{\oplus 2}$, which is called the* K3 *lattice.*

Proof. By the universal coefficient theorem for cohomology (see [Hat02, Theorem 3.2]) we have that

$$H^2(S; \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_2(S; \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_1(S; \mathbb{Z}), \mathbb{Z}).$$

The homology group $H_1(S; \mathbb{Z})$ is isomorphic to $\{0\}$ because $H_1(S; \mathbb{Z})$ is the abelianization of $\pi_1(S) \cong \{0\}$ (see Proposition 4.5). Therefore $H^2(S; \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_2(S; \mathbb{Z}), \mathbb{Z})$, which tells us that $H^2(S; \mathbb{Z})$ is torsion free. We can conclude that $H^2(S; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{22}$ as a $\mathbb{Z}$-module because $b_2(S) = 22$ by Proposition 4.5. The lattice $(H^2(S; \mathbb{Z}), \smile)$, which is obviously nondegenerate, is unimodular by Poincaré duality and even by Wu's formula (see [MS74]). Finally, let (t_+, t_-) be the signature of $(H^2(S; \mathbb{Z}), \smile)$. Then the Hodge-Riemann bilinear relations (see [GH94, Chapter 0, Section 7])

imply that

$$t_+ - t_- = \sum_{p+q \text{ is even}} (-1)^p h^{p,q} = h^{0,0} + h^{2,0} - h^{1,1} + h^{0,2} + h^{2,2} = -16.$$

Since $t_+ + t_- = 22$, we argue that $(t_+, t_-) = (3, 19)$. We can conclude that $H^2(S; \mathbb{Z})$ is isometric to $U^{\oplus 3} \oplus E_8^{\oplus 2}$ by Theorem 2.10. $\qquad \square$

4.3 Enriques surfaces

Definition 4.8. An *Enriques surface* is a surface S such that $2K_S \sim 0$ and $p_g(S) = q(S)$.

Proposition 4.9. *Let S be an Enriques surface. Then S is minimal, $\kappa(S) = 0$, $p_g(S) = q(S) = 0$, $K_S \not\sim 0$, $b_1(S) = 0$, and $b_2(S) = 10$.*

Proof. S is minimal by Observation 4.1. $\kappa(S) = 0$ follows by the definition of Kodaira dimension because $P_n(S) = 1$ for n even and $P_n(S) \leq 1$ for n odd (by contradiction, if $P_n(S) > 1$ for n odd, then $P_{2n}(S) > 1$). By the classification of surfaces of Kodaira dimension 0 (see [Bea96, Theorem VIII.2]), the only possibility is $p_g(S) = q(S) = 0$. In particular, $K_S \not\sim 0$, otherwise $p_g(S) = 1$. The first Betti number $b_1(S)$ is equal to $2q(S) = 0$. Noether's formula gives that $12 = 2b_0(S) - 2b_1(S) + b_2(S)$, hence $b_2(S) = 10$. $\qquad \square$

Proposition 4.10. *Let S be an Enriques surface and let $\pi \colon X \to S$ be the étale double cover corresponding to the 2-torsion sheaf ω_S. Then X is a K3 surface. In particular, the universal cover of S is a K3 surface with a fixed-point-free involution and $\pi_1(S) \cong \mathbb{Z}_2$. Conversely, the quotient of a K3 surface X by a fixed-point-free involution is an Enriques surface.*

Proof. First observe that $\omega_X = \pi^* \omega_S = \mathcal{O}_X$. Moreover, $\chi_{\text{top}}(X) = 2\chi_{\text{top}}(S) = 24$, implying that $\chi(\mathcal{O}_X) = 2 = 1 - q(X) + p_g(X)$. In conclusion, $q(X) = 0$ and X is a K3 surface. X is the universal cover of S because X is simply connected. Since π has degree 2, we have that $\pi_1(S)$ has cardinality 2, hence $\pi_1(S) \cong \mathbb{Z}_2$.

For the converse, let $\iota \colon X \to X$ be a fixed-point-free involution and denote by S the quotient X/ι. Let $\pi \colon X \to S$ be the quotient map. $\chi_{\text{top}}(S) = \frac{1}{2}\chi_{\text{top}}(X) = 12$, hence $\chi(\mathcal{O}_S) = 1$, which implies that $p_g(S) = q(S)$. Finally, $\mathcal{O}_X = \omega_X = \pi^* \omega_S$, which gives $\omega_S^{\otimes 2} = \pi_* \pi^* \omega_S = \pi_* \mathcal{O}_X = \mathcal{O}_S$. Therefore, S is an Enriques surface. $\qquad \square$

Example 4.11. Let X be the vanishing locus in $(\mathbb{P}^1)^3$ of the following multidegree $(2, 2, 2)$ equation:

$$\sum_{\substack{i,j,k=0,1,2 \\ i+j+k\equiv 0(\mathrm{mod}\ 2)}} d_{ijk} X_0^{2-i} X_1^i Y_0^{2-j} Y_1^j Z_0^{2-k} Z_1^k.$$

For a general choice of the coefficients d_{ijk}, we can assume that X is smooth. By Example 4.4 we know that X is a K3 surface. The involution $(\mathbb{P}^1)^3 \to (\mathbb{P}^1)^3$ given by

$$([X_0 : X_1], [Y_0 : Y_1], [Z_0 : Z_1]) \mapsto ([X_0 : -X_1], [Y_0 : -Y_1], [Z_0 : -Z_1]),$$

restricts to an involution $\iota\colon X \to X$, which is fixed-point-free if we assume the coefficients of the monomials $X_a^2 Y_b^2 Z_c^2$, $a, b, c \in \{0, 1\}$, to be nonzero. Under this assumption, X/ι is an Enriques surface by Proposition 4.10.

Proposition 4.12. *For an Enriques surface S, we have that the natural maps $\mathrm{Pic}(S) \to \mathrm{NS}(S) \hookrightarrow H^2(S; \mathbb{Z})$ are isomorphisms.*

Proof. Considering the long exact sequence in cohomology associated to the exponential short exact sequence

$$\ldots \to H^1(S, \mathcal{O}_S) \to \mathrm{Pic}(S) \to H^2(S; \mathbb{Z}) \to H^2(S, \mathcal{O}_S) \to \ldots,$$

we obtain the claimed isomorphisms because $p_g(S) = q(S) = 0$. $\qquad\square$

Observation 4.13. For an Enriques surface S, we have that $\mathrm{Num}(S) \ncong \mathrm{Pic}(S)$ because $\omega_S \equiv 0$, but $\omega_S \ncong \mathcal{O}_S$.

Proposition 4.14. *Let S be an Enriques surface. Then the torsion part T of $H^2(S; \mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$, and the quotient $H^2(S; \mathbb{Z})_f = H^2(S; \mathbb{Z})/T$ endowed with the cup product is a lattice isometric to $U \oplus E_8$, which is called the* Enriques lattice.

Proof. By the universal coefficient theorem for cohomology we have that

$$H^2(S; \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_2(S; \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_1(S; \mathbb{Z}), \mathbb{Z}).$$

The homology group $H_1(S; \mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$ because $H_1(S; \mathbb{Z})$ is the abelianization of $\pi_1(S) \cong \mathbb{Z}_2$ (see Proposition 4.10). Therefore, since $b_2(S) = 10$, we have that $H^2(S; \mathbb{Z}) \cong \mathbb{Z}^{10} \oplus \mathbb{Z}_2$,

showing that $T \cong \mathbb{Z}_2$. The $\mathbb{Z}$-module $H^2(S;\mathbb{Z})_f$ endowed with the cup product is a nondegenerate unimodular lattice of rank 10. It is also even by the Riemann-Roch formula: any class $C \in H^2(S;\mathbb{Z})_f$ is algebraic because $\text{Pic}(S) \cong H^2(S;\mathbb{Z})$ (see Proposition 4.12), and hence

$$\chi(\mathcal{O}_S(C)) = \chi(\mathcal{O}_S) + \frac{1}{2}(C^2 - C \cdot K_S) \implies C^2 \text{ is even.}$$

The signature of $H^2(S;\mathbb{Z})_f$ is $(1,9)$ by the Hodge-Riemann bilinear relations, or by the Hodge index theorem (see [BHPV04, Chapter IV, (2.16) Corollary]). In conclusion, $H^2(S;\mathbb{Z})_f$ is isometric to $U \oplus E_8$ by Theorem 2.10. $\qquad\square$

4.4 Half-fibers and the canonical class of an Enriques surface

Proposition 4.15. *Let S be an Enriques surface and let $f\colon S \to \mathbb{P}^1$ be a genus 1 pencil. Then*

(i) f has exactly two multiple fibers, and their multiplicities are equal to 2;

(ii) If F_1, F_2 are these double fibers, then $K_S \sim F_1 - F_2$.

Proof. The following proof is taken from [BHPV04, Chapter VIII, (17.1) Lemma]. Let $F_1, \ldots, F_k$ be the multiple fibers of f, and assume they have multiplicities $r_1, \ldots, r_k \geq 2$ respectively. If p is a general point in $\mathbb{P}^1$, then the formula for the canonical class of a genus 1 fibration (see [BHPV04, Chapter V, (12.3) Corollary]) gives that

$$K_S \sim f^*(-p) + \sum_{i=1}^{k}(r_i - 1)F_i. \tag{4.1}$$

But K_S is 2-torsion, therefore

$$0 \sim f^*(-2p) + \sum_{i=1}^{k}(2r_i - 2)F_i. \tag{4.2}$$

Fix an arbitrary $j \in \{1, \ldots, k\}$. Restricting (4.2) to F_j, we obtain that

$$0 \sim (2r_j - 2)F_j|_{F_j}.$$

Since $F_j|_{F_j}$ has order r_j (see [BHPV04, Chapter II, (8.3) Lemma]), we have that $r_j \mid (2r_j - 2)$. This implies that $(2 - h)r_j = 2$ for $h = 0$ or 1. But $r_j > 1$, so $h = 1$ and $r_j = 2$. We conclude that $r_1, \ldots, r_k = 2$, because j was arbitrary. In particular, (4.2) becomes

$$0 \sim f^*(-2p) + \sum_{i=1}^{k} 2F_i \implies f^*(2p) = f^*(kp) \implies k = 2,$$

which completes the proof of (i). To prove (ii), now (4.1) gives that

$$K_S \sim f^*(-p) + F_1 + F_2,$$

which combined with $f^*(p) \sim 2F_2$ implies that $K_S \sim F_1 - F_2$. $\qquad \square$

Definition 4.16. The fibers F_1, F_2 of Proposition 4.15 are called the *half-fibers* of the genus 1 pencil $f \colon S \to \mathbb{P}^1$.

Chapter 5

Convex Geometry and the Secondary Polytope

In this chapter we define the notion of secondary polytope, and we illustrate it with two examples: the Stasheff polytope and the secondary polytope of the unit cube. Our main references are [GKZ08, DLRS10].

5.1 Basic definitions in convex geometry

Definition 5.1. Let A be a subset of $\mathbb{R}^k$. Then the *convex hull* of A is the intersection of all the closed half-spaces containing A. Using open half-spaces instead gives the same result.

Definition 5.2. A *polytope* $Q \subset \mathbb{R}^k$ is the convex hull of a given finite set of points $A \subset \mathbb{R}^k$. The pair (Q, A) is called *marked polytope*.

Definition 5.3. Let $Q \subset \mathbb{R}^k$ be a polytope, $H \subset \mathbb{R}^k$ an affine hyperplane, and denote by H^+, H^- its corresponding open half spaces. Assume that $Q \cap H^- = \emptyset$ and $Q \cap H \neq \emptyset$. Then we call $Q \cap H$ a *face* of Q. The polytope Q itself is considered a face of Q. The faces of Q are also polytopes. The *vertices* of Q are the faces of Q that consist of a single point.

Definition 5.4. Let $Q \subset \mathbb{R}^k$ be a polytope, and let v_0 be one of its vertices. Then define the *dimension* of Q, which we denote by $\dim(Q)$, as the dimension of the $\mathbb{R}$-vector space spanned by the vectors $v - v_0$ as v varies among the vertices of Q. This definition is independent from the choice of v_0.

Observation 5.5. Let $Q \subset \mathbb{R}^k$ be a polytope and let F be one of its faces. Then $0 \leq \dim(F) \leq \dim(Q) \leq k$, and F is a vertex if and only if $\dim(F) = 0$.

Definition 5.6. Given a polytope, the faces of dimension 1 are called *edges*. Codimension 1 faces are called *facets*.

Definition 5.7. Let Q be a polytope. If $\dim(Q) = \#\text{vertices} - 1$, then Q is called a *simplex*.

5.2 Polyhedral subdivisions

Definition 5.8. A *polyhedral subdivision* of a given a marked polytope (Q, A) is a finite collection $\mathcal{Q} = \{Q_i\}_i$ of polytopes such that:

- $\cup_i Q_i = Q$;

- $Q_i \cap Q_j$ is either empty, or a face of both Q_i and Q_j for all i, j;

- The vertices of Q_i are contained in A for all i;

- If $Q_i \in \mathcal{Q}$ and F is a face of Q_i, then $F \in \mathcal{Q}$.

A polyhedral subdivision which only uses simplices is called *triangulation*. See Figure 5.1 for some concrete examples.
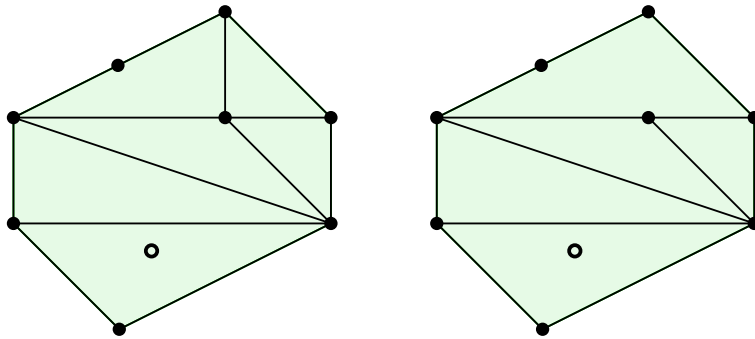
Figure 5.1: On the left an example of triangulation. On the right a non-example of polyhedral subdivision

Definition 5.9. Let (Q, A) be a marked polytope with $Q \subset \mathbb{R}^k$ and let $h\colon A \to \mathbb{R}$ be a function (we call h a *height function* on A). Let $Q^+ \subset \mathbb{R}^k \times \mathbb{R}$ be the convex hull of the half lines $(a, h(a) + \mathbb{R}_{\geq 0})$ for $a \in A$. The images of the faces of Q^+ under the projection $\mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$ give a

polyhedral subdivision of (Q, A), which we call the *polyhedral subdivision induced by the height function h*. Figure 5.2 illustrates this construction for a one-dimensional marked polytope. A polyhedral subdivision of (Q, A) is *regular* if it is induced by some height function on A. In the literature, regular subdivisions are sometimes also called *coherent* or *convex*.
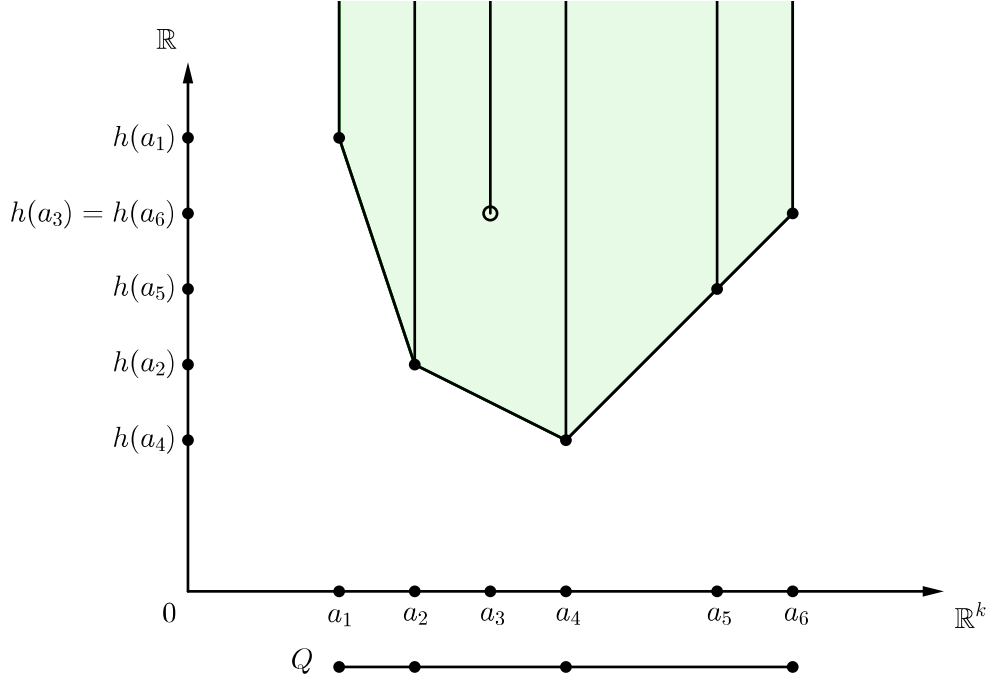
Figure 5.2: Polyhedral subdivision of Q induced by a height function h

Example 5.10. The simplest example of polyhedral subdivision which is not regular is given in Figure 5.3. To show this subdivision is not regular, assume by contradiction it is induced by a height function h. We can assume without loss of generality that $h(a_4) = h(a_5) = h(a_6) = 0$ because the polytope with vertices a_4, a_5, a_6 is part of the subdivision. We have that $h(a_1) > 0$, otherwise we would not see the edge with vertices a_1, a_4. Observe that $h(a_1) \neq h(a_2)$, otherwise we would not have the edge with vertices a_2, a_4. Therefore, assume that $h(a_1) > h(a_2)$ (a similar argument holds for the opposite inequality). But the symmetry of the figure implies that $h(a_2) > h(a_3) > h(a_1)$, which is a contradiction.
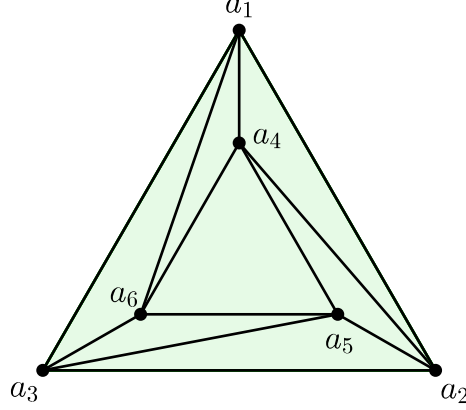
Figure 5.3: Example of nonregular polyhedral subdivision

5.3 The secondary polytope

Let (Q, A) be a marked polytope and assume $Q \subset \mathbb{R}^k$ is nondegenerate (i.e. $\dim(Q) = k$). We are ready to define the *secondary polytope* of the marked polytope (Q, A). Let $\mathcal{T} = \{Q_i\}_i$ be a triangulation of (Q, A). Then define

$$\varphi_{\mathcal{T}} \colon A \to \mathbb{R}$$

$$a \mapsto \sum_{Q_i \colon\, a \in Q_i \ \text{is a vertex}} \mathrm{Vol}(Q_i).$$

Observe that, if $A = \{a_1, \ldots, a_n\}$, then $\varphi_{\mathcal{T}}$ can be identified with the point $(\varphi_{\mathcal{T}}(a_1), \ldots, \varphi_{\mathcal{T}}(a_n))$ in $\mathbb{R}^{|A|}$.

Definition 5.11. The *secondary polytope* $\Sigma(A)$ of the marked polytope (Q, A) is the convex hull of the set $\{\varphi_{\mathcal{T}} | \mathcal{T} \text{ is a triangulation of } (Q, A)\} \subset \mathbb{R}^{|A|}$. If A coincide with the set of vertices of Q, then we denote $\Sigma(A)$ by $\Sigma(Q)$ instead.

Theorem 5.12 ([GKZ08, Chapter 7, Theorem 1.7]). *Let (Q, A) be a marked polytope. Then*

$$\dim(\Sigma(A)) = |A| - \dim(Q) - 1.$$

Theorem 5.13 ([GKZ08, Chapter 7, Theorem 2.4])**.** *The faces of the secondary polytope $\Sigma(A)$ are in bijection with the regular subdivisions of (Q, A). Moreover, containment of faces of $\Sigma(A)$ corresponds to refinement of subdivisions of (Q, A). In particular, the vertices of $\Sigma(A)$ correspond to the regular triangulations of (Q, A).*

Example 5.14. One of the first interesting examples of secondary polytopes is the *Stasheff polytope*. For $n \geq 3$, let A_n be the set of vertices of a planar n-gon. Then, the *n-th Stasheff polytope* is the secondary polytope $\Sigma(A_n)$. It is easy to observe that $\Sigma(A_3)$ is a point, $\Sigma(A_4)$ is a segment, and $\Sigma(A_5)$ is a planar pentagon (see Figure 5.4). For $n \geq 6$, the n-th Stasheff polytope is more complicated. We just remark that $\Sigma(A_n)$ has dimension $n - 3$ by Theorem 5.12, and $\frac{1}{n-1}\binom{2n-4}{n-2}$ vertices (see [GKZ08, Chapter 7, Section 3, B]).
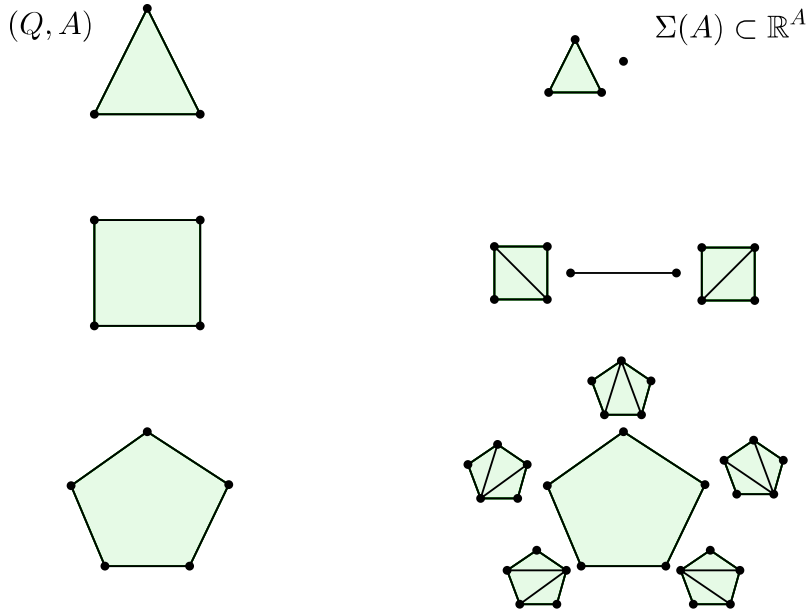

Figure 5.4: The Stasheff polytopes $\Sigma(A_n)$ for $n = 3, 4, 5$

5.4 Polyhedral subdivisions of the unit cube

In this section we fix Q to be the unit cube $[0, 1]^3 \subset \mathbb{R}^3$. The marking of Q is chosen to be its set of vertices $Q \cap \mathbb{Z}^3$ (and therefore it is omitted). Let us enumerate all the possible polyhedral subdivisions of $(Q, Q \cap \mathbb{Z}^3)$.

We start by listing all the possible subpolytopes of Q with vertices in $Q \cap \mathbb{Z}^3$. These are represented in Figure 5.5 with the corresponding *lattice volume*. For a nondegenerate polytope $P \subset \mathbb{R}^k$, recall that the *lattice volume* of P is defined to be $k! \cdot \text{Vol}(P)$. We can combine the polytopes in Figure 5.5 to form a subdivision of $(Q, Q \cap \mathbb{Z}^3)$, knowing that the sum of the lattice volumes of the polytopes chosen has to be 6.

A complete list of all the possible polyhedral subdivisions of $(Q, Q \cap \mathbb{Z}^3)$ up to symmetry can be found in Figure 5.6. A polyhedral subdivision of $(Q, Q \cap \mathbb{Z}^3)$ is represented by a graph whose vertices correspond to the maximal dimensional polytopes in the subdivision and there is an edge between two vertices if and only if the corresponding polytopes share a facet. When this graph is not enough to distinguish two distinct polyhedral subdivisions, we associate a number to each vertex which represents the number of facets of the corresponding polytope. If this is not enough, instead we draw a dotted edge between two vertices if the corresponding polytopes share an edge and not a facet. The sum in the top left corner of each box in Figure 5.6 represents the lattice volume of the polytopes used in the subdivisions. In the next proposition we observe that all these subdivisions are regular.

Proposition 5.15. *All the polyhedral subdivisions of the unit cube Q are regular.*

Proof. We know from [San01, Corollary 2.9] that a marked polytope with a nonregular subdivision has a nonregular triangulation. But all the triangulations of $(Q, Q \cap \mathbb{Z}^3)$ are regular (see [DL96, Theorem 3.2] or [DLRS10, Theorem 6.3.10]). These triangulations are pictured in Figure 5.7 up to symmetry with a corresponding height function. □

For more information about $\Sigma(Q)$, we refer to [Pfe00]. Here we just observe that $\Sigma(Q)$ has dimension 4 and 74 vertices (the number of vertices can easily be counted from Figure 5.7). The secondary polytope $\Sigma(Q)$ has an important role in Chapter 8.
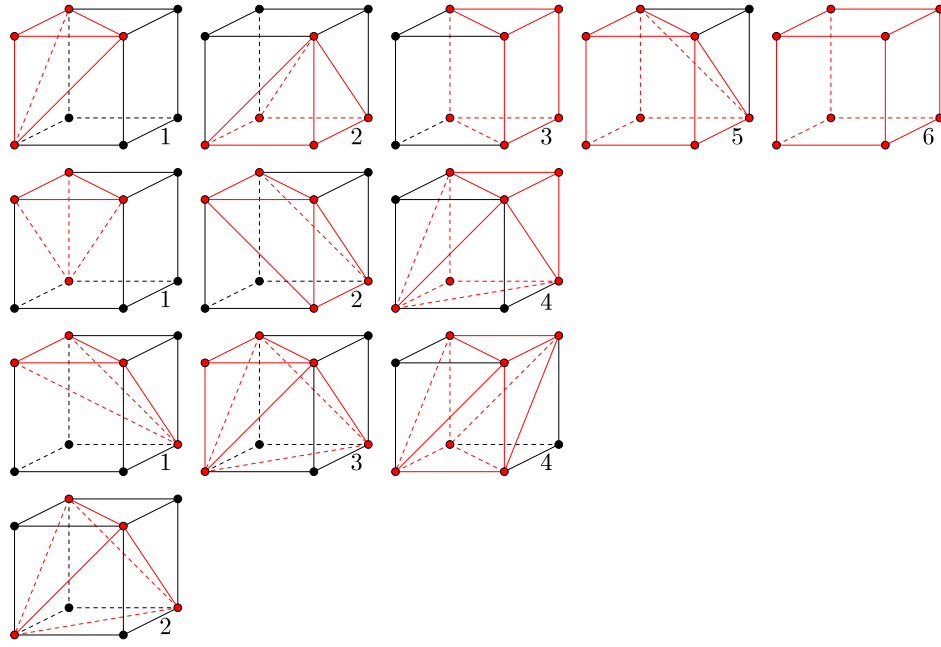
Figure 5.5: List of all the possible subpolytopes of Q up to symmetry with vertices in $Q \cap \mathbb{Z}^3$ and their corresponding lattice volume

Figure 5.6: List of all the polyhedral subdivisions of $(Q, Q \cap \mathbb{Z}^3)$, $Q = [0,1]^3$, up to symmetry
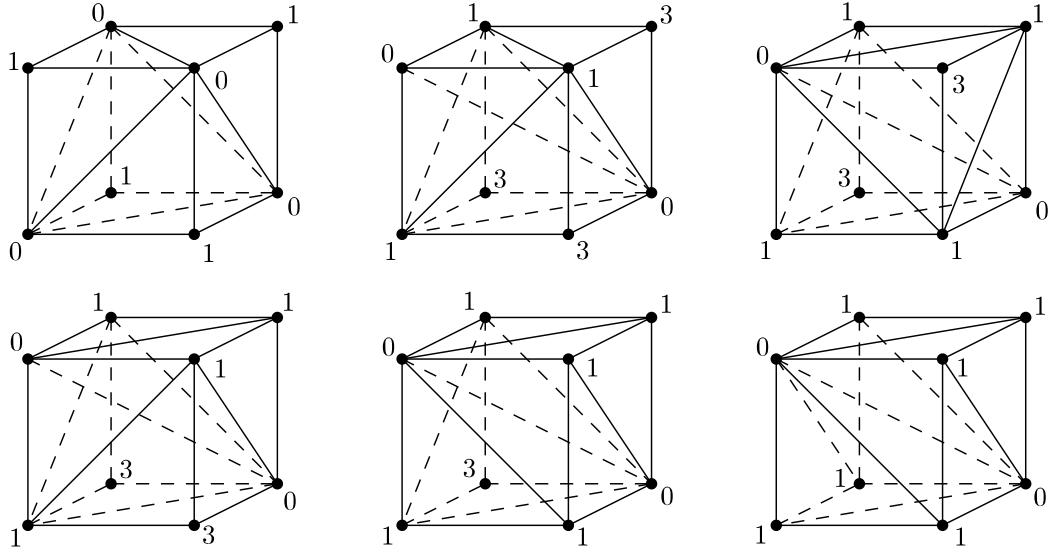
Figure 5.7: The triangulations of the unit cube Q up to symmetry and a corresponding height functions on the vertices. This figure is a reproduction of [DLRS10, Figure 1.33])

Chapter 6

Moduli of Stable Toric Pairs

In this chapter we survey the theory of stable toric pairs and their moduli. Our main references are [Ale02, Ale06, Ale15].

6.1 Stable toric pairs

For us, a *variety* is a connected and reduced scheme of finite type over an arbitrary algebraically closed field $\Bbbk$ of characteristic zero (in particular, a variety need not be irreducible). T denotes a fixed (split) torus over $\Bbbk$ and let M be its character lattice. $M_\mathbb{R}$ denotes the tensor product $M \otimes_\mathbb{Z} \mathbb{R}$. We want to remark that the theory of stable toric pairs we are about to review works in any characteristic.

Definition 6.1. Let X be a variety with a T-action. We say that X is a *stable toric variety* if X is seminormal and its irreducible components are toric varieties under the T-action. The *toric boundary* of a stable toric variety is defined to be the sum of the boundary divisors of each irreducible component which are not in common with other irreducible components. If X is projective and $\mathscr{L}$ is an ample and T-linearized invertible sheaf on X, we say that the pair $(X, \mathscr{L})$ is a *polarized stable toric variety*.

Remark 6.2. Assume we have a polarized stable toric variety $(X, \mathscr{L})$. Then we can associate to each irreducible component X_i of X a lattice polytope P_i. These polytopes can be glued to one another in the same way as X is the union of its irreducible components. This results into a topological space $\cup_i P_i$ which is called the *topological type* of X. The topological type comes together with a finite map $\cup_i P_i \to M_\mathbb{R}$, called the *reference map*, which embeds each P_i as a lattice

polytope in $M_\mathbb{R}$. The set of the faces of all the polytopes P_i, together with the identifications coming from the gluing, forms what is called a *complex of polytopes*. Up to isomorphism, there is a 1-to-1 correspondence between polarized stable toric varieties (for a fixed torus T) and the following data:

(a) A complex of polytopes $\mathcal{P}$;

(b) A reference map $\cup_{P \in \mathcal{P}} P \to M_\mathbb{R}$;

(c) An element of the cohomology group we are about to describe. For each $P \in \mathcal{P}$, let C_P be the saturated sublattice of $\mathbb{Z} \oplus M$ generated by $(1, P)$, and let T_P be the torus $\mathrm{Hom}(C_P, \Bbbk^*)$. The tori T_P for $P \in \mathcal{P}$ define a sheaf $\underline{T}$ of abelian groups on the poset $\mathcal{P}$ with the order topology. So we want to consider an element of $H^1(\mathcal{P}, \underline{T})$, which describes the way the irreducible components of X are glued together.

For more details about this, see [Ale06, Section 4.3] or [Ale02, Theorem 1.2.6].

Definition 6.3. Let $(X, \mathscr{L})$ be a polarized stable toric variety and let $Q \subset M_\mathbb{R}$ be a lattice polytope. We say that X has *type* $\leq Q$ if the complex of polytopes $\mathcal{P}$ associated to X is a polyhedral subdivision of the marked polytope $(Q, Q \cap M)$. Observe that in this case the reference map is implicitly given by the inclusion of Q in $M_\mathbb{R}$. Furthermore, the toric boundary of X is given by the sum of the divisors corresponding to the facets in $\mathcal{P}$ contained in the boundary of Q.

Definition 6.4. A *stable toric pair* is the datum of a polarized stable toric variety $(X, \mathscr{L})$ together with an effective Cartier divisor B which is the divisor of zeros of a global section of $\mathscr{L}$. Also, we require that B does not contain any torus fixed point (or equivalently any T-orbit). We denote a stable toric pair simply by (X, B) because it is understood that $\mathscr{L} \cong \mathcal{O}_X(B)$. Two stable toric pairs (X, B) and (X', B') are isomorphic if there exists an isomorphism $f \colon X \to X'$ that preserves the T-action and such that $f^* B' = B$. We say that a stable toric pair has *type* $\leq Q$ if the corresponding polarized stable toric variety has type $\leq Q$.

Remark 6.5. Let (X, B) be a stable toric pair and let $\mathcal{P}$ be the complex of polytopes corresponding to the polarized stable toric variety $(X, \mathcal{O}_X(B))$. If X_i is any irreducible component of X, then the restriction $B|_{X_i}$ can be described combinatorially as follows (see [Ale02, Theorem 1.2.7] in combination with [Ale02, Lemma 2.2.7, part 2]). Consider the marking on the corresponding lattice

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polytope P_i given by $P_i \cap M$. Observe that these lattice points in P_i correspond to monomials (recall that M is the character lattice of the fixed torus T). Now, $B|_{X_i}$ is determined (not uniquely) by a function $f \colon P_i \cap M \to \Bbbk^*$, which assigns to each monomial a corresponding coefficient (which we want to be nonzero because B does not contain any torus fixed point).

6.2   The stack of stable toric pairs

By $\mathbf{Sch}_\Bbbk$ we denote the category of locally noetherian schemes over our fixed base field $\Bbbk$.

Definition 6.6. Given $S \in \mathrm{Ob}(\mathbf{Sch}_\Bbbk)$, a *family of stable toric pairs* $(\mathfrak{X}, \mathfrak{B})/S$ is the datum of a proper and flat morphism of schemes $\pi \colon \mathfrak{X} \to S$, a compatible $T_S = T \times_\Bbbk S$ action on $\mathfrak{X}$, and an effective Cartier divisor $\mathfrak{B} \subset \mathfrak{X}$ such that $\pi|_\mathfrak{B}$ is flat and the fiber $(\mathfrak{X}_s, \mathfrak{B}_s)$ over every geometric point $s \to S$ is a stable toric pair with the action induced by T_S. Two families of stable toric pairs over the same base are *isomorphic* if there exists an isomorphism of pairs over S preserving the torus action. Given a lattice polytope Q, we say that a family of stable toric pairs has *type* $\leq Q$ if every geometric fiber has type $\leq Q$.

Remark 6.7. Let $(\mathfrak{X}, \mathfrak{B})/S$ be a family of stable toric pairs. Denote by π the morphism $\mathfrak{X} \to S$ and let $\mathscr{L} = \mathcal{O}_\mathfrak{X}(\mathfrak{B})$. Following [Ale02, Proof of Lemma 2.10.1], for $d \geq 0$ the sheaves $\pi_* \mathscr{L}^{\otimes d}$ are locally free, and the torus action gives a decomposition $\pi_* \mathscr{L}^{\otimes d} = \bigoplus_{m \in M} R_{(d,m)}$ into sheaves that are also locally free. This results into a locally free $(\mathbb{Z} \oplus M)$-graded $\mathcal{O}_S$-algebra $R = \bigoplus_{(d,m) \in \mathbb{Z} \oplus M} R_{(d,m)}$ together with a $\mathbb{Z}$-degree 1 section θ of R corresponding to $\mathfrak{B}$. Conversely a pair (R, θ), where R is a locally free graded $\mathcal{O}_S$-algebra and θ a degree 1 section, uniquely determines a family of stable toric pairs up to isomorphism (see [Ale02, Proof of Theorem 2.10.8] or [Ale06, Section 4]).

Remark 6.8. After defining families of stable toric pairs we automatically have a notion of stack over $\Bbbk$. Given a lattice polytope Q, denote by $\mathscr{M}_Q$ the category of families of stable toric pairs of type $\leq Q$, where a morphism $f = (f_t, f_b) \colon (\mathfrak{X}', \mathfrak{B}')/S' \to (\mathfrak{X}, \mathfrak{B})/S$ is a pullback diagram

$$
\begin{array}{ccc}
(\mathfrak{X}', \mathfrak{B}') & \xrightarrow{\;f_t\;} & (\mathfrak{X}, \mathfrak{B}) \\
\downarrow & & \downarrow \\
S' & \xrightarrow[f_b]{} & S.
\end{array}
$$

Let $\mathscr{M}_Q \to \mathbf{Sch}_{\Bbbk}$ be the functor sending $(\mathfrak{X}, \mathfrak{B})/S$ to S and a morphism f to f_b. Then $\mathscr{M}_Q$ is a stack over $\Bbbk$ parametrizing families of stable toric pairs of type $\leq Q$.

The main result about the stack $\mathscr{M}_Q$ is the following theorem due to Alexeev.

Theorem 6.9 ([Ale02, Theorem 1.2.15])**.** *Let Q be a lattice polytope and let $\mathscr{M}_Q$ be the stack of stable toric pairs of type $\leq Q$. Then the following hold:*

(i) *$\mathscr{M}_Q$ is a proper quotient stack (and therefore an Artin stack) with finite stabilizers[1];*

(ii) *It has a coarse moduli space $\overline{M}_Q$ which is a projective scheme;*

(iii) *$\overline{M}_Q$ is naturally stratified, and every stratum corresponds in a 1-to-1 way to a polyhedral subdivision of $(Q, Q \cap M)$;*

(iv) *The normalization of the main irreducible component of $(\overline{M}_Q)_{\mathrm{red}}$ is isomorphic to the toric variety associated to the secondary polytope $\Sigma(Q \cap M)$.*

Remark 6.10. In Theorem 6.9 (iv), one could define the main irreducible component of $(M_Q)_{\mathrm{red}}$ as the irreducible component of $(\overline{M}_Q)_{\mathrm{red}}$ satisfying the claimed property. Equivalently, the main irreducible component can be characterized by the fact that on a dense open subset it parametrizes stable toric pairs whose corresponding polyhedral subdivision of $(Q, Q \cap M)$ is the trivial one.

6.3   An explicit computation

The goal of this section is to explicitly compute a cohomology group $H^1(\mathcal{P}, \underline{T})$ for a given polytope Q and polyhedral subdivision $\mathcal{P}$ (this is explained in Remark 6.2). The example we analyze is fundamental later in Chapter 8.

Theorem 6.11. *Let Q be the unit cube and let $\mathcal{P}$ be a polyhedral subdivision of $(Q, Q \cap \mathbb{Z}^3)$. Then $H^1(\mathcal{P}, \underline{T}) = \{1\}$.*

To prove Theorem 6.11 we need some general preliminaries.

Observation 6.12. Let $\mathcal{P}$ be a complex of polytopes obtained by gluing finitely many polytopes $\{P_i\}_{i \in I}$ of the same dimension. Let $\cup_{P \in \mathcal{P}} P \to M_{\mathbb{R}}$ be a reference map. Recall that Λ_{P_i} (which

---

[1]Since we are working in characteristic 0, we have that $\mathscr{M}_Q$ is actually a Deligne-Mumford stack.

we denote for brevity Λ_i) is the saturated sublattice of $\mathbb{Z} \oplus M$ generated by $(1, P_i)$. For any $\{i_1, \ldots, i_s\} \subset I$, let $\Lambda_{i_1 \ldots i_s}$ be the intersection $\Lambda_{i_1} \cap \ldots \cap \Lambda_{i_s}$. An element of $H^1(\mathcal{P}, \underline{T})$ is a tuple $(\ldots, t_{ij}, \ldots)$ indexed by pairs in I, with $t_{ij} \in \text{Hom}(\Lambda_{ij}, \mathbb{k}^*)$ satisfying the *cocycle condition* and *modulo coboundaries*. The cocycle condition is the following: if $i, j, k \in I$ is any triple, then

$$t_{ij}|_{\Lambda_{ijk}} t_{jk}|_{\Lambda_{ijk}} = t_{ik}|_{\Lambda_{ijk}}.$$

A coboundary is a tuple $(\ldots, t_i|_{\Lambda_{ij}} t_j|_{\Lambda_{ij}}^{-1}, \ldots)$ for $t_i \in \text{Hom}(\Lambda_i, \mathbb{k}^*)$ and $t_j \in \text{Hom}(\Lambda_j, \mathbb{k}^*)$. Therefore, we have an equivalence relation on the cocycles given by

$$(\ldots, t_{ij}, \ldots) \sim (\ldots, t_{ij} t_i|_{\Lambda_{ij}}^{-1} t_j|_{\Lambda_{ij}}, \ldots).$$

Lemma 6.13. *Let $\mathcal{P}$ be a connected complex of polytopes obtained by gluing finitely many polytopes of the same dimension. Let P be one of these polytopes and assume P has exactly one facet in common with some other maximal dimensional polytope in $\mathcal{P}$. Let $\mathcal{P}'$ be the complex of polytopes obtained from $\mathcal{P}$ by eliminating P. Then $H^1(\mathcal{P}, \underline{T}) \cong H^1(\mathcal{P}', \underline{T})$.*

Proof. Label P by 1 and the polytope adjacent to P by 2. Consider an arbitrary cocycle $c = (t_{12}, \ldots)$. Let $t_1 \in \text{Hom}(\Lambda_1, \mathbb{k}^*)$ be such that $t_1|_{\Lambda_{12}} = t_{12}$ (observe that there can be different choices for t_1). If we consider the coboundary $(t_1|_{\Lambda_{12}} 1|_{\Lambda_{12}}^{-1}, 1, \ldots)$, then $c \sim (1, \ldots)$, giving the required isomorphism. $\square$

Definition 6.14. Let P be a polytope as in the statement of Lemma 6.13. We call P a *hanging polytope* of $\mathcal{P}$.

Lemma 6.15. *Let $\mathcal{P}$ be a connected complex of polytopes obtained by gluing finitely many polytopes $\{P_i\}_{i=1}^n$ of the same dimension. Assume that all these polytopes share a fixed face of codimension 2 and that they are organized around it as follows: for $i = 1, \ldots, n-1$, P_i and P_{i+1} share a facet, and also P_1 and P_n do. Then $H^1(\mathcal{P}, \underline{T}) = \{1\}$.*

Proof. Let $c = (t_{12}, t_{23}, \ldots, t_{n-1,n}, t_{n1})$ be a cocycle and consider a coboundary with $t_n|_{\Lambda_{n1}} = t_{n1}$ and the other t_i equal to 1. Then $c \sim (t'_{12}, t'_{23}, \ldots, t'_{n-1,n}, 1)$. We can iterate this strategy until we obtain $c \sim (\tau, 1, \ldots, 1)$ for some $\tau \in \text{Hom}(\Lambda_{12}, \mathbb{k}^*)$. Now consider a coboundary with $t_1|_{\Lambda_{12}} = \tau$,

$t_1|_{\Lambda_{n1}} = 1$ and the other t_i equal to 1. This makes sense because $\tau|_{\Lambda_{n12}} = 1$ by the cocycle condition on $(\tau, 1, \ldots, 1)$. We can conclude that $c \sim (\tau, 1, \ldots, 1) \sim (1, \ldots, 1)$. □

proof of Theorem 6.11. Let $\mathcal{P}$ be any polyhedral subdivision of $(Q, Q \cap \mathbb{Z}^3)$ where Q is the unit cube. Applying Lemma 6.13, we can reduce our problem to compute $H^1(\mathcal{P}', \underline{T})$ with $\mathcal{P}'$ without hanging polytopes (see Definition 6.14). After enumerating all the possibilities for $\mathcal{P}$, one can observe that either $\mathcal{P}'$ is generated by a single maximal dimensional polytope, or it is as in the statement of Lemma 6.15. In both cases, we can conclude that $H^1(\mathcal{P}, \underline{T}) \cong H^1(\mathcal{P}', \underline{T}) \cong \{1\}$. The complete list of all the possibilities for $\mathcal{P}$ is in Figure 5.6 (see Section 5.4). □

Chapter 7

Moduli of Stable Pairs

This chapter is devoted to the study of stable pairs. We start with the definition of weighted stable curve (which is the very first example of stable pair). Then we review the definition of log canonical singularities and its extension to the non-normal case, i.e. semi-log canonical singularities. After this, we define stable pairs and their corresponding Viehweg's moduli functor. We follow [Ale15, Kol13b, KM98]. Recall that we work over an algebraically closed field $\Bbbk$ of characteristic zero.

7.1 Weighted stable curves

Definition 7.1. Let $n \in \mathbb{Z}_{\geq 0}$ and fix $b_1, \ldots, b_n \in (0, 1] \cap \mathbb{Q}$. A *weighted stable curve for the weight* $\underline{b} = (b_1, \ldots, b_n)$ is a pair $(C, B = \sum_{i=1}^{n} b_i p_i)$ such that the following hold:

(i) C is a nodal curve;

(ii) The points $p_1, \ldots, p_n$ lie on the smooth locus of C, and if $p_{i_1}, \ldots, p_{i_m}$ coincide, then $b_{i_1} + \ldots + b_{i_m} \leq 1$;

(iii) Let $C = \cup_j C_j$ be the decomposition of C into its irreducible components and let C_j^ν be the normalization of C_j. Let d_j be the cardinality of the preimage of the nodes of C under the composition $C_j^\nu \to C_j \hookrightarrow C$. Then for all j we require that

$$2p_g(C_j^\nu) - 2 + d_j + \sum_{p_i \in C_j} b_i > 0.$$

Observation 7.2. Condition (iii) of Definition 7.1 is equivalent to $(K_C + B)|_{C_j}$ ample for all j, which is equivalent to $K_C + B$ ample.

Theorem 7.3 ([Has03, Theorem 2.1]). *For any $n, g \in \mathbb{Z}_{\geq 0}$ and collection of weights $\underline{b} = (b_1, \ldots, b_n)$, there exists a connected, smooth, and proper Deligne-Mumford stack $\overline{\mathcal{M}}_{g,\underline{b}}$ parametrizing weighted stable curves for the weight $\underline{b}$ of arithmetic genus g. The corresponding coarse moduli scheme $\overline{M}_{g,\underline{b}}$ is projective. (These results also hold over $\mathbb{Z}$.)*

7.2 Singularities of pairs

Definition 7.4. Let X be a normal variety and let B be a $\mathbb{Q}$-divisor on X with coefficients in $(0, 1]$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $f \colon Y \to X$ be a birational morphism. If $\{E_i\}_i$ is the set of irreducible exceptional divisors of f, then there exist rational numbers $a(E_i, X, B)$ such that

$$K_Y + f_*^{-1} B \sim_{\mathbb{Q}} f^*(K_X + B) + \sum_i a(E_i, X, B) E_i,$$

where $f_*^{-1} B$ denotes the strict transform of B under f. We define the *discrepancy* of the pair (X, B) as

$$\mathrm{discr}(X, B) = \inf\{a(E, X, B)\},$$

where we take the infimum over all birational morphisms $f \colon Y \to X$ and all irreducible exceptional divisors $E \subset Y$.

Definition 7.5. Let X be a normal variety and let B be a $\mathbb{Q}$-divisor on X with coefficients in $(0, 1]$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. We say that the singularities of the pair (X, B) are

- *terminal* if $\mathrm{discrep}(X, B) > 0$;

- *canonical* if $\mathrm{discrep}(X, B) \geq 0$;

- *Kawamata log terminal* (or *klt* for short) if $\mathrm{discrep}(X, B) > -1$ and $\lfloor B \rfloor = 0$;

- *purely log terminal* if $\mathrm{discrep}(X, B) > -1$;

- *log canonical* if $\mathrm{discrep}(X, B) \geq -1$.

In general, it is not very easy to compute the discrepancy of a pair (X, B), especially because we need to consider all birational morphisms $Y \to X$. However, it is possible to tell whether (X, B) has log canonical or klt singularities just by considering a *log resolution* of (X, B).

Theorem 7.6 ([Hir64])**.** *Let X be a scheme and B a $\mathbb{Q}$-divisor on X. Then there exists a proper birational morphism $\sigma \colon Y \to X$ such that Y is smooth, $\mathrm{Exc}(\sigma)$ is a divisor, and $\mathrm{Exc}(\sigma) \cup \mathrm{Supp}(\sigma_*^{-1} B)$ is simple normal crossing. We call σ a* log resolution *of (X, B).*

Proposition 7.7 ([Kol13b, Corollary 2.13])**.** *Let X be a normal variety and let B be a $\mathbb{Q}$-divisor on X with coefficients in $(0, 1]$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $\sigma \colon Y \to X$ be a log resolution of (X, B), so that*

$$K_Y \sim_{\mathbb{Q}} \sigma^*(K_X + B) + \sum_i a_i E_i.$$

(Observe that, writing in this way, a divisor $E_i \subset Y$ is not necessarily exceptional.) Then

- *(X, B) is log canonical if and only if $a_i \geq -1$ for all i;*

- *(X, B) is klt if and only if $a_i > -1$ for all i.*

Observation 7.8. In Proposition 7.7 the condition $a_i \geq 1$ for E_i not exceptional is automatically satisfied because the coefficients of the divisor B are assumed to be in the interval $(0, 1]$.

Example 7.9. Let $X = \mathbb{A}^2$, $C = Z(y^2 - x^3)$, and $\beta \in (0, 1] \cap \mathbb{Q}$. Let us compute the singularities of the pair $(X, \beta C)$ using Proposition 7.7. For simplicity of notation, when blowing up at a point, we denote the strict transform of a divisor D by the same letter D. A log resolution of $(X, \beta C)$ is given by $\sigma = \sigma_1 \circ \sigma_2 \circ \sigma_3$, where

- $\sigma_1 \colon X' \to X$ is the blow up of X at $(0, 0)$ with exceptional divisor E_1;

- $\sigma_2 \colon X'' \to X'$ is the blow up of X' at $C \cap E_1$ with exceptional divisor E_2;

- $\sigma_3 \colon Y \to X''$ is the blow up of X'' at $C \cap E_1 \cap E_2$ with exceptional divisor E_3.

We have that

$$K_Y = \sigma_3^*(K_{X''}) + E_3 = \sigma_3^*(\sigma_2^*(K_{X'}) + E_2) + E_3 = \sigma_3^*(\sigma_2^*(K_{X'})) + E_2 + 2E_3$$

$$= \sigma_3^*(\sigma_2^*(\sigma_1^*(K_X) + E_1)) + E_2 + 2E_3 = \sigma_3^*(\sigma_2^*(E_1)) + E_2 + 2E_3$$

$$= \sigma_3^*(E_1 + E_2) + E_2 + 2E_3 = E_1 + 2E_2 + 4E_3,$$

$$\sigma^* C = \sigma_3^* \sigma_2^* \sigma_1^* C = \sigma_3^* \sigma_2^* (C + 2E_1) = \sigma_3^* (C + 2E_1 + 3E_2) = C + 2E_1 + 3E_2 + 6E_3.$$

It follows that

$$K_Y = \sigma^* (K_X + \beta C) + \sum_{i=1}^{3} a_i E_i - \sigma_*^{-1} (\beta C)$$

$$\implies E_1 + 2E_2 + 4E_3 = \beta(C + 2E_1 + 3E_2 + 6E_3) + \sum_{i=1}^{3} a_i E_i - \beta C$$

$$\implies \begin{cases} a_1 = 1 - 2\beta, \\ a_2 = 2 - 3\beta, \\ a_3 = 4 - 6\beta. \end{cases}$$

Therefore, by Proposition 7.7, we have that $(X, \beta C)$ has log canonical (resp. klt) singularities if and only if $1 - 2\beta, 2 - 3\beta, 4 - 6\beta, -\beta \geq -1$ (resp. > -1), which is true if and only if $\beta \leq \frac{5}{6}$ (resp. $\beta < \frac{5}{6}$).

7.3 Extension to the non-normal case: semi-log canonical singularities

7.3.1 Demi-normal schemes

Definition 7.10. Let X be a scheme. A point $p \in X$ is called a *node* if $\mathcal{O}_{X,p} \cong R/(f)$ where $(R, \mathfrak{m})$ is a regular local ring of dimension 2, $f \in \mathfrak{m}^2$, and f is not a square in $\mathfrak{m}^2/\mathfrak{m}^3$.

Example 7.11. Let $X = \mathrm{Spec}(\Bbbk[x,y]/(xy))$. We want to verify that $p = (x,y)/(xy) \in X$ is a node. Let $R = \Bbbk[x,y]_{(x,y)}$, $\mathfrak{m} = (x,y)\Bbbk[x,y]_{(x,y)}$, and $f = xy$. Then $\mathcal{O}_{X,p} \cong R/(f)$, $(R, \mathfrak{m})$ is a regular local ring of dimension 2, $f \in \mathfrak{m}^2$, and f is not a square in $\mathfrak{m}^2/\mathfrak{m}^3$.

Definition 7.12. Let X be an equidimensional noetherian scheme. Then X is called *demi-normal* if X is S_2 and, given any $p \in X$ with $\dim \mathcal{O}_{X,p} = 1$, then p is regular or a node.

Observation 7.13. By definition, a demi-normal scheme X is Gorenstein in codimension 1. Therefore, it makes sense to consider the canonical class K_X.

7.3.2 Normalization of a demi-normal scheme

We start with some algebra preliminaries. The following lemma is a straightforward application of Zorn's lemma.

Lemma 7.14. *Let $A \subseteq B$ be an extension of rings. Then there exists a unique largest ideal of A which is also an ideal of B. We denote this ideal by $I_{A,B}$.*

Proposition 7.15. *Let $A \subseteq B$ be an extension of rings. Then $I_{A,B} = \mathrm{Ann}_A(B/A)$.*

Proof. Let us first show that $I_{A,B} \supseteq \mathrm{Ann}_A(B/A)$. By Lemma 7.14 it is enough to show that $\mathrm{Ann}_A(B/A)$, which is obviously an ideal of A, is also an ideal of B. So let $a \in \mathrm{Ann}_A(B/A)$ and $b \in B$. We want to show that $ab \in \mathrm{Ann}_A(B/A)$. So take any $b' + A \in B/A$. We have that $ab(b' + A) = a(bb' + A) = 0 + A$ because $bb' \in B$ and $a \in \mathrm{Ann}_A(B/A)$. Therefore $ab \in \mathrm{Ann}_A(B/A)$.

For the other containment, let $a \in I_{A,B}$ and take any $b + A \in B/A$. Then $a(b + A) = 0 + A$ because $I_{A,B}$ is an ideal in B and $ab \in I_{A,B} \subseteq A$. Hence $a \in \mathrm{Ann}_A(B/A)$. $\qquad\square$

Definition 7.16. Let A be a ring and let $\overline{A}$ be the integral closure of A in its total ring of fractions $\mathrm{Frac}(A)$. Then $I_{A,\overline{A}} = \mathrm{Ann}_A(\overline{A}/A)$ is called the *conductor ideal* of A.

Now we translate into sheaf theory the notion of conductor ideal.

Definition 7.17. Let X be a reduced scheme and let $\nu \colon X^{\nu} \to X$ be its normalization. Define the *conductor ideal sheaf* of X as:

$$\mathrm{cond}_X := \mathrm{Ann}_{\mathscr{O}_X}(\nu_* \mathscr{O}_{X^{\nu}}/\mathscr{O}_X).$$

By our previous discussion, cond_X is also an ideal sheaf on X^{ν}. When we look at cond_X as an ideal sheaf on X^{ν}, we denote it by $\mathrm{cond}_{X^{\nu}}$.

Definition 7.18. Let X be a reduced scheme and let $\nu \colon X^{\nu} \to X$ be its normalization. Define the *conductor subschemes* $D \subseteq X$ and $D^{\nu} \subseteq X^{\nu}$ as:

$$D := \mathrm{Spec}_X(\mathscr{O}_X/\mathrm{cond}_X), \ D^{\nu} := \mathrm{Spec}_{X^{\nu}}(\mathscr{O}_{X^{\nu}}/\mathrm{cond}_{X^{\nu}}).$$

In the case of demi-normal schemes, the conductor subschemes have the following properties.

Proposition 7.19 ([Kol13b, Section 5.1])**.** *Let X be a demi-normal scheme, $\nu\colon X^\nu \to X$ its normalization, and D, D^ν the corresponding conductor subschemes. Then D, D^ν are both reduced subschemes of pure codimension 1.*

Proposition 7.20 ([Kol13b, Section 5.1])**.** *Let X be a demi-normal projective scheme, $\nu\colon X^\nu \to X$ its normalization, and D, D^ν the corresponding conductor subschemes. Let B be an effective $\mathbb{Q}$-divisor such that $K_X + B$ is $\mathbb{Q}$-Cartier. Then*

$$\nu^*(K_X + B) \sim_{\mathbb{Q}} K_{X^\nu} + \nu_*^{-1}B + D^\nu.$$

7.3.3 Semi-log canonical singularities

Definition 7.21. Let X be a variety and let B be a $\mathbb{Q}$-divisor on X with coefficients in $(0,1]$. Then the pair (X, B) is *semi-log canonical* if the following conditions are satisfied:

(i) X is demi-normal;

(ii) If $\nu\colon X^\nu \to X$ is the normalization with conductors $D \subset X$ and $D^\nu \subset X^\nu$, then the support of B does not contain any irreducible component of D;

(iii) $K_X + B$ is $\mathbb{Q}$-Cartier;

(iv) The pair $(X^\nu, D^\nu + \nu_*^{-1}B)$ is log canonical (i.e. for each connected component Z of X^ν the pair $(Z, (D^\nu + \nu_*^{-1}B)|_Z)$ is log canonical).

7.4 Stable pairs and Viehweg's moduli functor

7.4.1 Stable pairs

Definition 7.22. A pair (X, B) is *stable* if the following conditions are satisfied:

(i) *On singularities:* (X, B) is a semi-log canonical pair;

(ii) *Numerical:* $K_X + B$ is ample.

Example 7.23. It is immediate from the definitions that a weighted stable curve (C, B) (see Definition 7.1) is an example of stable pair.

Example 7.24. Another fundamental example of stable pair is given by $(S, 0)$ where S is the canonical model (see [KM98, Section 3.8]) of a surface of general type (i.e. $\kappa(S) = 2$). We have that S is normal, K_S is $\mathbb{Q}$-Cartier and ample, and S has canonical singularities.

7.4.2 The Viehweg's moduli stack

Definition 7.25. The Viehweg's moduli stack $\overline{\mathcal{M}}_{d,N,C,\underline{b}}$ is defined as follows. Let us fix constants $d, N \in \mathbb{Z}_{>0}$, $C \in \mathbb{Q}_{>0}$, and $\underline{b} = (b_1, \ldots, b_n)$ with $b_i \in (0,1] \cap \mathbb{Q}$ and $Nb_i \in \mathbb{Z}$ for all $i = 1, \ldots, n$. For any scheme S over $\Bbbk$, $\overline{\mathcal{M}}_{d,N,C,\underline{b}}(S)$ is the set of proper flat families $\mathfrak{X} \to S$ together with a divisor $\mathfrak{B} = \sum_i b_i \mathfrak{B}_i$ satisfying the following properties:

- For all $i = 1, \ldots, n$, $\mathfrak{B}_i$ is a codimension 1 closed subscheme which is flat over S;

- Every geometric fiber (X, B) is a stable pair of dimension d with $(K_X + B)^d = C$;

- There exists an invertible sheaf $\mathscr{L}$ on $\mathfrak{X}$ such that for every geometric fiber (X, B) one has
  $$\mathscr{L}|_X \cong \mathcal{O}_X(N(K_X + B)).$$

Remark 7.26. Whether the stack $\overline{\mathcal{M}}_{d,N,C,\underline{b}}$ is coarsely represented by a projective scheme is a subtle matter. In our case (see Section 8.1.5) it is. More details can be found in [Ale15, Vie95].

Chapter 8

The KSBA Compactification of the Moduli Space of $D_{1,6}$-polarized Enriques Surfaces

In the current chapter we discuss the original results of this dissertation.

8.1 $D_{1,6}$-polarized Enriques surfaces and the choice of the divisor

8.1.1 The $D_{1,6}$ polarization

We refer to Chapter 4 for the definition and the main properties of Enriques surfaces.

Definition 8.1. Recall from Section 2.3 that the $D_{1,6}$ lattice is the index 2 sublattice of $\mathbb{Z}^{1,6}$ (see Examples 2.9 (a)) consisting of the vectors with even square (or equivalently, we can require the sum of the coordinates to be even). If $e_0, e_1, \ldots, e_6$ is the canonical basis of $\mathbb{Z}^{1,6}$, then we distinguish the following vectors in $D_{1,6}$: $2e_0, e_1 \pm e_2, e_3 \pm e_4, e_5 \pm e_6$.

Definition 8.2. A $D_{1,6}$-polarized Enriques surface is an Enriques surface S whose Picard group contains a primitively embedded copy of $D_{1,6}$ (this means that $\mathrm{Pic}(S)/D_{1,6}$ is torsion free) such that:

(a) The distinguished vector $2e_0$ corresponds to a nef divisor class H (observe that H is also big because $H^2 = (2e_0)^2 = 4$);

(b) The distinguished vectors $e_1 \pm e_2, e_3 \pm e_4, e_5 \pm e_6$ correspond to six irreducible curves $R_1^\pm, R_2^\pm, R_3^\pm$ respectively (these curves are (-2)-curves, i.e. isomorphic to $\mathbb{P}^1$ and with self-intersection -2).

The next proposition is well known from [Oud10, Section 3]. However, given its importance for us, we briefly sketch its proof.

Proposition 8.3. *Let S be a $D_{1,6}$-polarized Enriques surface. Then the linear system $|H - R_i^+ - R_i^-|$ is a genus 1 pencil, $i = 1, 2, 3$.*

Proof. The linear system $|H|$ is base point free and 2-dimensional (see [Oud10, Proposition 3.1]). Moreover, $|H|$ contracts the curves $R_i^\pm$, $i = 1, 2, 3$, because $H \cdot R_i^\pm = 0$. On the other hand, $|H - R_i^+ - R_i^-|$ is a proper linear subsystem of $|H|$ (for instance, it does not contract $R_i^\pm$ and contracts $R_j^\pm$ for $j \neq i$). Therefore, $|H - R_i^+ - R_i^-|$ is a pencil and a curve C in this linear system has (arithmetic) genus 1 because $C^2 = (H - R_i^+ - R_i^-)^2 = 0$ (now use [Bea96, I.15 The genus formula]). $\qquad\square$

Observation 8.4. Every genus 1 fibration on an Enriques surface has two half-fibers (see Section 4.4). Therefore, for a $D_{1,6}$-polarized Enriques surface S, denote by E_i and E_i' the two half-fibers in the genus 1 pencil $|H - R_i^+ - R_i^-|$, $i = 1, 2, 3$. Then the divisor $\sum_{i=1}^3 (E_i + E_i')$ is canonically associated to the $D_{1,6}$-polarization, and its intrinsic nature leads us to consider moduli of pairs $\left(S, \epsilon \cdot \sum_{i=1}^3 (E_i + E_i')\right)$ for $0 < \epsilon \ll 1$, $\epsilon \in \mathbb{Q}$. Observe also that the divisor $\sum_{i=1}^3 E_i$ is a degree 6 polarization which gives the classical construction of Enriques: it realizes S as the normalization of a degree 6 surface in $\mathbb{P}^3$ with only double crossing singularities along the edges of the coordinate tetrahedron.

Remark 8.5. If S is a $D_{1,6}$-polarized Enriques surface, then the divisor $C = H + \sum_{i=1}^3 (R_i^+ + R_i^-)$ is divisible by 2 in $\mathrm{Pic}(S)$. The $\mathbb{Z}_2$-cover of S branched along C has six (-1)-curves. Blowing down these curves we obtain a Campedelli surface with (topological) fundamental group $\mathbb{Z}_2^3$ (these were considered in [AP09]). Conversely, such a Campedelli surface X can be realized as the $\mathbb{Z}_2^3$-cover of $\mathbb{P}^2$ branched along seven lines. The minimal desingularization of the quotient of X by the involution fixing pointwise the preimage of one of these lines is a $D_{1,6}$-polarized Enriques surface.

Remark 8.6. By [Cos85, Theorem 3] we know that every Enriques surface contains three genus 1 pencils $|2E_1|, |2E_2|, |2E_3|$ such that $E_1 \cdot E_2 = E_2 \cdot E_3 = E_3 \cdot E_1 = 1$. Moreover, observe that

Enriques surfaces with a $D_{1,6}$-polarization are a specialization of the family of Enriques surfaces considered in [Ver83].

8.1.2 $D_{1,6}$-polarized Enriques surfaces as $\mathbb{Z}_2^2$-covers

Notation 8.7. Let $\mathrm{Bl}_3 \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $[1:0:0], [0:1:0], [0:0:1]$. $\mathrm{Bl}_3 \mathbb{P}^2$ comes with three genus 0 pencils $\pi_1, \pi_2, \pi_3 \colon \mathrm{Bl}_3 \mathbb{P}^2 \to \mathbb{P}^1$, and denote by ℓ_i, ℓ_i' two distinct irreducible elements in the i-th pencil, $i = 1, 2, 3$.

Proposition 8.8. *Let S be a $D_{1,6}$-polarized Enriques surface. Then there exists a divisor $\sum_{i=1}^3 (\ell_i + \ell_i')$ on $\mathrm{Bl}_3 \mathbb{P}^2$ (see Notation 8.7) without triple intersection points and a morphism $\pi \colon S \to \mathrm{Bl}_3 \mathbb{P}^2$ such that*

(i) π is a $\mathbb{Z}_2^2$-cover of $\mathrm{Bl}_3 \mathbb{P}^2$ ramified at $\sum_{i=1}^3 (E_i + E_i')$ and branched along $\sum_{i=1}^3 (\ell_i + \ell_i')$;

(ii) The linear system $|2E_i|$ is the pullback of $|\ell_i|$, $i = 1, 2, 3$.

Conversely, given any divisor $\sum_{i=1}^3 (\ell_i + \ell_i')$ on $\mathrm{Bl}_3 \mathbb{P}^2$ without triple intersection points, an appropriate $\mathbb{Z}_2^2$-cover of $\mathrm{Bl}_3 \mathbb{P}^2$ branched along $\sum_{i=1}^3 (\ell_i + \ell_i')$ is an Enriques surface with exactly two choices of $D_{1,6}$-polarization: H is the pullback of a general line in $\mathrm{Bl}_3 \mathbb{P}^2$, and the six lines $R_i^\pm$, $i = 1, 2, 3$, are the preimages of the exceptional divisors in $\mathrm{Bl}_3 \mathbb{P}^2$, or the other three (-1)-curves. These two choices are exchanged by the Cremona involution of $\mathrm{Bl}_3 \mathbb{P}^2$ (see Observation 8.9).

Proof. (i) and (ii) are discussed in [Oud10, Section 3]. The divisor $\sum_{i=1}^3 (\ell_i + \ell_i')$ cannot have triple intersection points otherwise S would be singular (see [Par91, Proposition 3.1]).

For the inverse construction, we use the general theory of abelian covers developed in [Par91]. Let $\mathbb{Z}_2^2 = \{e, a, b, c\}$ where e is the identity element, and let $\{\chi_0, \chi_1, \chi_2, \chi_3\}$ be the characters of $\mathbb{Z}_2^2$ with $\chi_0 = 1$ and $\chi_1(b) = \chi_1(c) = \chi_2(a) = \chi_2(c) = \chi_3(a) = \chi_3(b) = -1$. Define $D_a = \ell_1 + \ell_1', D_b = \ell_2 + \ell_2', D_c = \ell_3 + \ell_3'$. Consider the building data (see [Par91, Definition 2.1]) consisting of the divisors D_a, D_b, D_c and the line bundles $L_{\chi_1}, L_{\chi_2}, L_{\chi_3}$ satisfying

$$2L_{\chi_1} = D_b + D_c, \ 2L_{\chi_2} = D_a + D_c, \ 2L_{\chi_3} = D_a + D_b.$$

This building data determines a $\mathbb{Z}_2^2$-cover $\pi \colon S \to \mathrm{Bl}_2 \mathbb{P}^2$ branched along $\sum_{i=1}^3 (\ell_i + \ell_i')$ which is unique up to isomorphism of $\mathbb{Z}_2^2$-covers (see [Par91, Theorem 2.1]). By [Par91, Proposition 3.1] we

have that S is smooth, and using [Par91, Proposition 4.2, formula (4.8)] one can easily compute that $\chi(\mathcal{O}_S) = 1$, which tells us that $h^0(S, \omega_S) = h^1(S, \mathcal{O}_S)$. If R denotes the ramification divisor of the cover, then $K_S \sim \pi^*(K_{\mathrm{Bl}_3 \mathbb{P}^2}) + R$ and $2R \sim \pi^*(\sum_{i=1}^3 (\ell_i + \ell'_i))$ imply that $2K_S \sim 0$, hence S is an Enriques surface. $\qquad\square$

Observation 8.9. In the statement of Proposition 8.8, the ambiguity for the choice of $D_{1,6}$-polarization on the $\mathbb{Z}_2^2$-cover of $\mathrm{Bl}_3 \mathbb{P}^2$ branched along $\sum_{i=1}^3 (\ell_i + \ell'_i)$ is not an issue for our purposes. This is because the Cremona involution on $\mathrm{Bl}_3 \mathbb{P}^2$ does not change the isomorphism class of the pair $(\mathrm{Bl}_3 \mathbb{P}^2, \sum_{i=1}^3 (\ell_i + \ell'_i))$, even though it changes the line configuration.

Lemma 8.10. *Let S be a $D_{1,6}$-polarized Enriques surface and let $\pi\colon S \to \mathrm{Bl}_3 \mathbb{P}^2$ be the corresponding $\mathbb{Z}_2^2$-cover ramified at $E = \sum_{i=1}^3 (E_i + E'_i)$ and branched along $L = \sum_{i=1}^3 (\ell_i + \ell'_i)$ (see Proposition 8.8). Then*

$$K_S + \epsilon E \sim_{\mathbb{Q}} \pi^* \left(K_{\mathrm{Bl}_3 \mathbb{P}^2} + \left(\frac{1+\epsilon}{2} \right) L \right).$$

In particular, $(S, \epsilon E)$ is stable if and only if $\left(\mathrm{Bl}_3 \mathbb{P}^2, \left(\frac{1+\epsilon}{2} \right) L \right)$ is stable.

Proof. We have that $K_S \sim \pi^*(K_{\mathrm{Bl}_3 \mathbb{P}^2}) + E$ and $2E \sim \pi^*(L)$. This implies that $K_S \sim_{\mathbb{Q}} \pi^*(K_{\mathrm{Bl}_3(\mathbb{P}^2)}) + \frac{1}{2}\pi^*(L)$, and by adding $\epsilon E \sim_{\mathbb{Q}} \frac{\epsilon}{2}\pi^*(L)$ to both sides we obtain what claimed. For the last statement about stability, we have that $(S, \epsilon E)$ is semi-log canonical if and only if $\left(\mathrm{Bl}_3 \mathbb{P}^2, \left(\frac{1+\epsilon}{2} \right) L \right)$ is semi-log canonical by [AP12, Lemma 2.3], and $K_S + \epsilon E$ is ample if and only if $K_{\mathrm{Bl}_3 \mathbb{P}^2} + \left(\frac{1+\epsilon}{2} \right) L$ is ample by [Laz04, Proposition 1.2.13 and Corollary 1.2.28]. $\qquad\square$

Remark 8.11. With the notation introduced in Lemma 8.10, we claim that studying the degenerations of the stable pairs $(S, \epsilon E)$ is equivalent to the study of the degenerations of the stable pairs $\left(\mathrm{Bl}_3 \mathbb{P}^2, \left(\frac{1+\epsilon}{2} \right) L \right)$. To prove this, let K be the field of fractions of a DVR $(A, \mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of A. Let $(\mathfrak{S}^\circ, \mathcal{E}^\circ)$ (resp. $(\mathfrak{B}^\circ, \mathcal{L}^\circ)$) be a family of stable pairs over $\mathrm{Spec}(K)$ with fibers isomorphic to $(S, \epsilon E)$ (resp. $\left(\mathrm{Bl}_3 \mathbb{P}^2, \left(\frac{1+\epsilon}{2} \right) L \right)$). Let $\mathfrak{S}^\circ \to \mathfrak{B}^\circ$ be the appropriate $\mathbb{Z}_2^2$-cover ramified at $\mathcal{E}^\circ$ and branched along $\mathcal{L}^\circ$. Let $(\mathfrak{S}, \mathcal{E})$ be the completion of $(\mathfrak{S}^\circ, \mathcal{E}^\circ)$ over $\mathrm{Spec}(A)$, or a ramified base change of it, having as central fiber a stable pair (see [Ale06, Theorem 2.1]). Denote by $(\mathfrak{S}_\mathfrak{m}, \mathcal{E}_\mathfrak{m})$ the central fiber of $(\mathfrak{S}, \mathcal{E})$. We adopt a similar notation for $(\mathfrak{B}^\circ, \mathcal{L}^\circ)$. What we want to show is that $(\mathfrak{S}_\mathfrak{m}, \mathcal{E}_\mathfrak{m})$ is a $\mathbb{Z}_2^2$-cover of $(\mathfrak{B}_\mathfrak{m}, \mathcal{L}_\mathfrak{m})$. This is automatic if we can show that the $\mathbb{Z}_2^2$-action on $\mathfrak{S}^\circ$ extends to $\mathfrak{S}$ (in this case, the quotient of $\mathfrak{S}$ by $\mathbb{Z}_2^2$ is isomorphic to $\mathfrak{B}$ by the uniqueness of

the completion of $\mathfrak{B}^\circ$ over $\mathrm{Spec}(A)$, and this implies what we want). Fix any $g \in \mathbb{Z}_2^2$ and consider its corresponding action $\alpha_g \colon \mathfrak{S} \dashrightarrow \mathfrak{S}$. Resolving the indeterminacies of α_g, we obtain a morphism $\alpha_g' \colon \mathfrak{S}' \to \mathfrak{S}$ where $\mathfrak{S}'$ is obtained by blowing up $\mathfrak{S}$. Then α_g' corresponds to a morphism $\overline{\alpha}_g$ from the log canonical model of $\mathfrak{S}'$ to the log canonical model of $\mathfrak{S}$, which are both isomorphic to $\mathfrak{S}$. Then $\overline{\alpha}_g \colon \mathfrak{S} \to \mathfrak{S}$ is the desired extension of α_g. Since this extension is unique, there is a unique way to build the $\mathbb{Z}_2^2$-cover $\mathfrak{S}_{\mathfrak{m}} \to \mathfrak{B}_{\mathfrak{m}}$ branched along $\mathcal{L}_{\mathfrak{m}}$, and this way is determined by $\mathfrak{S}^\circ \to \mathfrak{B}^\circ$.

Remark 8.12. As we already commented in Section 1.2, after reducing our problem to considering lines on $\mathrm{Bl}_3 \mathbb{P}^2$, we cannot use the hyperplane arrangements machinery from [Ale08, Ale15, AP09]. The reason is that the divisor $L = \sum_{i=1}^3 (\ell_i + \ell_i')$ does not contain any (-1)-curve, and therefore L is not the pullback from $\mathbb{P}^2$ of a line arrangement.

8.1.3 Reduction to stable toric pairs

Notation 8.13. Let $([X_0 : X_1], [Y_0 : Y_1], [Z_0 : Z_1])$ be coordinates in $(\mathbb{P}^1)^3$. We denote by Δ the toric boundary $V(X_0 X_1 Y_0 Y_1 Z_0 Z_1) \subset (\mathbb{P}^1)^3$ ($V(I)$ denotes the closed subscheme associated to the homogeneous ideal I).

Proposition 8.14. *Consider* $\mathrm{Bl}_3 \mathbb{P}^2$ *together with a divisor* $\sum_{i=1}^3 (\ell_i + \ell_i')$ *(see Notation 8.7) without triple intersection points. Then there exists* $B = V\left(\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k\right) \subset (\mathbb{P}^1)^3$ *with coefficients* $c_{ijk} \neq 0$ *such that* $\left(\mathrm{Bl}_3 \mathbb{P}^2, \sum_{i=1}^3 (\ell_i + \ell_i')\right)$ *is isomorphic to* $(B, \Delta|_B)$. *Moreover, such* $B \subset (\mathbb{P}^1)^3$ *is uniquely determined up to the action of* $\mathbb{G}_m^3 \rtimes \mathrm{Sym}(Q)$, *where* Q *is the unit cube.*

Proof. Consider the three projections $\pi_i \colon \mathrm{Bl}_3 \mathbb{P}^2 \to \mathbb{P}^1$, $i = 1, 2, 3$. Let $\ell_i = \pi_i^{-1}([a_{0i} : a_{1i}])$ and $\ell_i' = \pi_i^{-1}([a_{0i}' : a_{1i}'])$. The morphism $(\pi_1, \pi_2, \pi_3) \colon \mathrm{Bl}_3 \mathbb{P}^2 \to (\mathbb{P}^1)^3$ is an embedding whose image is a divisor of class $(1, 1, 1)$ (observe that the restriction of Δ to this divisor gives the six (-1)-curves, each one with multiplicity 2). For each one of the three copies of $\mathbb{P}^1$ (which we label with $i = 1, 2, 3$) choose an automorphism φ_i sending $[a_{0i} : a_{1i}], [a_{0i}' : a_{1i}']$ to $[1 : 0], [0 : 1]$ respectively. Let B be the image of the composition of the embedding (π_1, π_2, π_3) followed by the automorphism $(\varphi_1, \varphi_2, \varphi_3)$. Then, under this morphism, $\mathrm{Bl}_3 \mathbb{P}^2 \cong B$ and $\sum_{i=1}^3 (\ell_i + \ell_i')$ corresponds to $\Delta|_B$. Moreover, $B = V\left(\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k\right)$ where the coefficients c_{ijk} are nonzero (otherwise $\Delta|_B$ would have triple intersection points).

In the construction of B above we made some choices. There is a $\mathrm{Sym}(Q)$-action which comes from the fact that we can permute the three projections π_1, π_2, π_3 and for each i we can exchange $[a_{0i} : a_{1i}]$ with $[a'_{0i} : a'_{1i}]$ ($\mathrm{Sym}(Q)$ is isomorphic to the wreath product $\mathbb{Z}_2 \wr S_3$). There is also a $\mathbb{G}_m^3$-action due to the fact that each φ_i is uniquely determined up to $\mathbb{G}_m$. It is easy to observe that our construction of B is invariant under the action of $\mathrm{Aut}(\mathrm{Bl}_3 \mathbb{P}^2)$ on the line arrangement (see [Dol12, Theorem 8.4.2] for the description of $\mathrm{Aut}(\mathrm{Bl}_3 \mathbb{P}^2)$).

To conclude we need to show that any realization $B = V\left(\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k\right) \subset (\mathbb{P}^1)^3$ with coefficients $c_{ijk} \neq 0$ such that $\left(\mathrm{Bl}_3 \mathbb{P}^2, \sum_{i=1}^3 (\ell_i + \ell'_i)\right) \cong (B, \Delta|_B)$ can be obtained as we described up to $(\mathbb{G}_m^3 \rtimes \mathrm{Sym}(Q))$-action. But this is true because, up to $\mathbb{G}_m^3 \rtimes \mathrm{Sym}(Q)$, there is a unique way to realize $\mathrm{Bl}_3 \mathbb{P}^2$ in $(\mathbb{P}^1)^3$ so that the six (-1)-curves are given by the restriction of Δ, and this is given by $V(X_0 Y_0 Z_0 - X_1 Y_1 Z_1)$ (we omit the proof of this). $\qquad \square$

Summing up, starting from our pairs of interest $(S, \epsilon \cdot \sum_{i=1}^3 (E_i + E'_i))$ defined in Observation 8.4, the considerations we made so far (Proposition 8.8, Lemma 8.10, Remark 8.11, and Proposition 8.14) allow us to consider moduli of the pairs $\left(B, \left(\frac{1+\epsilon}{2}\right) \Delta|_B\right)$ with $B \subset (\mathbb{P}^1)^3$ as prescribed by Proposition 8.14 (observe that $\left(B, \left(\frac{1+\epsilon}{2}\right) \Delta|_B\right)$ is obviously stable). This approach in terms of $B \subset (\mathbb{P}^1)^3$ is convenient because $((\mathbb{P}^1)^3, B)$ is a stable toric pair of type $\leq Q$ where Q is the unit cube, and Theorem 6.9 gives an explicit description of the moduli space $\overline{M}_Q$ parametrizing $((\mathbb{P}^1)^3, B)$ and its degenerations.

Remark 8.15. Let $B \subset (\mathbb{P}^1)^3$ as in Proposition 8.14. An alternative construction of the $\mathbb{Z}_2^2$-cover of B branched along $\Delta|_B$ which gives a $D_{1,6}$-polarized Enriques surface is the following. Let $\widetilde{S} = V\left(\sum_{i,j,k=0,1} c_{ijk} X_i^2 Y_j^2 Z_k^2\right) \subset (\mathbb{P}^1)^3$. The restriction to $\widetilde{S}$ of the map $(\mathbb{P}^1)^3 \to (\mathbb{P}^1)^3$ given in an affine patch by $(x, y, z) \mapsto (x^2, y^2, z^2)$ is a $\mathbb{Z}_2^3$-cover of B branched along $\Delta|_B$. To show that $\widetilde{S}$ is smooth, we first observe that $\widetilde{S}$ is smooth in the complement of the ramification locus because B is smooth. If $\widetilde{S}$ is singular at a point in the ramification locus, then one can explicitly compute using the equation of $\widetilde{S}$ that one of the irreducible components of Δ restricts to B giving a reducible curve, which is against our assumptions. Therefore $\widetilde{S}$ is smooth, implying that $\widetilde{S}$ is a K3 surface (see Example 4.4). If ι is the involution $(\mathbb{P}^1)^3 \to (\mathbb{P}^1)^3$ given in an affine patch by $(x, y, z) \mapsto (-x, -y, -z)$, then $S = \widetilde{S}/\langle \iota \rangle$ is a $D_{1,6}$-polarized Enriques surface.

8.1.4 The moduli space $\overline{M}_Q$

Let $B \subset (\mathbb{P}^1)^3$ as in Proposition 8.14. Denote by Q the unit cube. Then, according to Definition 6.4, $((\mathbb{P}^1)^3, B)$ is a stable toric pair of type $\leq Q$. Let $\overline{M}_Q$ be the projective coarse moduli space parametrizing these pairs and their degenerations (see Theorem 6.9). In the next proposition we prove some first properties of $\overline{M}_Q$, and we can see how the geometry of $\overline{M}_Q$ interacts with the combinatorics of the polyhedral subdivisions of $(Q, Q \cap \mathbb{Z}^3)$.

Notation 8.16. Let X be a scheme. We denote by X^ν the normalization of X with the reduced scheme structure.

Proposition 8.17. *Let $Q \subset \mathbb{Z}^3$ be the unit cube. Then the coarse moduli space $\overline{M}_Q$ is irreducible and has dimension 4.*

Proof. The irreducibility of $\overline{M}_Q$ follows from the fact that, if $\mathcal{P}$ is any polyhedral subdivisions of $(Q, Q \cap \mathbb{Z}^3)$, then $\mathcal{P}$ is regular and $H^1(\mathcal{P}, \underline{T}) = \{1\}$ (see Proposition 5.15 and Section 6.3 respectively).

For the dimension count, we know that $\overline{M}_Q^\nu$ is the projective toric variety associated to the secondary polytope $\Sigma(Q \cap \mathbb{Z}^3)$ (see [Pfe00] for a complete description of this polytope), which has dimension $\#(Q \cap \mathbb{Z}^3) - \dim(Q) - 1 = 4$ (see Theorem 5.12). In conclusion, $\dim(\overline{M}_Q) = 4$. $\qquad\square$

8.1.5 The moduli space $\overline{M}_{D_{1,6}}$

With reference to Section 7.4.2, consider the moduli stack $\overline{\mathcal{M}}_C = \overline{\mathcal{M}}_{d,N,C,\underline{b}}$ with $d = 2$, $\underline{b} = (b_1, b_2, b_3) = \left(\frac{1+\epsilon}{2}, \frac{1+\epsilon}{2}, \frac{1+\epsilon}{2}\right)$ (because we want three pairs of divisors, and we do not distinguish divisors in the same pair) where $0 < \epsilon \ll 1$ is a fixed rational number and $C = 6\epsilon^2$ (given $\left(B, \left(\frac{1+\epsilon}{2}\right)\Delta|_B\right)$ as in Proposition 8.14, it is easy to compute that $(K_B + \left(\frac{1+\epsilon}{2}\right)\Delta|_B)^2 = 6\epsilon^2$). For a suitably chosen positive integer N depending on $d, \underline{b}$, and C (which does not need to be specified, see [Ale96b, 3.13]), the Viehweg's moduli functor $\overline{\mathcal{M}}_{6\epsilon^2}$ above is coarsely represented by a projective scheme, which we denote by $\overline{M}_{6\epsilon^2}$. This is true because we are working with surface pairs $(d = 2)$ and our coefficients are strictly greater than $\frac{1}{2}$ (see [Ale15, Theorem 1.6.1, case 2]).

Observation 8.18. The group S_3 has a natural action on the Viehweg moduli stack $\overline{\mathcal{M}}_{6\epsilon^2}$ given by permuting the labels of the three divisors $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3$. In particular, we have an induced S_3-action on the coarse moduli space $\overline{M}_{6\epsilon^2}$.

Now we want to identify inside $\overline{M}_{6\epsilon^2}/S_3$ the Zariski closed subscheme containing a dense open subset whose points parametrize the stable pairs given by $\mathrm{Bl}_3\,\mathbb{P}^2$ together with our three undistinguished pairs of lines of weight $\frac{1+\epsilon}{2}$. The next lemma allows us to do so.

Lemma 8.19. *The coarse moduli space $\overline{M}_Q$ contains a dense open subset $\mathcal{U}$ which is a fine moduli space parametrizing all the stable toric pairs (X, B) with $X \cong (\mathbb{P}^1)^3$. $\mathcal{U}$ is isomorphic to the dense open subtorus of the toric variety $\overline{M}_Q^\nu$. The universal family on $\mathcal{U}$ is of the form $((\mathbb{P}^1)^3 \times \mathcal{U}, \mathfrak{S})$.*

Proof. Let $\mathcal{P}$ be the complex of polytopes given by Q and its faces. Let C be the set of vertices of Q. Following [Ale02, Definition 2.6.6], let $\mathrm{MP}^{\mathrm{fr}}[\mathcal{P}, C](\Bbbk)$ be the groupoid of stable toric pairs $((\mathbb{P}^1)^3, B)$ over $\Bbbk$ with the linearized line bundle $\mathcal{O}_{(\mathbb{P}^1)^3}(B)$ in which the arrows are the isomorphisms identical on the torus T. By [Ale02, Lemma 2.6.7], we have that $\mathrm{MP}^{\mathrm{fr}}[\mathcal{P}, C](\Bbbk)$ is equivalent to the quotient stack $[\mathbb{G}_m^8/\mathbb{G}_m^4]$, where $\mathbb{G}_m^8$ represents the space of coefficients of the divisor $B = V\left(\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k\right)$ on $(\mathbb{P}^1)^3$ and $(\lambda, \mu_1, \mu_2, \mu_3) \in \mathbb{G}_m^4$ acts on $\mathbb{G}_m^8$ as follows:

$$(\lambda, \mu_1, \mu_2, \mu_3) \cdot (\ldots, c_{ijk}, \ldots) = (\ldots, \lambda\mu_1^i \mu_2^j \mu_3^k c_{ijk}, \ldots).$$

Observe that the quotient $\mathbb{G}_m^8/\mathbb{G}_m^4$ exists as a scheme and it is isomorphic to $\mathbb{G}_m^4$. The stabilizers of the action $\mathbb{G}_m^4 \curvearrowright \mathbb{G}_m^8$ are trivial. It follows that the quotient stack $[\mathbb{G}_m^8/\mathbb{G}_m^4]$ is finely represented by $\mathbb{G}_m^8/\mathbb{G}_m^4$. This gives us a dense open subset $\mathcal{U}$ of $\overline{M}_Q$ isomorphic to $\mathbb{G}_m^4$ with a universal family which can be realized as the quotient of the pair $\left((\mathbb{P}^1)^3 \times \mathbb{G}_m^8, V\left(\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k\right)\right)$ under the $\mathbb{G}_m^4$-action on the coefficients c_{ijk}. $\qquad\square$

Definition 8.20. Let $((\mathbb{P}^1)^3 \times \mathcal{U}, \mathfrak{S})$ be the universal family over $\mathcal{U}$ constructed in Lemma 8.19. Denote by $\mathcal{U}_{\mathrm{sm}}$ the open subset of $\mathcal{U}$ parametrizing stable toric pairs $((\mathbb{P}^1)^3, B)$ such that $(B, \Delta|_B)$ is as in Proposition 8.14. Let $\mathfrak{S}_{\mathrm{sm}}$ be the pullback of $\mathfrak{S}/\mathcal{U}$ along the inclusion $\mathcal{U}_{\mathrm{sm}} \hookrightarrow \mathcal{U}$. Then $\left(\mathfrak{S}_{\mathrm{sm}}, \left(\frac{1+\epsilon}{2}\right)(\Delta \times \mathcal{U}_{\mathrm{sm}})|_{\mathfrak{S}_{\mathrm{sm}}}\right)/\mathcal{U}_{\mathrm{sm}}$ is a family for the moduli functor $\overline{\mathcal{M}}_{6\epsilon^2}$, and hence we have an induced morphism $f: \mathcal{U}_{\mathrm{sm}} \to \overline{M}_{6\epsilon^2}/S_3$. We define $\overline{M}_{D_{1,6}}$ to be the closure of the image of f. In particular, by construction we have a dominant rational map $\overline{M}_Q^\nu \dashrightarrow \overline{M}_{D_{1,6}}$.

Observation 8.21. Let p be a point in $\mathcal{U}_{\mathrm{sm}}$ and let $((\mathbb{P}^1)^3, B)$ be the corresponding stable toric pair. Then the fiber of $f: \mathcal{U}_{\mathrm{sm}} \to \overline{M}_{D_{1,6}}$ over $f(p)$ is in bijection with the $\mathrm{Sym}(Q)$-orbit of $((\mathbb{P}^1)^3, B)$ (this follows from Proposition 8.14). In particular, f is a quasi-finite map. This, together with the

fact that f is dominant, implies that $\dim(\overline{M}_{D_{1,6}}) = \dim(\mathcal{U}_{\mathrm{sm}}) = \dim(\overline{M}_Q) = 4$. $\overline{M}_{D_{1,6}}$ is irreducible because $\mathcal{U}_{\mathrm{sm}}$ is irreducible. In conclusion, $\overline{M}_{D_{1,6}}$ is a 4-dimensional irreducible projective coarse moduli space whose points in a dense open subset parametrize isomorphism classes of $\mathrm{Bl}_3\,\mathbb{P}^2$ together with the three undistinguished pairs of undistinguished weighted lines (and hence Enriques surfaces with our choice of divisor after an appropriate $\mathbb{Z}_2^2$-cover).

8.2 Stable replacement in a one-parameter family

Let Q be the unit cube and let $(A, \mathfrak{m})$ be a DVR. Let $(\mathfrak{X}, \mathfrak{B})$ be a family of stable toric pairs of type $\leq Q$ over $\mathrm{Spec}(A)$. Define $X = \mathfrak{X}_{\mathfrak{m}}$, $B = \mathfrak{B}_{\mathfrak{m}}$, and assume that the fiber of $\mathfrak{X}$ over the generic point is isomorphic to $(\mathbb{P}^1)_K^3$ where K is the field of fractions of A. Observe that the pair $\left(B, \left(\frac{1+\epsilon}{2}\right)\Delta|_B\right)$ may be not stable: to start with, the restriction $\Delta|_B$ makes no sense if Δ is not $\mathbb{Q}$-Cartier (see Proposition 8.28). In this section we define a new family $(\mathfrak{X}^\bullet, \mathfrak{B}^\bullet)$ which is isomorphic to the original one in the complement of the central fiber, but the new central fiber $(X^\bullet, B^\bullet)$ is such that $\left(B^\bullet, \left(\frac{1+\epsilon}{2}\right)\Delta^\bullet|_{B^\bullet}\right)$ is a stable pair ($\Delta^\bullet$ denotes the toric boundary of $X^\bullet$). The stability of the pair $\left(B^\bullet, \left(\frac{1+\epsilon}{2}\right)\Delta^\bullet|_{B^\bullet}\right)$ is proved later in Section 8.3.

8.2.1 Families of stable toric pairs over a DVR

Let $(A, \mathfrak{m})$ be a DVR with field of fractions K, uniformizing parameter t, and residue field our fixed base field $\Bbbk$. Let T be a torus over $\Bbbk$ with character lattice M. Let $Q \subset M_\mathbb{R}$ be a lattice polytope. Define

$$\theta = \sum_{m \in Q \cap M} c_m(t) t^{h(m)} x^{(1,m)},$$

where, for all $m \in Q \cap M$, $c_m(t) \in A$, $c_m(0) \in \Bbbk^*$, and $h(m) \in \mathbb{Z}$. Observe that the map $m \mapsto h(m)$ gives a height function $h \colon Q \cap M \to \mathbb{Z}$. Let $Q^+ \subset M_\mathbb{R} \oplus \mathbb{R}$ be the convex hull of the half-lines $(m, h(m) + \mathbb{R}_{\geq 0})$, $m \in Q \cap M$, and let $\mathrm{Cone}(Q^+) \subset \mathbb{R} \oplus M_\mathbb{R} \oplus \mathbb{R}$ be the cone over $(1, Q^+)$ with vertex at the origin. Then h defines the following $(\mathbb{Z} \oplus M)$-graded A-algebra:

$$R = A[t^\ell x^{(d,m)} \mid (d, m, \ell) \in \mathrm{Cone}(Q^+) \cap (\mathbb{Z} \oplus M \oplus \mathbb{Z})].$$

Observe that $\theta \in R$ is an element of $\mathbb{Z}$-degree 1. Let $(\mathfrak{X}, \mathfrak{B})/\operatorname{Spec}(A)$ be the family of stable toric pairs associated to (R, θ) (see Remark 6.7). If η is the generic point of $\operatorname{Spec}(A)$, then $\mathfrak{X}_\eta = Y \times \operatorname{Spec}(K)$, where Y is the toric variety associated to the polytope Q. The central fiber $(\mathfrak{X}_{\mathfrak{m}}, \mathfrak{B}_{\mathfrak{m}})$ is a stable toric pair whose corresponding polyhedral subdivision of $(Q, Q \cap M)$ is induced by the height function h (hence, it is a regular subdivision). The equation of $\mathfrak{B}_{\mathfrak{m}}$ is given by

$$\sum_{m \in Q \cap M} c_m(0) x^{(1,m)} = 0.$$

For more details about this construction we refer to [Ale02, Section 2.8].

8.2.2 Corner cuts

Definition 8.22. Let Q be the unit cube. We call *corner cut* a subpolytope of Q which is equal to the convex hull of the points $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)$ up to a symmetry of Q. An example of corner cut can be found in Figure 8.1 on the left. We call *apex* the vertex of a corner cut which is at the intersection of three edges of the cube.

Notation 8.23. Let $\mathcal{P}$ be a polyhedral subdivision of a lattice polytope Q. We denote by $\mathcal{P}_i$ the set of i-dimensional faces in $\mathcal{P}$.

Definition 8.24. Let $\mathcal{P}$ be a polyhedral subdivision of $(Q, Q \cap \mathbb{Z}^3)$. We define a polyhedral subdivision $\mathcal{P}^\bullet$ of $(Q, Q \cap \mathbb{Z}^3)$ via the following algorithm:

(1) $\mathcal{R} = \mathcal{P}$;

(2) If $\mathcal{R}$ contains no corner cut, define $\mathcal{P}^\bullet = \mathcal{R}$ and stop. Otherwise, go to step (3);

(3) Let $P \in \mathcal{R}$ be a corner cut and let $R \in \mathcal{R}$ be that unique polytope sharing exactly a facet with P. Define
$$\mathcal{S} = (\mathcal{R}_3 \setminus \{P, R\}) \cup \{P \cup R\}.$$

Then redefine $\mathcal{R}$ to be the polyhedral subdivision of Q generated by $\mathcal{S}$. Go to step (2).

In Figure 8.1 we give an explicit example of $\mathcal{P}^\bullet$ given $\mathcal{P}$.
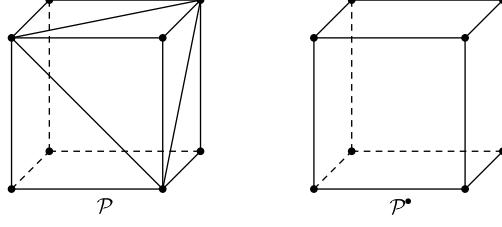
Figure 8.1: Modification $\mathcal{P}^\bullet$ of the polyhedral subdivision $\mathcal{P}$

8.2.3 The modified family $(\mathfrak{X}^\bullet, \mathfrak{B}^\bullet)$

Let us adopt the same notation introduced in Section 8.2.1, but set Q to be the unit cube. We define another $(\mathbb{Z} \oplus M)$-graded A-algebra induced by θ as follows. Denote by $\mathcal{P}$ the regular polyhedral subdivision of $(Q, Q \cap \mathbb{Z}^3)$ associated to $(\mathfrak{X}_\mathfrak{m}, \mathfrak{B}_\mathfrak{m})$. Assume $P \in \mathcal{P}_3$ is a corner cut and let $P' \in \mathcal{P}$ be that unique polytope sharing exactly a facet with P. Denote by L the unique hyperplane in $\mathbb{R}^3 \oplus \mathbb{R}$ containing the points $(m, h(m))$ for $m \in P' \cap \mathbb{Z}^3$. If m is the apex of the corner cut P, then there exists a unique positive rational number q_m such that $(m, h(m) - q_m) \in L$. Moreover, up to a ramified finite base change, we can assume that q_m is integral. Let us consider the height function

$$
h^\bullet(m) = \begin{cases} h(m) - q_m & \text{if } m \text{ is the apex of a corner cut,} \\ h(m) & \text{otherwise.} \end{cases}
$$

Define a new $(\mathbb{Z} \oplus M)$-graded A-algebra $R^\bullet$ as we did in Section 8.2.1, but using $h^\bullet$ in place of h. Observe that $R \subset R^\bullet$ is a degree preserving embedding of graded algebras. Therefore $\theta \in R^\bullet$ is an element of $\mathbb{Z}$-degree 1 and the pair $(R^\bullet, \theta)$ corresponds to a family $\mathfrak{X}^\bullet / \operatorname{Spec}(A)$ of stable toric varieties together with a Cartier divisor $\mathfrak{B}^\bullet \subset \mathfrak{X}^\bullet$ given by the vanishing of θ. Observe that $(\mathfrak{X}_\eta, \mathfrak{B}_\eta)$ and $((\mathfrak{X}^\bullet)_\eta, (\mathfrak{B}^\bullet)_\eta)$ are isomorphic over $\operatorname{Spec}(K)$ by construction. The central fiber $((\mathfrak{X}^\bullet)_\mathfrak{m}, (\mathfrak{B}^\bullet)_\mathfrak{m})$ is not a stable toric pair if and only if $\mathcal{P}$ contains corner cuts. However, $((\mathfrak{X}^\bullet)_\mathfrak{m}, \mathcal{O}((\mathfrak{B}^\bullet)_\mathfrak{m}))$ is always a polarized stable toric variety whose corresponding polyhedral subdivision of $(Q, Q \cap \mathbb{Z}^3)$ is $\mathcal{P}^\bullet$

(see Definition 8.24). The equation of $(\mathfrak{B}^\bullet)_\mathfrak{m}$ is given by

$$\sum_{\substack{m \in Q \cap M, \\ m \text{ is not an apex}}} c_m(0) x^{(1,m)} = 0.$$

Observe that if $\mathcal{P}$ has no corner cuts, then $(\mathfrak{X}, \mathfrak{B}) = (\mathfrak{X}^\bullet, \mathfrak{B}^\bullet)$.

Remark 8.25. With the same notation introduced above, denote by (X, B) (resp. $(X^\bullet, B^\bullet)$) the central fiber of $(\mathfrak{X}, \mathfrak{B})$ (resp. $(\mathfrak{X}^\bullet, \mathfrak{B}^\bullet)$). Then $(X^\bullet, B^\bullet)$ only depends on (X, B) and it is independent from the whole family $(\mathfrak{X}, \mathfrak{B})/\operatorname{Spec}(A)$.

Definition 8.26. Let (X, B) be a stable toric pair of type $\leq Q$. Define $(X^\bullet, B^\bullet)$ to be the central fiber of $(\mathfrak{X}^\bullet, \mathfrak{B}^\bullet)$, where $(\mathfrak{X}, \mathfrak{B})/\operatorname{Spec}(A)$ is any one-parameter family with central fiber (X, B) and smooth generic fiber (such a family exists because $\overline{M}_Q$ is irreducible, hence (X, B) is smoothable). The pair $(X^\bullet, B^\bullet)$ is well defined by Remark 8.25. Observe that, if $\mathcal{P}$ has no corner cuts, then $(X, B) = (X^\bullet, B^\bullet)$.

8.3 Analysis of stability

8.3.1 Preliminaries

Throughout this section let Q be the unit cube. We show that if (X, B) is a stable toric pair of type $\leq Q$, then $(B^\bullet, \left(\frac{1+\epsilon}{2}\right) \Delta^\bullet|_{B_P^\bullet})$ is stable. Before formally stating this result, we need some preliminaries.

Notation 8.27. Consider a stable toric pair (X, B) of type $\leq Q$ and let $\mathcal{P}$ be the corresponding polyhedral subdivision of $(Q, Q \cap \mathbb{Z}^3)$. Let $\amalg_{P \in \mathcal{P}_3} X_P \to X$ be the normalization of X, where X_P is the toric variety corresponding to the polytope P. Then we denote by Δ_P the toric boundary of X_P, by $D_P \subset X_P$ the conductor divisor, and by B_P the restriction to X_P of the preimage of B under the normalization morphism. Define $X_P^\bullet, \Delta_P^\bullet, D_P^\bullet, B_P^\bullet$ analogously with $(X^\bullet, B^\bullet)$ in place of (X, B).

Proposition 8.28. *Let (X, B) be a stable toric pair of type $\leq Q$ and let $\mathcal{P}$ be the associated polyhedral subdivision of $(Q, Q \cap \mathbb{Z}^3)$. If $\mathcal{P}$ does not contain a corner cut, then Δ is Cartier. If $\mathcal{P}$ contains a corner cut, then Δ is not $\mathbb{Q}$-Cartier.*

Proof. Define a piecewise linear function on the normal fan of each maximal dimensional polytope $P \in \mathcal{P}$ as follows: associate to the $\mathbb{Z}$-generators of the rays, negative the lattice distance between the facet of $2P$ normal to the ray and the lattice point $(1,1,1)$. Observe that this number is 0 if the facet contains $(1,1,1)$, and -1 otherwise. If $\mathcal{P}$ has no corner cuts, then this gives a Cartier divisor on X equal to Δ.

Now assume $\mathcal{P}$ contains a corner cut P. Denote with R that unique polytope in $\mathcal{P}$ sharing exactly a facet with P. Let ℓ_1, ℓ_2, ℓ_3 be the three edges of P which do not contain the apex. Observe that ℓ_i can be contained in two or three maximal dimensional polytopes in $\mathcal{P}$ (P and R included), $i = 1, 2, 3$.

If some ℓ_i is contained in three maximal dimensional polytopes, then take a point $x \in X$ lying on the torus invariant line corresponding to ℓ_i. If Δ is $\mathbb{Q}$-Cartier, then $m\Delta$ is given by the vanishing of one equation in an open neighborhood of x for some $m > 0$. However, the vanishing locus of this equation on X_R has codimension 2, which cannot be.

Assume that each ℓ_i is only contained in P and R. Denote by $\nu \colon X^\nu \to X$ the normalization. If Δ is $\mathbb{Q}$-Cartier, then $(\nu^*\Delta)|_{X_R} = \Delta_R - D_R$ is also $\mathbb{Q}$-Cartier. But this is a contradiction because there is no $\mathbb{Q}$-piecewise linear function on the normal fan of R corresponding to $\Delta_R - D_R$ (consider the normal cone to a vertex of R in common with P). $\square$

Theorem 8.29. *Let Q be the unit cube and let (X, B) be a stable toric pair of type $\leq Q$. Consider $(X^\bullet, B^\bullet)$ as in Definition 8.26. Then $\left(B^\bullet, \left(\frac{1+\epsilon}{2}\right)\Delta^\bullet|_{B^\bullet}\right)$ is a stable pair (it makes sense to consider $\Delta^\bullet|_{B^\bullet_P}$ by Proposition 8.28).*

8.3.2 Proof of Theorem 8.29

Let $\mathcal{P}$ be the polyhedral subdivision of $(Q, Q \cap \mathbb{Z}^3)$ associated to (X, B). We show that for all $P \in \mathcal{P}^\bullet$, the pair $\left(B^\bullet_P, D^\bullet_P|_{B^\bullet_P} + \left(\frac{1+\epsilon}{2}\right)(\Delta^\bullet_P - D^\bullet_P)|_{B^\bullet_P}\right)$ is stable. It is easy to observe that there are four possibilities for P up to symmetries of Q as shown in Figure 8.2.

Definition 8.30. We say that $P \in \mathcal{P}^\bullet$ has type (a) (resp. $(b), (c), (d)$) if P is equal to the polytope in Figure 8.2 (a) (resp. $(b), (c), (d)$) up to a symmetry of Q.

Proposition 8.31. *Given P of type (a), then the pair $\left(B^\bullet_P, D^\bullet_P|_{B^\bullet_P} + \left(\frac{1+\epsilon}{2}\right)(\Delta^\bullet_P - D^\bullet_P)|_{B^\bullet_P}\right)$ is a stable pair.*
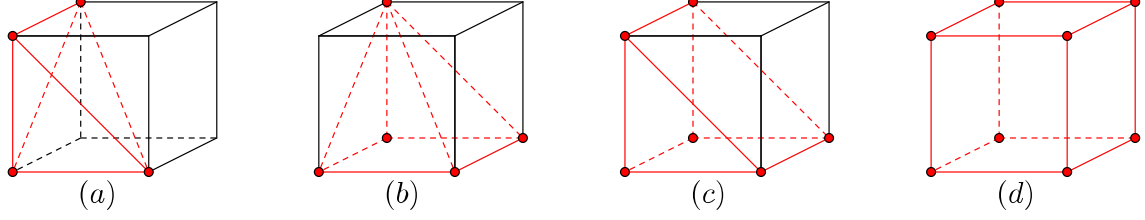
Figure 8.2: Possible maximal dimensional polytopes in $\mathcal{P}^\bullet$ up to symmetries of Q

Proof.

$$X_P^\bullet = \mathbb{P}^3 \ni [W_0 : \ldots : W_3],$$

$$D_P^\bullet = V(W_0) + V(W_1),$$

$$\Delta_P^\bullet - D_P^\bullet = V(W_2) + V(W_3),$$

$$B_P^\bullet = V(a_0 W_0 + \ldots + a_3 W_3),$$

where $a_i \neq 0$ for all $i = 0, \ldots, 3$. To find the equation of $B_P^\bullet$ we used Remark 6.5, and $a_i \neq 0$ for all i because there is no corner cut that can possibly be contained in P (see the construction of $(X^\bullet, B^\bullet)$ in Section 8.2.3). Then $B_P^\bullet$ is isomorphic to $\mathbb{P}^2$ and $\Delta_P^\bullet$ restricts to $B_P^\bullet$ giving four lines in general linear position, implying that $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$ is log canonical. Finally, if L denotes a general line in $B_P^\bullet$, then

$$K_{B_P^\bullet} + D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet} \sim \epsilon L,$$

which is ample. □

The next lemma is used in the analysis of the cases P of type $(b), (c), (d)$.

Lemma 8.32. *Let L_1, L_2, L_3 be three distinct lines in $\mathbb{A}^2$ through the point $(0,0)$. Then the pair $\left(\mathbb{A}^2, \left(\frac{1+\epsilon}{2}\right)(L_1 + L_2 + L_3)\right)$ is log canonical.*

Proof. Consider the blow up morphism $\sigma\colon \mathrm{Bl}_{(0,0)}\mathbb{A}^2 \to \mathbb{A}^2$ and let E be the exceptional divisor. Then we have that

$$K_{\mathrm{Bl}_{(0,0)}\mathbb{A}^2} + \sigma_*^{-1}\left(\left(\frac{1+\epsilon}{2}\right)\sum_{i=1}^{3} L_i\right) = \sigma^*\left(K_{\mathbb{A}^2} + \left(\frac{1+\epsilon}{2}\right)\sum_{i=1}^{3} L_i\right) + aE$$

$$\implies E = 3\left(\frac{1+\epsilon}{2}\right)E + aE$$

$$\implies a = -\frac{1+3\epsilon}{2} > -1.$$

Therefore, $\left(\mathbb{A}^2, \left(\frac{1+\epsilon}{2}\right)(L_1 + L_2 + L_3)\right)$ is log canonical because σ is a log resolution (see Proposition 7.7). $\qquad\square$

Proposition 8.33. *Given P of type (b), then the pair $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$ is a stable pair.*

Proof.

$$X_P^\bullet = V(W_0W_1 - W_2W_3) \subset \mathbb{P}^4 \ni [W_0 : \ldots : W_4],$$

$$D_P^\bullet = V(W_0, W_2) + V(W_0, W_3) = V(W_0W_1 - W_2W_3, W_0),$$

$$\Delta_P^\bullet - D_P^\bullet = V(W_1, W_2) + V(W_1, W_3) + V(W_0W_1 - W_2W_3, W_4)$$

$$= V(W_0W_1 - W_2W_3, W_1) + V(W_0W_1 - W_2W_3, W_4),$$

$$B_P^\bullet = V(a_0W_0 + \ldots + a_4W_4) \cap X_P^\bullet,$$

where $a_i \neq 0$ for $i = 1, \ldots, 4$ and a_0 can possibly vanish (we can have at most one corner cut contained in P). $B_P^\bullet \cong \mathbb{P}^1 \times \mathbb{P}^1$ because it is a hyperplane section of the projective cone $X_P^\bullet$ which does not pass through the vertex.

Now let us study the restrictions of $D_P^\bullet$ and $\Delta_P^\bullet - D_P^\bullet$ to $B_P^\bullet$. This boils down to understand how the coordinate hyperplanes $H_i = V(W_i)$, $i = 0, \ldots, 4$, restrict to $B_P^\bullet$. First of all, observe that H_i cuts on $B_P^\bullet$ a curve C of divisor class $(1, 1)$. To show this, denote by (a, b) the divisor class of $C = B_P^\bullet \cap H_i$. If $H' = V(a_0W_0 + \ldots + a_4W_4)$ and H is a general hyperplane in $\mathbb{P}^4$, then the self-intersection of C is given by

$$C^2 = H_i|_{B_P^\bullet} \cdot H_i|_{B_P^\bullet} = H_i \cdot H_i \cdot B_P^\bullet = H_i \cdot H_i \cdot X_P^\bullet \cdot H' = H_i \cdot H_i \cdot 2H \cdot H' = 2.$$

On the other hand, $C^2 = (a,b)^2 = 2ab = 2$, implying that $(a,b) = (1,1)$. For $i \neq 4$, $H_i \cap B_P^\bullet$ is always reducible, so it is given by two curves of divisor classes $(1,0)$ and $(0,1)$. If $a_0 \neq 0$, then it is easy to check that $H_4 \cap B_P^\bullet$ is smooth for a general choice of the coefficients, but it can possibly break into two curves. In any case, $\Delta_P^\bullet$ restricts to $B_P^\bullet$ giving a simple normal crossing divisor, implying that $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2} \right) (\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet} \right)$ is log canonical.

If $a_0 = 0$, then $H_4 \cap B_P^\bullet$ is irreducible, but it passes through the singular point of $H_1 \cap B_P^\bullet$. However, we have that $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2} \right) (\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet} \right)$ is log canonical by Lemma 8.32.

Finally, observe that

$$K_{B_P^\bullet} + D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2} \right) (\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet} \sim (-2,-2) + (1,1) + \left(\frac{1+\epsilon}{2} \right)(2,2) = \epsilon(1,1),$$

which is ample. $\square$

Proposition 8.34. *Given P of type (c), then the pair $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2} \right) (\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet} \right)$ is a stable pair.*

Proof.

$$X_P^\bullet = \mathbb{P}^2 \times \mathbb{P}^1 \ni ([X_0 : X_1 : X_2], [Y_0 : Y_1]),$$

$$D_P^\bullet = V(X_2),$$

$$\Delta_P^\bullet - D_P^\bullet = V(X_0 X_1 Y_0 Y_1),$$

$$B_P^\bullet = V((a_0 X_0 + a_1 X_1 + a_2 X_2)Y_0 + (b_0 X_0 + b_1 X_1 + b_2 X_2)Y_1).$$

where $a_0, a_1, b_0, b_1 \neq 0$ and exactly one among a_2 and b_2 can possibly be zero (there is at most one corner cut contained in P). Let us start by assuming that $a_2 b_2 \neq 0$.

If $B_P^\bullet$ is singular, assume without loss of generality that there is a singular point in the affine patch where $X_0 \neq 0$ and $Y_0 \neq 0$. Let $x_i = \frac{X_i}{X_0}$, $i = 1, 2$, and $y = \frac{Y_1}{Y_0}$ be the coordinates in this affine patch. Then the equation of $B_P^\bullet$ becomes

$$a_0 + a_1 x_1 + a_2 x_2 + (b_0 + b_1 x_1 + b_2 x_2)y = 0.$$

Therefore the following system of equations has a solution:

$$
\begin{cases}
a_0 + a_1 x_1 + a_2 x_2 + (b_0 + b_1 x_1 + b_2 x_2)y = 0, \\
a_1 + b_1 y = 0, \\
a_2 + b_2 y = 0, \\
b_0 + b_1 x_1 + b_2 x_2 = 0,
\end{cases}
\implies
\begin{cases}
y = -\frac{a_1}{b_1}, \\
y = -\frac{a_2}{b_2}, \\
a_1 x_1 + a_2 x_2 = -a_0, \\
b_1 x_1 + b_2 x_2 = -b_0.
\end{cases}
$$

This implies that $a_1 b_2 - a_2 b_1 = 0$, and therefore the two vectors (a_0, a_1, a_2) and (b_0, b_1, b_2) are proportional because the matrix $\left(\begin{smallmatrix} a_1 & a_2 & -a_0 \\ b_1 & b_2 & -b_0 \end{smallmatrix} \right)$ has rank 1. If $(b_0, b_1, b_2) = \lambda(a_0, a_1, a_2)$ for $\lambda \in \mathbb{k}^*$, then the equation of $B_P^\bullet$ becomes

$$
(a_0 X_0 + a_1 X_1 + a_2 X_2)(Y_0 + \lambda Y_1) = 0,
$$

where $V(Y_0 + \lambda Y_1) \cong \mathbb{P}^2$ and $V(a_0 X_0 + a_1 X_1 + a_2 X_2) \cong \mathbb{P}^1 \times \mathbb{P}^1$ are glued along a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ and the line $a_0 X_0 + a_1 X_1 + a_2 X_2 = 0$ in $\mathbb{P}^2$. The restrictions of $D_P^\bullet$ and $\Delta_P^\bullet - D_P^\bullet$ to these two irreducible components are described in Remark 8.37 $(c3)$. In this case, to conclude that $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2} \right) (\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet} \right)$ is stable, we reduce the question to each connected component of the normalization of $B_P^\bullet$ and we apply what we already proved in the $(a), (b)$ cases (see the proofs of Propositions 8.31 and 8.33).

Now let us assume that $B_P^\bullet$ is smooth (and hence irreducible). By the discussion above, the two vectors (a_0, a_1, a_2) and (b_0, b_1, b_2) are not proportional. Denote by p the point of intersection of the two lines $a_0 X_0 + a_1 X_1 + a_2 X_2 = 0$ and $b_0 X_0 + b_1 X_1 + b_2 X_2 = 0$ in $\mathbb{P}^2$. If $\pi \colon B_P^\bullet \to \mathbb{P}^2$ is the restriction to $B_P^\bullet$ of the usual projection map $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$, observe that π is an isomorphism if restricted to the complement of $\pi^{-1}(p)$ and $\pi^{-1}(p) \cong \mathbb{P}^1$. This proves that $B_P^\bullet \cong \mathbb{F}_1$. In this case, let us explain how $D_P^\bullet|_{B_P^\bullet}$ depends on the coefficients a_i, b_j. The restriction $D_P^\bullet|_{B_P^\bullet}$ has equation

$$
\begin{cases}
X_2 = 0, \\
(a_0 X_0 + a_1 X_1)Y_0 + (b_0 X_0 + b_1 X_1)Y_1 = 0.
\end{cases}
$$

By using an argument completely analogous to what we just did for $B_P^\bullet$, we have that this restriction is irreducible if and only if (a_0, a_1) and (b_0, b_1) are not proportional. In this case, $D_P^\bullet|_{B_P^\bullet}$ is a section of $B_P^\bullet$ with self-intersection 1. If (a_0, a_1) and (b_0, b_1) are proportional, then the equation of $D_P^\bullet|_{B_P^\bullet}$

becomes

$$
\begin{cases}
X_2 = 0, \\
(a_0 X_0 + a_1 X_1)(Y_0 + \lambda Y_1) = 0.
\end{cases}
$$

for some $\lambda \in \Bbbk^*$. The irreducible component $V(X_2, a_0 X_0 + a_1 X_1)$ (resp. $V(X_2, Y_0 + \lambda Y_1)$) is the exceptional section (resp. a fiber) of $B_P^\bullet$. We are left with understanding $(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}$, and for this we need to study how $V(X_i), V(Y_i)$, $i = 0, 1$, restrict to $B_P^\bullet$. But $V(Y_i)$ restricts giving a fiber, and we can study $V(X_i)|_{B_P^\bullet}$ in the same way we did for $D_P^\bullet|_{B_P^\bullet}$. Observe that at most one among $V(X_0)|_{B_P^\bullet}, V(X_1)|_{B_P^\bullet}, D_P^\bullet|_{B_P^\bullet}$ can be reducible (otherwise $B_P^\bullet$ would be reducible). We conclude that $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2} \right) (\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet} \right)$ is log canonical.

Now we consider the case where exactly one among a_2 and b_2 is zero. It is easy to check that $B_P^\bullet$ is automatically smooth and a similar description to the one above applies, with the only difference that the restriction $(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}$ can acquire a triple intersection point. But then we can apply Lemma 8.32 to conclude that $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2} \right) (\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet} \right)$ is log canonical.

For the ampleness condition, on $B_P^\bullet \cong \mathbb{F}_1$ denote by h a section of self-intersection 1 and by f a fiber. Then

$$
K_{B_P^\bullet} + D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2} \right) (\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet} \sim -2h - f + h + \left(\frac{1+\epsilon}{2} \right)(2h + 2f) = \epsilon(h + f),
$$

which is ample. $\qquad\square$

To deal with the last case (type (d)), we need an additional preliminary lemma.

Lemma 8.35. *Let H_1, H_2, H_3, and H be four distinct planes in $\mathbb{A}^3$ through the origin in general position. Then $\left(\mathbb{A}^3, \left(\frac{1+\epsilon}{2} \right) (H_1 + H_2 + H_3) + H \right)$ is log canonical.*

Proof. Consider the blow up morphism $f \colon \mathrm{Bl}_{(0,0,0)} \mathbb{A}^3 \to \mathbb{A}^3$ and let E be the exceptional divisor. If $\alpha = \frac{1+\epsilon}{2}$, then we have that

$$
K_{\mathrm{Bl}_{(0,0,0)} \mathbb{A}^3} + f_*^{-1}(\alpha(H_1 + H_2 + H_3) + H) = f^*(K_{\mathbb{A}^3} + \alpha(H_1 + H_2 + H_3) + H) + aE
$$

$$
\implies 2E = (3\alpha + 1)E + aE
$$

$$
\implies a = (-3\alpha + 1) > -1.
$$

This implies that $\left(\mathbb{A}^3, \left(\frac{1+\epsilon}{2}\right)(H_1 + H_2 + H_3) + H\right)$ is log canonical because f is a log resolution of singularities (see Proposition 7.7). $\qquad \square$

Proposition 8.36. *Given P of type (d), then the pair $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$ is a stable pair.*

Proof.

$$X_P^\bullet = (\mathbb{P}^1)^3 \ni ([X_0 : X_1], [Y_0 : Y_1], [Z_0 : Z_1]),$$

$$D_P^\bullet = \emptyset \text{ (there is no conductor divisor in this case)},$$

$$\Delta_P^\bullet = V(X_0 X_1 Y_0 Y_1 Z_0 Z_1),$$

$$B_P^\bullet = V\left(\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k\right),$$

where any two distinct coefficients c_{ijk} and $c_{i'j'k'}$ cannot be simultaneously zero if (i, j, k) and (i', j', k') are vertices of the same edge of the cube (this is because inside Q we cannot fit two corner cuts with apices lying on the same edge).

Let us first assume that $B_P^\bullet$ is smooth (hence irreducible), which happens for a general choice of the coefficients c_{ijk} by Bertini's Theorem (see [Har77, Chapter II, Theorem 8.18]). Then the anticanonical class $-K_{B_P^\bullet} = -(K_{(\mathbb{P}^1)^3} + B_P^\bullet)|_{B_P^\bullet} = (1, 1, 1)|_{B_P^\bullet}$ is ample and $K_{B_P^\bullet}^2 = 6$, implying that $B_P^\bullet \cong \mathrm{Bl}_3 \mathbb{P}^2$ (see [Bea96, Exercise V.21(1)]). If all the c_{ijk} are nonzero, then the restriction $\Delta_P^\bullet|_{B_P^\bullet}$ can be as in Proposition 8.14 (general case), or some of these lines can break into the union of two incident (-1)-curves. If some coefficients c_{ijk} are zero, then the lines configuration $\Delta_P^\bullet|_{B_P^\bullet}$ acquires triple intersection points. In any case, the pair $\left(B_P^\bullet, \left(\frac{1+\epsilon}{2}\right)\Delta_P^\bullet|_{B_P^\bullet}\right)$ is log canonical (here we use Lemma 8.32 in case of triple intersection points).

Now assume that $B_P^\bullet$ is irreducible but singular. Let $p \in B_P^\bullet$ be a singular point. We prove that p is a singularity of type A_1 and the only singular point of $B_P^\bullet$. We can assume that p is in the form $([1 : a], [1 : b], [1 : c])$. Consider the invertible change of coordinates $X_0' = X_0$, $X_1' = X_1 - aX_0$ and so on. This automorphism sends $B_P^\bullet$ to an isomorphic surface B_P' which is singular at $p' = ([1 : 0], [1 : 0], [1 : 0])$. If we set $x' = \frac{X_1'}{X_0'}, y' = \frac{Y_1'}{Y_0'}, z' = \frac{Z_1'}{Z_0'}$, then the equation of B_P'

in this affine patch is in the form

$$c_0 x'y' + c_1 x'z' + c_2 y'z' + c_3 x'y'z' = 0. \tag{8.1}$$

The coefficients c_0, c_1, c_2 are nonzero because B'_P is irreducible. Therefore it is clear that the singularity is of type A_1. Now we show that p' is the only singularity of B'_P, whose equation is given by

$$c_0 X_1 Y_1 Z_0 + c_1 X_1 Y_0 Z_1 + c_2 X_0 Y_1 Z_1 + c_3 X_1 Y_1 Z_1 = 0, \tag{8.2}$$

where we intentionally dropped the "prime" signs for simplicity of notation. It is enough to check the affine patches $X_0 Y_0 Z_0 \neq 0, X_0 Y_0 Z_1 \neq 0, X_0 Y_1 Z_1 \neq 0, X_1 Y_1 Z_1 \neq 0$ because of the symmetries of our equation.

- In $X_0 Y_0 Z_0 \neq 0$ we have

$$\begin{cases} c_0 xy + c_1 xz + c_2 yz + c_3 xyz = 0, \\ c_0 y + c_1 z + c_3 yz = 0, \\ c_0 x + c_2 z + c_3 xz = 0, \\ c_1 x + c_2 y + c_3 xy = 0. \end{cases}$$

Let us show that this system has no solution other than $(0, 0, 0)$. Observe that $x = 0, y = 0$, or $z = 0$ implies that $x = y = z = 0$. Therefore, assume x, y, z nonzero. But if we multiply the last equation of the system by z, then the first equation implies that $c_0 xy = 0$, which is a contradiction.

- In $X_0 Y_0 Z_1 \neq 0$ we have to study the solutions of

$$\begin{cases} c_0 xyz + c_1 x + c_2 y + c_3 xy = 0, \\ c_0 yz + c_1 + c_3 y = 0, \\ c_0 xz + c_2 + c_3 x = 0, \\ c_0 xy = 0. \end{cases}$$

But this system has no solutions because $x = 0$ or $y = 0$ implies $c_2 = 0$ or $c_1 = 0$ respectively.

- In $X_0 Y_1 Z_1 \neq 0$ consider

$$\begin{cases} c_0 xz + c_1 xy + c_2 + c_3 x = 0, \\[2mm] c_0 z + c_1 y + c_3 = 0, \\[2mm] c_1 x = 0, \\[2mm] c_0 x = 0, \end{cases}$$

which has no solutions because $x = 0$ implies $c_2 = 0$.

- In the affine patch $X_1 Y_1 Z_1 \neq 0$, we obviously have no singular points because the dehomo-geneization of equation (8.2) is linear.

It is easy to see that the singular point p can lie on at most one irreducible component of $\Delta_P^\bullet|_{B_P^\bullet}$ (in checking this, it is fundamental that if two distinct coefficients $c_{ijk}, c_{i'j'k'}$ are both zero, then (i, j, k) and (i', j', k') are not vertices of the same edge of the cube). To prove that $\left(B_P^\bullet, \left(\frac{1+\epsilon}{2} \right) \Delta_P^\bullet|_{B_P^\bullet} \right)$ is log canonical, we show that $\left(X_P^\bullet, \left(\frac{1+\epsilon}{2} \right) \Delta_P^\bullet + B_P^\bullet \right)$ is log canonical and then we use inversion of adjunction (see [Kaw07]). This is done in two steps.

- $\left(X_P^\bullet, \left(\frac{1+\epsilon}{2} \right) \Delta_P^\bullet + B_P^\bullet \right)$ is log canonical in an neighborhood of a quadruple intersection point p of $\Delta_P^\bullet + B_P^\bullet$. Assume without loss of generality that $p = ([1 : 0], [1 : 0], [1 : 0])$, an therefore $c_{000} = 0$. In an affine neighborhood of p the equation of $B_P^\bullet$ becomes

$$c_{100}x + c_{010}y + c_{001}z + c_{110}xy + c_{101}xz + c_{011}yz + c_{111}xyz = 0.$$

where we must have $c_{100}, c_{010}, c_{001}$ nonzero. The affine equations for $\Delta_P^\bullet$ at p are $x = 0, y = 0, z = 0$. Therefore, locally at p, the four irreducible components of $\Delta_P^\bullet + B_P^\bullet$ are equivalent to hyperplanes in general linear position. Now we use Lemma 8.35;

- $\left(X_P^\bullet, \left(\frac{1+\epsilon}{2} \right) \Delta_P^\bullet + B_P^\bullet \right)$ is log canonical in the complement of the quadruple intersection points. This is true in the complement of $\Delta_P^\bullet$ because $B_P^\bullet$ has at most an A_1 singularity. Let $H \subset \Delta_P^\bullet$ be an irreducible component. We show that $(X_P^\bullet, \Delta_P^\bullet + B_P^\bullet)$ is log canonical in a neighborhood of H away from the quadruple intersection points. But this follows from inversion of adjunction because it is easy to observe that $(H, (\Delta_P^\bullet - H + B_P^\bullet)|_H)$ is log canonical away from the quadruple intersection points (more precisely, $H \cong \mathbb{P}^1 \times \mathbb{P}^1$, $(\Delta_P^\bullet - H)|_H$ gives the

toric boundary, and $B_P^\bullet|_H$ is a $(1,1)$-curve with no components in common with the toric boundary).

We are left with the case $B_P^\bullet$ reducible. Decompositions into two irreducible components are given by

$$(aX_0Y_0 + bX_0Y_1 + cX_1Y_0 + dX_1Y_1)(eZ_0 + fZ_1),$$

for any choice of nonzero coefficients such that $ad \neq bc$. It is easy to see that these two components are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and how they are glued together. Decompositions into three irreducible components are given by

$$(aX_0 + bX_1)(cY_0 + dY_1)(eZ_0 + fZ_1),$$

for any choice of nonzero coefficients. Up to $\mathrm{Aut}((\mathbb{P}^1)^3)$ there is only one choice of coefficients. In both cases $\left(B_P^\bullet, \left(\frac{1+\epsilon}{2}\right) \Delta_P^\bullet|_{B_P^\bullet} \right)$ is semi-log canonical.

To conclude, $K_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right) \Delta_P^\bullet|_{B_P^\bullet}$ is ample because it is the pullback to $B_P^\bullet$ of the following divisor:

$$K_{X_P^\bullet} + B_P^\bullet + \left(\frac{1+\epsilon}{2}\right) \Delta_P^\bullet \sim \epsilon(1,1,1),$$

which is ample (we used the adjunction formula to say that $K_{B_P^\bullet} \sim (K_{X_P^\bullet} + B_P^\bullet)|_{B_P^\bullet}$). $\qquad\square$

This completes the proof of Theorem 8.29. $\qquad\square$

Remark 8.37. The proof of Theorem 8.29 gives us an explicit description of the possible stable pairs $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B^\bullet} \right)$ for all the stable toric pairs (X, B) of type $\leq Q$. Here we want to summarize these possibilities. In Figure 8.3, a triangle (resp. trapezoid, parallelogram) means $B_P^\bullet \cong \mathbb{P}^2$ (resp. $\mathbb{F}^1$, $\mathbb{P}^1 \times \mathbb{P}^1$). $D_P^\bullet|_{B_P^\bullet}$ is represented by the thickened segments and $(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}$ by the colored segments. First, let us assume that $\mathcal{P}$ has no corner cuts, so that $(X, B) = (X^\bullet, B^\bullet)$. In all the cases that follow, $\Delta_P|_{B_P}$ is simple normal crossing.

(a) $B_P \cong \mathbb{P}^2$ and $D_P|_{B_P}$ (resp. $(\Delta_P - D_P)|_{B_P}$) consists of two lines;

(b) $B_P \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $D_P|_{B_P}$ consists of two incident rulings. $(\Delta_P - D_P)|_{B_P}$ is given by two incident rulings and a curve of divisor class $(1,1)$ which can possibly be reducible;

(c1) $B_P \cong \mathbb{F}_1$ and $D_P|_{B_P}$ is a line disjoint from the exceptional divisor. $(\Delta_P - D_P)|_{B_P}$ is given by two fibers and two lines disjoint from the exceptional divisor. Exactly one of these last two lines can possibly break into the union of the exceptional divisor and a fiber;

(c2) $B_P \cong \mathbb{F}_1$ and $D_P|_{B_P}$ is the union of the exceptional divisor and a fiber. $(\Delta_P - D_P)|_{B_P}$ is given by two fibers and two lines disjoint from the exceptional divisor;

(c3) B_P is isomorphic to the union of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ glued along a line in $\mathbb{P}^2$ and a ruling in $\mathbb{P}^1 \times \mathbb{P}^1$. $D_P|_{B_P}$ consists of a line in $\mathbb{P}^2$ and a ruling in $\mathbb{P}^1 \times \mathbb{P}^1$. $(\Delta_P - D_P)|_{B_P}$ is given by two lines on the $\mathbb{P}^2$ component and four rulings on $\mathbb{P}^1 \times \mathbb{P}^1$ arranged as shown in Figure 8.3 (c3);

(d1) $B_P \cong \mathrm{Bl}_3 \mathbb{P}^2$ and $D_P|_{B_P} = \emptyset$. $\Delta_P|_{B_P}$ is as in Proposition 8.8 (this is the general case), or some lines can possibly break into two intersecting (-1)-curves;

(d1′) B_P is a singular del Pezzo surface of degree 6 with exactly one A_1 singularity. This singularity can lie on at most one irreducible component of $\Delta_P|_{B_P}$;

(d2) B_P is isomorphic to the union of two copies of $\mathbb{P}^1 \times \mathbb{P}^1$ glued along a ruling and an irreducible curve of divisor class $(1, 1)$. $\Delta_P|_{B_P}$ is given by four rulings on one component and six rulings on the other. These are arranged as shown in Figure 8.3 (d2);

(d3) B_P is isomorphic to the union of three copies of $\mathbb{P}^1 \times \mathbb{P}^1$ glued along rulings as shown in Figure 8.3 (d3). $\Delta_P|_{B_P}$ consists of four rulings on each component as shown in Figure 8.3 (d3).

Now assume that $\mathcal{P}$ has a corner cut and let $P \in \mathcal{P}^\bullet$. In this case, the possibilities for the pair $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$ are as above, with the difference that now $(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}$ is allowed to have triple intersection points.

8.4 KSBA compactification for $D_{1,6}$-polarized Enriques surfaces

8.4.1 Map to the KSBA compactification

In Theorem 8.40 we describe a surjective (algebraic) morphism $\overline{M}_Q^\nu \to \overline{M}_{D_{1,6}}$ (see Notation 8.16 for the meaning of X^ν for a given scheme X) which gives a geometric interpretation of the $\Bbbk$-points
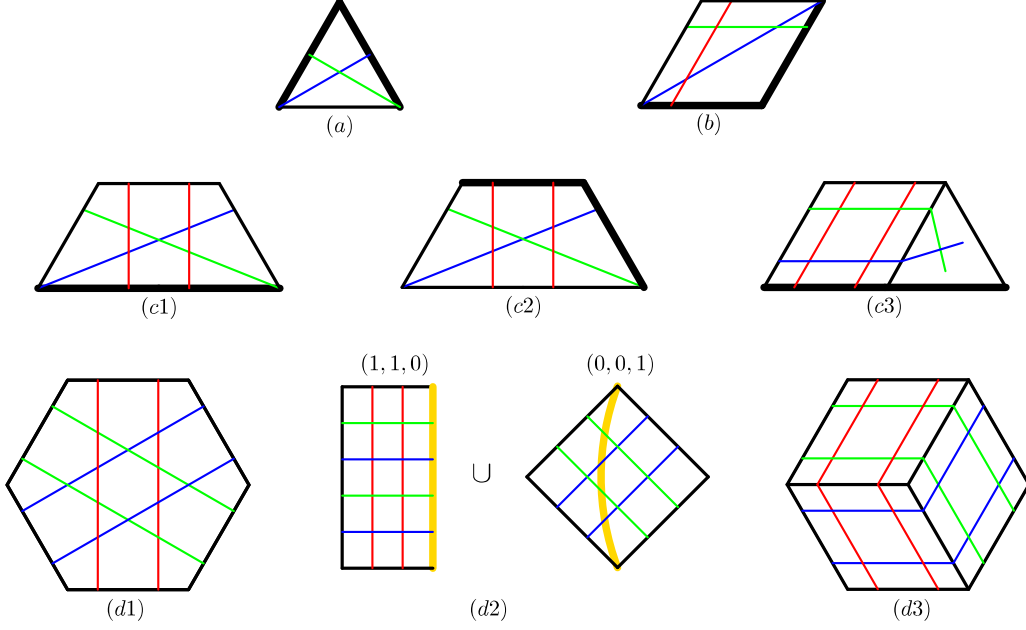
Figure 8.3: The pictures represent $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$ for $P \in \mathcal{P}^\bullet$ (see Remark 8.37). The restriction $(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}$ is in blue, green, and red, $D_P^\bullet|_{B_P^\bullet}$ is thickened. Two irreducible components are glued together along the yellow curves

of $\overline{M}_{D_{1,6}}$ (the global geography of $\overline{M}_{D_{1,6}}$ is described in Section 8.4.2). We need some preliminaries from [GG14].

Notation 8.38. Consider a map $g\colon \operatorname{Spec}(K) \to Y$, where K is the field of fractions of a DVR $(A, \mathfrak{m})$ with residue field our fixed base field $\Bbbk$. Let Y be a proper scheme over a noetherian scheme S. By the valuative criterion of properness, g extends in a unique way to a map $\overline{g}\colon \operatorname{Spec}(A) \to Y$. We denote by $\lim g$ the point $\overline{g}(\mathfrak{m})$.

Theorem 8.39 ([GG14, Theorem 7.3]). *Suppose X_1 and X_2 are proper schemes over a noetherian scheme S with X_1 normal. Let $U \subseteq X_1$ be an open dense subset and $f\colon U \to X_2$ an S-morphism. Then f extends to an S-morphism $\overline{f}\colon X_1 \to X_2$ if and only if for any point $x \in X_1$, any DVR $(A, \mathfrak{m})$ as in Notation 8.38, and any morphism $g\colon \operatorname{Spec}(K) \to U$ such that $\lim g = x$, the point $\lim(f \circ g)$ of X_2 only depends on x, and not on the choice of g.*

Theorem 8.40. *Let Q be the unit cube. Then there is a surjective (algebraic) morphism $\overline{M}_Q^\nu \to \overline{M}_{D_{1,6}}$ which is given on $\Bbbk$-points by*

$$(X, B) \mapsto \left(B^\bullet, \left(\frac{1+\epsilon}{2} \right) \Delta^\bullet|_{B^\bullet} \right).$$

Proof. Set $X_1 = \overline{M}_Q^\nu, X_2 = \overline{M}_{D_{1,6}}, U = \mathcal{U}_{\mathrm{sm}}$ (which was constructed in Definition 8.20), and let $S = \mathrm{Spec}(\Bbbk)$. The claimed morphism $X_1 \to X_2$ is obtained by extending $f \colon \mathcal{U}_{\mathrm{sm}} \to X_2$ to the whole X_1 by means of Theorem 8.39.

Let $(A, \mathfrak{m})$ be any DVR and consider a map $g \colon \mathrm{Spec}(K) \to \mathcal{U}_{\mathrm{sm}}$. The point $\lim g \in X_1$ corresponds to a stable toric pair (X, B) of type $\leq Q$. Denote by Δ the toric boundary of X. If we prove that $\lim(f \circ g)$ corresponds to the stable pair $\left(B^\bullet, \left(\frac{1+\epsilon}{2} \right) \Delta^\bullet|_{B^\bullet} \right)$, then we are done by Theorem 8.39, because this shows that $\lim(f \circ g)$ only depends on $\lim g$, and not from the way we approach it (see Remark 8.25).

Let $((\mathbb{P}^1)^3 \times \mathcal{U}_{\mathrm{sm}}, \mathfrak{S}_{\mathrm{sm}})/\mathcal{U}_{\mathrm{sm}}$ be as in Definition 8.20. Let $((\mathbb{P}_K^1)^3, \mathfrak{B}^\circ)$ be the pullback of $((\mathbb{P}^1)^3 \times \mathcal{U}_{\mathrm{sm}}, \mathfrak{S}_{\mathrm{sm}})$ under the map g and denote by $(\mathfrak{X}, \mathfrak{B})$ its completion over $\mathrm{Spec}(A)$, or a finite ramified base change of it, as a family of stable toric pairs (so that the central fiber is (X, B)). Consider $(\mathfrak{X}^\bullet, \mathfrak{B}^\bullet)$ and, if D denotes the toric boundary of $(\mathbb{P}^1)^3$, let $\overline{D}_K$ be the closure of $D_K = D \times \mathrm{Spec}(K)$ in $\mathfrak{X}^\bullet$. Then the central fiber of $\left(\mathfrak{B}^\bullet, \left(\frac{1+\epsilon}{2} \right) \overline{D}_K|_{\mathfrak{B}^\bullet} \right)$, which is $\left(B^\bullet, \left(\frac{1+\epsilon}{2} \right) \Delta^\bullet|_{B^\bullet} \right)$ by construction, is the stable pair corresponding to $\lim(f \circ g)$. $\qquad\square$

Theorem 8.41. *Let $\overline{M}_Q^\nu \to \overline{M}_{D_{1,6}}$ be the morphism in Theorem 8.40. Then the induced morphism $\overline{M}_Q^\nu / \mathrm{Sym}(Q) \to \overline{M}_{D_{1,6}}^\nu$ is an isomorphism.*

Proof. The group $\mathrm{Sym}(Q)$ acts on $\overline{M}_Q^\nu$ because $\overline{M}_Q^\nu$ is the projective toric variety associated to the secondary polytope of $(Q, Q \cap \mathbb{Z}^3)$. The modular interpretation of the action is the following: $\mathrm{Sym}(Q)$ acts on the stable toric pair (X, B) by changing the torus action on it. In particular, the morphism $\overline{M}_Q^\nu \to \overline{M}_{D_{1,6}}$ is $\mathrm{Sym}(Q)$-equivariant. Therefore, we have an induced morphism $\overline{M}_Q^\nu / \mathrm{Sym}(Q) \to \overline{M}_{D_{1,6}}$ which lifts to the normalization of $\overline{M}_{D_{1,6}}$.

The fibers of the restriction of $\overline{M}_Q^\nu \to \overline{M}_{D_{1,6}}$ to $\mathcal{U}_{\mathrm{sm}}$ are exactly $\mathrm{Sym}(Q)$-orbits (see Definition 8.20 and Observation 8.21), and hence the morphism $\overline{M}_Q^\nu / \mathrm{Sym}(Q) \to \overline{M}_{D_{1,6}}^\nu$ is bijective on a dense open subset. In what follows, we show that $\overline{M}_Q^\nu \to \overline{M}_{D_{1,6}}$ is quasi-finite, which implies that
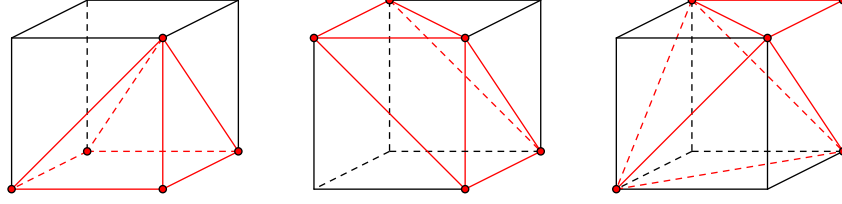
Figure 8.4: Subpolytopes of Q which can be subdivided further only once (see the proof of Theorem 8.41)

$\overline{M}^{\nu}_Q / \operatorname{Sym}(Q) \to \overline{M}^{\nu}_{D_{1,6}}$ is an isomorphism by the Zariski Main Theorem (see [Mum99, Chapter III, Section 9]).

To prove that $\overline{M}^{\nu}_Q \to \overline{M}_{D_{1,6}}$ is quasi-finite, it is enough to check that no 1-dimensional boundary stratum of $\overline{M}^{\nu}_Q$ is contracted by $\overline{M}^{\nu}_Q \to \overline{M}_{D_{1,6}}$. These strata correspond to the minimal elements of the poset of regular polyhedral subdivisions of $(Q, Q \cap \mathbb{Z}^3)$ which are not triangulations (all the polyhedral subdivisions of $(Q, Q \cap \mathbb{Z})$ are regular, see the proof of Proposition 8.17). If $\mathcal{P}$ is one of these polyhedral subdivisions, then it has to contain a subpolytope of Q with vertices in $Q \cap \mathbb{Z}^3$ which can be subdivided further only once. An easy enumeration shows that $\mathcal{P}$ has to contain one of the polytopes listed in Figure 8.4. Denote by P one of such polytopes, and let (X, B) be any stable toric pair with $\mathcal{P}$ as corresponding polyhedral subdivision of $(Q, Q \cap \mathbb{Z}^3)$. Then, as (X, B) varies among the stable toric pairs parametrized by the 1-dimensional boundary stratum corresponding to $\mathcal{P}$, $(B^\bullet, \left(\frac{1+\epsilon}{2}\right) \Delta^\bullet|_{B^\bullet})$ describes a 1-dimensional family of stable pairs (to see this, if $P^\bullet \in \mathcal{P}^\bullet$ is the polytope corresponding to $P \in \mathcal{P}$, it is enough to consider the irreducible component of $(B^\bullet, \left(\frac{1+\epsilon}{2}\right) \Delta^\bullet|_{B^\bullet})$ corresponding to $P^\bullet$). In conclusion, the 1-dimensional boundary stratum corresponding to $\mathcal{P}$ is not contracted. $\qquad\square$

8.4.2 Description of the boundary of $\overline{M}^{\nu}_{D_{1,6}}$

Definition 8.42. The *boundary* of the moduli space $\overline{M}^{\nu}_{D_{1,6}}$ is the closed subset whose $\Bbbk$-points parametrize stable pairs (B, D) with B reducible. Let (B, D) be a stable pair parametrized by the boundary, and consider the locus of points in $\overline{M}^{\nu}_{D_{1,6}}$ parametrizing stable pairs (B', D') such that $B \cong B'$. We call the closure of such locus a *stratum*.

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Theorem 8.43. *The boundary of $\overline{M}^\nu_{D_{1,6}}$ is stratified as shown in Figure 8.6. The strata are organized from bottom to top in increasing order of dimension. Going from left to right in Figure 8.6, call the 0-dimensional strata even, odd of type 1, and odd of type 2 (this terminology is borrowed from [Oud10]). Then the strata containing the even 0-dimensional stratum correspond bijectively to the polyhedral subdivisions of $(Q, Q \cap \mathbb{Z}^3)$ without corner cuts up to symmetries of Q, where Q is the unit cube.*

Proof. By Theorem 8.41 any stable pair parametrized by $\overline{M}^\nu_{D_{1,6}}$ is in the form $\left(B^\bullet, \left(\frac{1+\epsilon}{2}\right) \Delta^\bullet|_{B^\bullet}\right)$ for some stable toric pair (X, B) parametrized by $\overline{M}^\nu_Q$. In Remark 8.37 we listed all the possibilities for the pairs $\left(B^\bullet_P, D^\bullet_P|_{B^\bullet_P} + \left(\frac{1+\epsilon}{2}\right) (\Delta^\bullet_P - D^\bullet_P)|_{B^\bullet_P}\right)$, which can be glued together for $P \in \mathcal{P}^\bullet$ to recover $\left(B^\bullet, \left(\frac{1+\epsilon}{2}\right) \Delta^\bullet|_{B^\bullet}\right)$, where $\mathcal{P}^\bullet$ is a polyhedral subdivisions of $(Q, Q \cap \mathbb{Z}^3)$ without corner cuts. These subdivisions are shown in Figure 8.5 up to symmetries of Q. The end result is shown in Figure 8.6, where for each stratum we picture the stable pair parametrized by a general point in it. The three colors (blue, green, and red) used to draw the divisor have the following meaning: lines sharing the same color come from the same pair of lines on the original $\mathrm{Bl}_3 \mathbb{P}^2$. We want to remark that, even though we use three different colors, we do not distinguish the three pairs because we took the quotient by S_3 in our definition of $\overline{M}_{D_{1,6}}$. The yellow color is used to indicate lines along which two irreducible components are glued together. The claimed combinatorial interpretation of the strata containing the even 0-dimensional stratum follows immediately after comparing Figure 8.5 and Figure 8.6. □

8.4.3 Degenerations of Enriques surfaces

After describing all the degenerations of $\mathrm{Bl}_3 \mathbb{P}^2$ together with the three undistinguished pairs of undistinguished weighted lines parametrized by $\overline{M}^\nu_{D_{1,6}}$, we want to discuss what are the overlying $\mathbb{Z}_2^2$-covers which correspond to actual degenerations of Enriques surfaces. We describe such $\mathbb{Z}_2^2$-covers for the stable pairs parametrized by a general point in each boundary stratum of $\overline{M}^\nu_{D_{1,6}}$. In what follows, we refer to the $\mathbb{Z}_2^2$-covers with the appropriate building data that comes from the subdivision into three pairs of lines (which we kept track of). In some cases, the double locus is part of the branch locus.
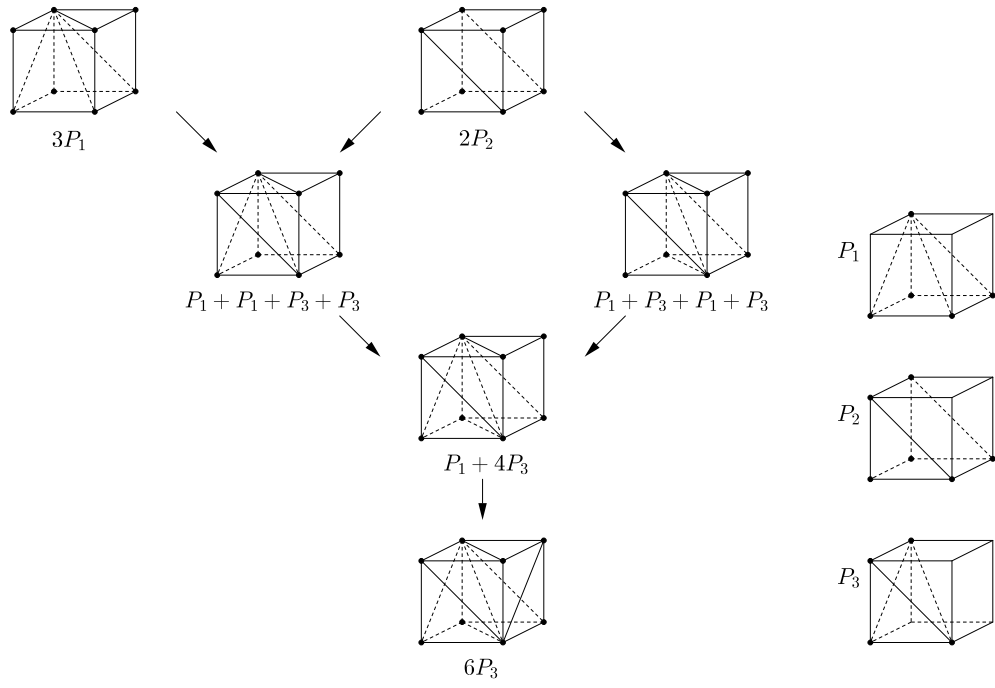
Figure 8.5: Polyhedral subdivisions of $(Q, Q \cap \mathbb{Z}^3)$ without corner cuts up to symmetry and ordered by refinement
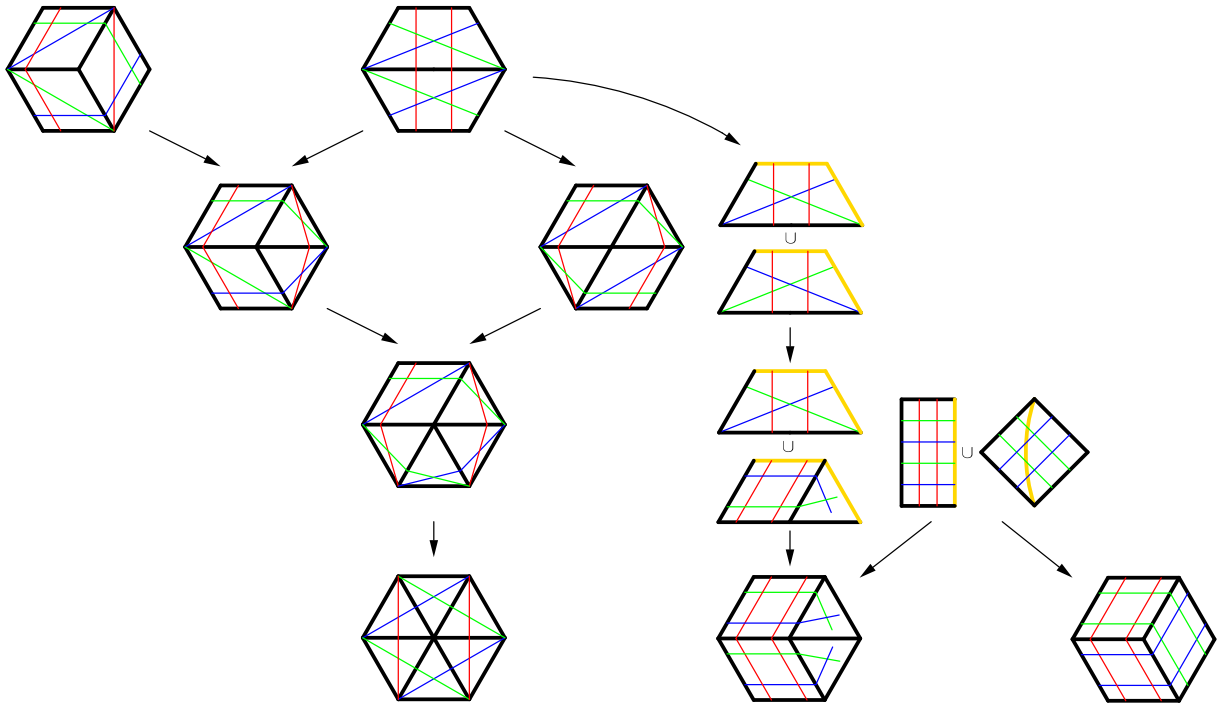

Figure 8.6: Stratification of the boundary of $\overline{M}_{D_{1,6}}^{\nu}$

(a) Consider the stable pair parametrized by the odd 0-dimensional stratum of type 2. Then its $\mathbb{Z}_2^2$-cover is isomorphic to the quotient of

$$V((X_0^2 - X_1^2)(Y_0^2 - Y_1^2)(Z_0^2 - Z_1^2)) \subset (\mathbb{P}^1)^3,$$

by the involution ι (see Remark 8.15 and the proof of Proposition 8.36). This quotient consists of three copies of $\mathbb{P}^1 \times \mathbb{P}^1$ glued along rulings in such a way that its dual complex gives a triangulation of the real projective plane.

(b) The stable pair parametrized by a general point in the maximal 1-dimensional stratum of $\overline{M}_{D_{1,6}}^\nu$ has $\mathbb{Z}_2^2$-cover isomorphic to the quotient of

$$V((X_0^2 - X_1^2)(Y_0^2 Z_0^2 + Y_1^2 Z_0^2 + Y_0^2 Z_1^2 + \lambda Y_1^2 Z_1^2)) \subset (\mathbb{P}^1)^3, \lambda \neq 0, 1,$$

by ι (again, see Remark 8.15 and the proof of Proposition 8.36). If we define

$$E = V(Y_0^2 Z_0^2 + Y_1^2 Z_0^2 + Y_0^2 Z_1^2 + \lambda Y_1^2 Z_1^2) \subset \mathbb{P}^1 \times \mathbb{P}^1,$$

then the irreducible components of this $\mathbb{Z}_2^2$-cover are a copy of $\mathbb{P}^1 \times \mathbb{P}^1$, and an elliptic fibration F over $\mathbb{P}^1$ with fibers isomorphic to E and two double fibers. These surfaces are glued along $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ and a reduced fiber of F.

(c) Consider the stable pair parametrized by a general point in the irreducible boundary divisor containing two 0-dimensional strata. Let us describe the $\mathbb{Z}_2^2$-cover X of one of the two irreducible components, which are both isomorphic to $\mathbb{F}_1$. Let h be a section of self-intersection 1 and f a fiber. Then the building data for the cover $\pi \colon X \to \mathbb{F}_1$ is given by $D_a \sim 2f + h, D_b \sim D_c \sim h$, implying that

$$K_X \sim_\mathbb{Q} \pi^* \left(K_{\mathbb{F}_1} + \frac{1}{2}(D_a + D_b + D_c) \right) \sim_\mathbb{Q} -\frac{1}{2}\pi^*(h).$$

This shows that $-K_X$ is big, nef, and $K_X^2 = 1$. Therefore, X is a weak del Pezzo surface of degree 1. The total degeneration is given by two of such weak del Pezzo surfaces glued along a genus 1 curve.

(d) Let X be the $\mathbb{Z}_2^2$-cover of the $\mathbb{P}^1 \times \mathbb{P}^1$ in Figure 8.3 (b). Denote by ℓ_1 and ℓ_2 two incident rulings. Then the building data for the cover $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ is given by $D_a \sim D_b \sim D_c \sim \ell_1 + \ell_2$. From this we obtain that $K_X \sim_{\mathbb{Q}} -\frac{1}{2}\pi^*(\ell_1 + \ell_2)$. It follows that $-K_X$ is ample and $K_X^2 = 2$. Hence, X is a del Pezzo surface of degree 2.

(e) Let us describe the $\mathbb{Z}_2^2$-cover of the $\mathbb{P}^2$ in Figure 8.3 (a). If ℓ denotes a line in $\mathbb{P}^2$, then the building data for the cover $\pi\colon X \to \mathbb{P}^2$ is $D_a \sim D_b \sim 2\ell, D_c \sim 0$. Therefore, $K_X \sim_{\mathbb{Q}} -\pi^*(\ell)$, implying that $-K_X$ is ample and $K_X^2 = 4$. Hence, X is a del Pezzo surface of degree 4.

(f) Let us describe the $\mathbb{Z}_2^2$-cover of the $\mathbb{P}^1 \times \mathbb{P}^1$-component in Figure 8.3 $(c3)$. If ℓ_1 and ℓ_2 denote two incident rulings, then the building data for the cover $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ is given by $D_a \sim 2\ell_1 + \ell_2, D_b \sim D_c \sim \ell_2$. Therefore, $K_X \sim_{\mathbb{Q}} -\frac{1}{2}\pi^*(2\ell_1 + \ell_2)$, $-K_X$ is ample, $K_X^2 = 4$, and X is a del Pezzo surface of degree 4.

Remark 8.44. A Coble surface is a smooth rational projective surface X with $|-K_X| = \emptyset$ and $|-2K_X| \neq \emptyset$ (see [DZ01]). These are related to our degenerations of Enriques surfaces as follows. Let S be the appropriate $\mathbb{Z}_2^2$-cover of $\mathrm{Bl}_3\,\mathbb{P}^2$ branched along $\sum_{i=1}^{3}(\ell_i + \ell'_i)$ (see Notation 8.7), and assume that this lines configuration has exactly one triple intersection point. Then S has a quotient singularity of type $\frac{(1,1)}{4}$ over this triple intersection point, and the minimal resolution $\widetilde{S}$ of S is a Coble surface. This follows from Castelnuovo rationality criterion, and from $K_{\widetilde{S}} = -\frac{1}{2}E$, where E is the exceptional divisor over the quotient singularity. The Zariski closure in $\overline{M}^\nu_{D_{1,6}}$ of the locus of points parametrizing these surfaces S defines a divisor.

8.5 Relation with the Baily-Borel compactification

In what follows we construct a morphism from the KSBA compactification $\overline{M}^\nu_{D_{1,6}}$ to the Baily-Borel compactification of $\mathcal{D}/\Gamma$, where $\mathcal{D}$ is the period domain parametrizing $D_{1,6}$-polarized Enriques surfaces (details in Section 8.5.2). Section 8.5.1 contains a technical result which is fundamental to construct such morphism. In Section 8.5.3 we show that the combinatorics of the boundary strata of $\overline{M}^\nu_{D_{1,6}}$ has an interpretation in terms of Vinberg diagrams. Let $\Bbbk = \mathbb{C}$ be our base field.

8.5.1 Generalized type of degenerations of stable K3 surface pairs

Remark 8.45. In Section 8.5 our focus moves from Enriques surfaces to K3 surfaces. The reason for this is that in Theorem 8.52 we compute limits of one-parameter families of $D_{1,6}$-polarized Enriques surfaces in the Baily-Borel compactification of $\mathcal{D}/\Gamma$. This is done by using the corresponding K3 covers.

Let Δ be the unit disk $\{t \in \mathbb{C} \mid |t| < 1\}$ and let $\Delta^* = \Delta \setminus \{0\}$. We are interested in proper flat families $\mathfrak{X}^* \to \Delta^*$ with $\mathfrak{X}^*$ smooth and where $\mathfrak{X}^*_t$ is a smooth K3 surface for all $t \in \Delta^*$. Equip the family $\mathfrak{X}^*$ with an effective relative Cartier divisor $\mathcal{H}^*$ such that $(\mathfrak{X}^*, \mathcal{H}^*) \to \Delta^*$ is a family of stable pairs. Let $\mathfrak{X}$ be a semistable degeneration with $K_{\mathfrak{X}} \sim 0$ (or simply Kulikov degeneration for short) completing $\mathfrak{X}^*$ over Δ (see [Kul77, PP81]). Recall that the central fiber $\mathfrak{X}_0$ can be of type I, II or III (see [Kul77, Theorem II]). In type II, denote by $j(\mathfrak{X}_0)$ the j-invariant of one of the mutually isomorphic genus 1 double curves on $\mathfrak{X}_0$. Then define $\mathcal{H}$ to be the closure of $\mathcal{H}^*$ inside $\mathfrak{X}$ ($\mathcal{H}$ is flat over Δ). We can define a second completion of $(\mathfrak{X}^*, \mathcal{H}^*)$ over Δ, which we denote by $(\mathfrak{X}', \mathcal{H}')$, such that $(\mathfrak{X}'_0, \epsilon\mathcal{H}'_0)$ is a stable pair for all $0 < \epsilon \ll 1$.

Definition 8.46. With the notation introduced above, define the *dual graph of $\mathfrak{X}'_0$* as follows. Draw a vertex v_i for each irreducible component V_i of $\mathfrak{X}'_0$. Then, given any two distinct irreducible components V_i and V_j, draw one edge between v_i and v_j for each irreducible curve in $V_i \cap V_j$. If an irreducible component V_i self-intersects along a curve C, then draw one loop on v_i for each irreducible component of C. Denote by $G(\mathfrak{X}'_0)$ the dual graph of $\mathfrak{X}'_0$.

Definition 8.47. Let $\mathfrak{X}'$ as defined above. We say that $\mathfrak{X}'_0$ has *generalized type I, II*, or *III* if the following hold:

- Type I: $G(\mathfrak{X}'_0)$ consists of one vertex and $\mathfrak{X}'_0$ has at worst Du Val singularities;

- Type II: $G(\mathfrak{X}'_0)$ is a chain and $\mathfrak{X}'_0$ has at worst elliptic singularities. If there are at least two vertices and the double curves are mutually isomorphic genus 1 curves, then denote by $j(\mathfrak{X}'_0)$ the j-invariant of one of these;

- Type III: otherwise.

The proof of the following theorem, which builds upon the proof of [Laz16, Theorem 2.9], was communicated to me by Valery Alexeev.

Theorem 8.48. *With the notation introduced above, $\mathfrak{X}_0$ and $\mathfrak{X}_0'$ have the same type. In addition, if $\mathfrak{X}_0, \mathfrak{X}_0'$ have type II and $j(\mathfrak{X}_0')$ can be defined, then $j(\mathfrak{X}_0) = j(\mathfrak{X}_0')$.*

Proof. The proof of [Laz16, Theorem 2.9] describes a procedure to construct the unique stable model $(\mathfrak{X}', \epsilon\mathcal{H}')$ by modifying $(\mathfrak{X}, \mathcal{H})$. This procedure consists mainly of the following two steps:

- Step 1: Replace $\mathfrak{X}$ with another Kulikov degeneration such that $\mathcal{H}$ is nef and it does not contain double curves or triple points (this may involve, among other things, base changes);

- Step 2: For $n \geq 4$, the line bundle $\mathcal{O}_{\mathfrak{X}}(n\mathcal{H})$ induces a birational morphism $(\mathfrak{X}, \mathcal{H}) \to (\mathfrak{X}', \mathcal{H}')$ where $\mathfrak{X}' = \mathrm{Proj}_\Delta \left(\bigoplus_{n \geq 0} \mathcal{O}_{\mathfrak{X}}(n\mathcal{H}) \right)$ (see [SB83, Theorem 2, part (i)]). This birational morphism consists of the contraction of some components or curves in the central fiber $\mathfrak{X}_0$.

In step 1, the new Kulikov model is obtained from the one we started by applying elementary modifications of type 0, I, II (see [FM83, pages 12–15] for their definitions), base changes, and blow ups of $\mathfrak{X}_0$ along double curves or triple points. The elementary modifications and the blow ups obviously do not change the type of the central fiber. A description of how the central fiber is modified after a base change can be found in [Fri83], and also in this case the type does not change. In step 2, it follows from our definition of generalized type that $\mathfrak{X}_0'$ has the same type as $\mathfrak{X}_0$.

This shows that if $\mathfrak{X}_0$ has type I, II, or III, then $\mathfrak{X}_0'$ has type I, II, or III respectively. The converse follows from this and from the uniqueness of the stable model. The claim about the j-invariants also follows from our discussion. $\square$

8.5.2 Map to the Baily-Borel compactification

The period domain $\mathcal{D}_L = \mathcal{D}$ parametrizing $D_{1,6}$-polarized Enriques surfaces (where L is the lattice $\mathbb{Z}^2(2) \oplus \mathbb{Z}^4(-1)$, see [Oud10, Section 3.4.2] for the details) and its Baily-Borel compactification $\overline{\mathcal{D}/\Gamma}^{BB}$ (Γ denotes the isometry group of L) are studied in [Oud10]. Oudompheng shows that the boundary of $\overline{\mathcal{D}/\Gamma}^{BB}$ consists of two rational 1-cusps and three 0-cusps (see [Oud10, Proposition 7.7]). The 1-cusps are labeled even and odd. The 0-cusps are labeled even, odd of type 1, and odd of type 2. These are arranged as shown in Figure 8.7. In what follows we construct a birational morphism $\overline{M}_{D_{1,6}}^\nu \to \overline{\mathcal{D}/\Gamma}^{BB}$ which highlights a correspondence between the boundary strata of the two compactifications.
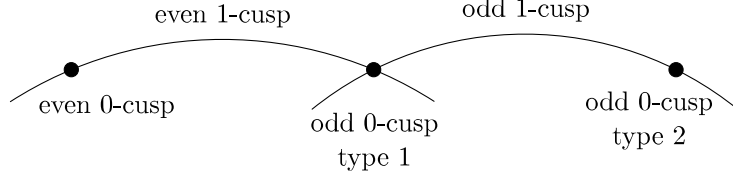
Figure 8.7: Boundary of the Baily-Borel compactification $\overline{\mathcal{D}/\Gamma}^{BB}$

Observation 8.49. The two rational 1-cusps of the Baily-Borel compactification $\overline{\mathcal{D}/\Gamma}^{BB}$ are degree 3 covers of the modular curve $X(1) \cong \mathbb{P}^1$ (see [Oud10, Section 7.3]).

Lemma 8.50. *There exists a compactification $\overline{\mathcal{D}/\Gamma}'$ of $\mathcal{D}/\Gamma$ obtained from $\overline{\mathcal{D}/\Gamma}^{BB}$ by gluing the three 0-cusps together and by gluing the two 1-cusps to a rational curve with one node whose smooth points correspond to isomorphism classes of elliptic curves. Moreover, $\overline{\mathcal{D}/\Gamma}^{BB}$ is the normalization of $\overline{\mathcal{D}/\Gamma}'$.*

Proof. The main technique we use for the proof is [Fer03, Theorem 5.4]. Let us briefly recall it. Let X' be a scheme, Y' a closed subscheme of X', and $Y' \to Y$ a finite morphism. Consider the ringed space $X = X' \amalg_{Y'} Y$ and the cocartesian square

$$
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X' & \longrightarrow & X.
\end{array}
$$

Let us assume that any finite sets of points in X' (resp. Y) are contained in an open affine subset of X' (resp. Y). Then X is a scheme verifying the same property on finite sets of points, the above diagram is cartesian, $Y \to X$ is a closed immersion, the morphism $X' \to X$ is finite, and it induces an isomorphism $X' \setminus Y' \cong X \setminus Y$.

Back to our case, denote by $C_{\text{even}}, C_{\text{odd}}$ the two 1-cusps of $\overline{\mathcal{D}/\Gamma}^{BB}$. We want to first consider the finite morphism $C_{\text{even}} \to X(1)$ to build the projective scheme $X_1 = \overline{\mathcal{D}/\Gamma}^{BB} \amalg_{C_{\text{even}}} X(1)$ (the hypothesis on finite sets of points is satisfied because $\overline{\mathcal{D}/\Gamma}^{BB}$ and $X(1)$ are projective). Repeat the gluing on X_1 by considering now $C_{\text{odd}} \to X(1)$ to obtain X_2. Now glue together the two copies of $X(1)$ on X_2 to obtain X_3. Finally, identify the images in X_3 of the three 0-cusps to obtain the claimed compactification $\overline{\mathcal{D}/\Gamma}'$. The isomorphism $\overline{\mathcal{D}/\Gamma}^{BB} \cong (\overline{\mathcal{D}/\Gamma}')^{\nu}$ follows from the Zariski Main Theorem because $\overline{\mathcal{D}/\Gamma}^{BB}$ is normal and the morphism $\overline{\mathcal{D}/\Gamma}^{BB} \to \overline{\mathcal{D}/\Gamma}'$ is finite and birational. □

Definition 8.51. Recall the family $\mathfrak{S}_{\mathrm{sm}}/\mathcal{U}_{\mathrm{sm}}$ in Definition 8.20. We denote by $\mathfrak{S}_{\mathrm{sm}}^{\mathrm{K3}}$ the appropriate $\mathbb{Z}_2^3$-cover of $\mathfrak{S}_{\mathrm{sm}}$ branched along $(\Delta \times \mathcal{U}_{\mathrm{sm}})|_{\mathfrak{S}_{\mathrm{sm}}}$, which is a family of K3 surfaces. Observe that $\mathfrak{S}_{\mathrm{sm}}^{\mathrm{K3}}$, like $\mathfrak{S}_{\mathrm{sm}}$, is embedded in $(\mathbb{P}^1)^3 \times \mathcal{U}_{\mathrm{sm}}$.

Theorem 8.52. *There exists a birational morphism $\overline{M}_{D_{1,6}}^{\nu} \to \overline{\mathcal{D}/\Gamma}^{BB}$ which maps the boundary of $\overline{M}_{D_{1,6}}^{\nu}$ to the boundary of $\overline{\mathcal{D}/\Gamma}^{BB}$.*

Proof. The GIT interpretation of $\overline{\mathcal{D}/\Gamma}^{BB}$ in [Oud10] as quotient of the Grassmannian $\mathrm{Gr}(3, 6)$ gives us a birational morphism $\overline{\mathcal{D}/\Gamma}^{BB} \dashrightarrow \overline{M}_{D_{1,6}}^{\nu}$ (this morphism is defined at points corresponding to arrangements of six lines in $\mathbb{P}^2$ without triple intersection points). Given the open subset $\mathcal{U}_{\mathrm{sm}} \subset \overline{M}_Q^{\nu}$, consider the composition $\mathcal{U}_{\mathrm{sm}} \to \overline{M}_{D_{1,6}}^{\nu} \dashrightarrow \overline{\mathcal{D}/\Gamma}^{BB}$ (which is regular). We show that this morphism extends to $\overline{M}_Q^{\nu}$ giving a $\mathrm{Sym}(Q)$-equivariant morphism. This extension is induced by the universal property of the normalization after we extend to $\overline{M}_Q^{\nu}$ the composition $\mathcal{U}_{\mathrm{sm}} \to \overline{\mathcal{D}/\Gamma}^{BB} \to \overline{\mathcal{D}/\Gamma}'$, which we denote by ρ (the compactification $\overline{\mathcal{D}/\Gamma}'$ is constructed in Lemma 8.50). To extend ρ we use Theorem 8.39. So let K be the field of fractions of a DVR $(A, \mathfrak{m})$ and consider any $g\colon \mathrm{Spec}(K) \to \mathcal{U}_{\mathrm{sm}}$. We show that $\lim(\rho \circ g) \in \overline{\mathcal{D}/\Gamma}'$ can be computed using only $\lim g \in \overline{M}_Q^{\nu}$.

Let (X, B) be the stable toric pair parametrized by $\lim g \in \overline{M}_Q^{\nu}$ and consider the corresponding stable pair $\left(B^\bullet, \left(\frac{1+\epsilon}{2}\right) \Delta^\bullet|_{B^\bullet}\right)$. We distinguish the following three cases:

- Case I: $B^\bullet$ is irreducible;

- Case II: $B^\bullet$ has exactly two irreducible components glued along an irreducible curve;

- Case III: otherwise.

Denote by p_0 (resp. C_0) the image of the 0-cusps (resp. 1-cusps) under the morphism $\overline{\mathcal{D}/\Gamma}^{BB} \to \overline{\mathcal{D}/\Gamma}'$. The point $\lim(\rho \circ g)$ can be computed as follows. $\mathcal{U}_{\mathrm{sm}}$ carries the family of K3 surfaces $\left(\mathfrak{S}_{\mathrm{sm}}^{\mathrm{K3}}, \epsilon(\Delta \times \mathcal{U}_{\mathrm{sm}})|_{\mathfrak{S}_{\mathrm{sm}}^{\mathrm{K3}}}\right)$, where Δ is the toric boundary of $(\mathbb{P}^1)^3$. Let $\mathcal{Y}'$ be the completion over $\mathrm{Spec}(A)$, or a finite ramified base change of it, of the restriction of $\mathfrak{S}_{\mathrm{sm}}^{\mathrm{K3}}$ to $\mathrm{Spec}(K)$ as a family of stable pairs (we omit the divisor for simplicity of notation). In particular, $\mathcal{Y}_{\mathfrak{m}}'$ is the $\mathbb{Z}_2^3$-cover of $B^\bullet$ branched along $\Delta^\bullet|_{B^\bullet}$ and it is the stable model of a degeneration of smooth K3 surface pairs. Let $\mathcal{Y}$ be a birational modification of $\mathcal{Y}'$ which is a Kulikov degeneration. Then $\mathcal{Y}_{\mathfrak{m}}$ determines a unique point in $\overline{\mathcal{D}/\Gamma}'$, which depends on the type of $\mathcal{Y}_{\mathfrak{m}}$ as follows:

- Type I: $\lim(\rho \circ g) \in \overline{\mathcal{D}/\Gamma}^{BB}$ is the image under the quotient $\mathcal{D} \to \mathcal{D}/\Gamma$ of the period point corresponding to $\mathcal{Y}_{\mathfrak{m}}$;

- Type II: $\lim(\rho \circ g) \in C_0$ and it corresponds to $j(\mathcal{Y}_{\mathfrak{m}})$;

- Type III: $\lim(\rho \circ g) = p_0$.

It is easy to observe that $B^\bullet$ falls into case I (resp. II, III) if and only if $\mathcal{Y}'_{\mathfrak{m}}$ has generalized type I (resp. II, III) (by Remark 8.37 $(d1), (d1')$, if $B^\bullet$ falls into case I, then $\mathcal{Y}'_{\mathfrak{m}}$ is smooth or it has isolated singularities of type A_1 or A_3). Moreover, the generalized type of $\mathcal{Y}'_{\mathfrak{m}}$ equals the type of $\mathcal{Y}_{\mathfrak{m}}$ by Theorem 8.48. In addition, if we are in case II, then it makes sense to define the j-invariant $j(\mathcal{Y}'_{\mathfrak{m}})$, and this equals $j(\mathcal{Y}_{\mathfrak{m}})$ again by Theorem 8.48. In conclusion, we proved that $\lim(\rho \circ g)$ only depends on $\lim g$. $\qquad \square$

Remark 8.53. Consider the three 0-cusps of $\overline{\mathcal{D}/\Gamma}^{BB}$ (even, odd of type 1, and odd of type 2). From the way we constructed the morphism $\overline{M}^\nu_{D_{1,6}} \to \overline{\mathcal{D}/\Gamma}^{BB}$, it is not clear to which one of these 0-cusps is a given 0-dimensional stratum mapped to. Let us briefly show that a 0-dimensional boundary stratum maps to the 0-cusp with the same label (recall the terminology for the 0-dimensional boundary strata in Theorem 8.43). It is enough to show that a given point in the interior of the maximal 1-dimensional stratum of $\overline{M}^\nu_{D_{1,6}}$ is mapped to the 1-cusp C_{odd}. Consider a smooth one-parameter family with fibers isomorphic to $\mathrm{Bl}_3 \, \mathbb{P}^2$. Equip this family with a divisor of coefficient $\frac{1+\epsilon}{2}$ cutting on each fiber our usual configuration of three pairs of lines without triple points, exception made for the central fiber where two lines belonging to the same pair come together and the other four lines are general. It is easy to compute that the limit of this one-parameter family in $\overline{M}^\nu_{D_{1,6}}$ lies on the maximal 1-dimensional stratum (to show this, one has to compute the stable replacement of the central fiber, but this is done by blowing up the double line and then contracting the strict transforms of the two (-1)-curves intersecting the double line). The limit of the same family in $\overline{\mathcal{D}/\Gamma}^{BB}$ belongs to C_{odd}, and this can be deduced using the GIT interpretation of $\overline{\mathcal{D}/\Gamma}^{BB}$ in terms of degenerations of line arrangements in $\mathrm{Bl}_3 \, \mathbb{P}^2$ (see [Oud10]). This shows what we claimed.

8.5.3 Comparison between the boundaries of $\overline{M}^\nu_{D_{1,6}}$ and $\overline{\mathcal{D}/\Gamma}^{BB}$

We want to analyze the preimages of the three 0-cusps of $\overline{\mathcal{D}/\Gamma}^{BB}$ under the morphism $\overline{M}^\nu_{D_{1,6}} \to \overline{\mathcal{D}/\Gamma}^{BB}$ of Theorem 8.52.

Observation 8.54. It follows from the proof of Theorem 8.52 and from Remark 8.53 that the preimage of the even 0-cusp (resp. odd of type 1, odd of type 2) consists of the union of the strata of $\overline{M}^\nu_{D_{1,6}}$ containing the even 0-dimensional stratum (resp. odd of type 1, odd of type 2), and parametrizing degenerate stable pairs whose $\mathbb{Z}_2^3$-coves has generalized type III (see Definition 8.47). In what follows, we refer to the stable pairs $(a), (b), (c2), (c3), (d3)$ in Figure 8.3. Observe that $(b), (c2)$ have 1-dimensional moduli, and $(a), (c3), (d3)$ are unique up to isomorphism.

Even 0-cusp

The preimage of the even 0-cusp is the union of the boundary strata whose points parametrize stable pairs obtained by gluing copies of $(a), (b)$ as shown in Figure 8.6. In particular, this preimage consists of the union of a divisor and a codimension 2 boundary stratum. In Section 8.6 we show that the boundary of $\overline{M}^\nu_{D_{1,6}}$ has a toroidal behavior in a neighborhood of the preimage of the even 0-cusp.

Odd 0-cusp of type 1

The preimage of the odd 0-cusp of type 1 is a codimension 2 stratum. A point in this stratum parametrizes one of the following stable pairs:

- Glue two $(c2)$;

- Glue $(c2)$ and $(c3)$;

- Glue two $(c3)$ (this is the stable pair parametrized by the odd 0-stratum of type 1).

The behavior of the compactification $\overline{M}^\nu_{D_{1,6}}$ in a neighborhood of the preimage of this 0-cusp is neither toroidal nor Baily-Borel (more about this is discussed in Section 8.6).

Odd 0-cusp of type 2

By the stratification of the boundary of $\overline{M}^\nu_{D_{1,6}}$ in Figure 8.6, we can see that the preimage of the odd 0-cusp of type 2 consists only of the point parametrizing the stable pair $(d3)$.

Observation 8.55. The compactification $\overline{M}^\nu_{D_{1,6}}$ is isomorphic to the Baily-Borel compactification in a neighborhood of the preimage of the odd 0-cusp of type 2. To prove this isomorphism, just

observe that the restriction of the morphism $\overline{M}^{\nu}_{D_{1,6}} \to \overline{\mathcal{D}/\Gamma}^{BB}$ to this neighborhood is birational and finite, hence an isomorphism by the Zariski Main Theorem because $\overline{\mathcal{D}/\Gamma}^{BB}$ is normal.

8.6 Relation with Looijenga's semitoric compactifications

The behavior of the boundary of $\overline{M}^{\nu}_{D_{1,6}}$ suggests that $\overline{M}^{\nu}_{D_{1,6}}$ can possibly be isomorphic to a semitoric compactification $\overline{\mathcal{D}/\Gamma}^{\Sigma}$. These semitoric compactifications were introduced in [Loo03], and they depend on the choice of Σ, which is called an *admissible decomposition of the conical locus of $\mathcal{D}$* (see Definition 8.58). In what follows, we provide a natural choice of Σ such that the boundary strata of the corresponding semitoric compactification $\overline{\mathcal{D}/\Gamma}^{\Sigma}$ are in bijection with the boundary strata of $\overline{M}^{\nu}_{D_{1,6}}$. These considerations make us conjecture that $\overline{\mathcal{D}/\Gamma}^{\Sigma}$ is actually isomorphic to $\overline{M}^{\nu}_{D_{1,6}}$. We will address this conjecture in future work.

8.6.1 Admissible decomposition of the conical locus

Notation 8.56. We adopt the same notation as in [Loo03]. Let $V = L \otimes_{\mathbb{Z}} \mathbb{C}$, where the lattice $L = \mathbb{Z}^2(2) \oplus \mathbb{Z}^4(-1)$ was introduced in Section 8.5.2. We call $\mathbb{Q}$-*isotropic* an isotropic subspace $W \subset V$ which is defined over $\mathbb{Q}$. Denote by I (resp. J) a $\mathbb{Q}$-isotropic line (resp. plane) in V. Denote by $C_I \subset (I^{\perp}/I)(\mathbb{R})$ (resp. $C_J \subset \bigwedge^2 J(\mathbb{R})$) one of the two connected components of $\{x \in (I^{\perp}/I)(\mathbb{R}) \mid x \cdot x > 0\}$ (resp. $\bigwedge^2 J(\mathbb{R}) \setminus \{0\}$). There is a canonical choice for C_I and C_J if we specify a connected component of $\mathcal{D}$, as it is explained in [Loo03, Sections 1.1 and 1.2] (in this case, we let Γ be the subgroup of isometries of L which preserve the chosen connected component). Denote by $C_{I,+}$ (resp. $C_{J,+}$) the convex hull of the $\mathbb{Q}$-vectors in $\overline{C}_I$ (resp. $\overline{C}_J$).

Definition 8.57 ([Loo03, Definition 2.1]). The *conical locus* of $\mathcal{D}$ is defined as

$$C(\mathcal{D}) = \coprod_{\substack{W \subset V, \\ W \ \mathbb{Q}\text{-isotropic}}} C_W.$$

Definition 8.58 ([Loo03, Definition 6.1]). An *admissible decomposition* of $C(\mathcal{D})$ is a Γ-invariant locally rational decomposition of $C_{I,+}$ for all $\mathbb{Q}$-isotropic lines I such that, for all $\mathbb{Q}$-isotropic planes J, the support space of $C_{J,+} \subset (I^{\perp}/I)(\mathbb{R})$ (see [Loo03, Lemma-Definition 4.5]) is independent of I when we regard that support space as a subspace of $J^{\perp}$ containing J.

We are ready to define our choice of admissible decomposition of $C(\mathcal{D})$.

Definition 8.59. For any $\mathbb{Q}$-isotropic line I, consider the decomposition of $C_{I,+}$ induced by the mirrors of the reflections with respect to the vectors of square -1 in the hyperbolic lattice $(I^\perp/I)(\mathbb{Z})$ (recall that these vectors are called (-1)-vectors). Let Σ be the decomposition of $C(\mathcal{D})$ induced by these mirrors. This is an admissible decomposition of $C(\mathcal{D})$, as we show in Corollary 8.65 after the considerations in Section 8.6.2.

8.6.2 Study of the decomposition Σ

There are three choices of $\mathbb{Q}$-isotropic lines $I \subset V$ up to isometries of L. In each case, the hyperbolic lattice $(I^\perp/I)(\mathbb{Z})$ is computed in [Oud10, Proposition 7.5]. Let Γ_I be the discrete reflection group generated by the reflections with respect to the (-1)-vectors in $(I^\perp/I)(\mathbb{Z})$.

Even 0-cusp

For I corresponding to the even 0-cusp, we have that $(I^\perp/I)(\mathbb{Z}) \cong \mathbb{Z}^{1,3}$. Running Vinberg's algorithm to determine a fundamental domain for Γ_I (see [Vin75, Section 1] and [Vin75, Theorem 2.6 bis] for the stopping condition), we obtain the *Coxeter diagram* (see [Vin75, Section 1] or [Sca87, Table 2.2.8]) shown on the left in Figure 8.8. For the computations, we chose $(1,0,0,0)$ as our initial vector.

Observation 8.60. There is a bijection between the strata of the boundary of $\overline{M}^\nu_{D_{1,6}}$ containing the even 0-dimensional stratum (see Figure 8.6) and the elliptic and maximal parabolic subdiagrams of the Coxeter diagram up to isometries of $\mathbb{Z}^{1,3}$ (on the right in Figure 8.8).

Remark 8.61. Let J be a $\mathbb{Q}$-isotropic plane containing I. Consider the hyperplanes $v^\perp \subset (I^\perp/I)(\mathbb{R})$ where $v \in (I^\perp/I)(\mathbb{Z})$ is a (-1)-vector and $C_{J,+} \subset v^\perp$. All these hyperplanes intersect C_I nontrivially and their intersection is $C_{J,+}$. Therefore, the support space of $C_{J,+}$ is equal to J when we regard it as a subspace of $J^\perp$ containing J.

Odd 0-cusp of type 1

If I corresponds to the odd 0-cusp of type 1, then $(I^\perp/I)(\mathbb{Z}) \cong \mathbb{Z}^{1,1} \oplus \mathbb{Z}^2(-2)$. We run Vinberg's algorithm to determine a fundamental domain for Γ_I. Choose $v_0 = (1,-1,0,0)$ as initial vector.
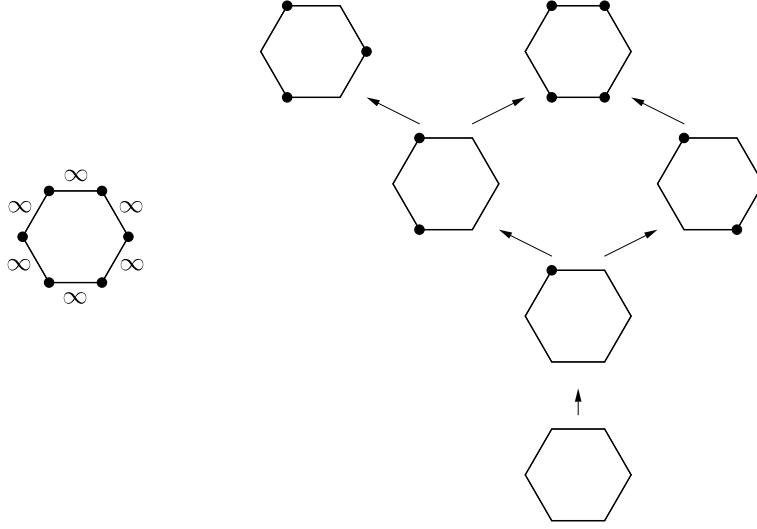
Figure 8.8: On the left, we show the Coxeter diagram for the discrete reflection group Γ_I, where I corresponds to the even 0-cusp. On the right are its elliptic and maximal parabolic subdiagrams, up to isometries of $\mathbb{Z}^{1,3}$

In step 0 of the algorithm, we first determine all the (-1)-vectors which are orthogonal to v_0, and there are no such vectors. In step 1, we determine all the (-1)-vectors x such that $v_0 \cdot x = 1$. These vectors are given by

$$\alpha_{(a,b)} = (a^2 + b^2, 1 - a^2 - b^2, a, b), \text{ with } (a,b) \in \mathbb{Z}^2.$$

At step $n \geq 2$ no new vectors are added. More precisely, if n is even, then there are no (-1)-vectors such that $v_0 \cdot x = n$. If n is odd, let x be a (-1)-vector such that $v_0 \cdot x = n$. Then x is not accepted by the algorithm because it is easy to check that $x \cdot \alpha_{(a,b)} < 0$ if we choose $(a,b) \in \mathbb{Z}^2$ such that $\left| a - \frac{x_3}{n} \right|, \left| b - \frac{x_4}{n} \right| \leq \frac{1}{2}$. Therefore, $\mathbb{Z}^2$ is the set of vertices of the Coxeter diagram for Γ_I, where $(a,b) \in \mathbb{Z}^2$ corresponds to $\alpha_{(a,b)}$. Observe that

$$\alpha_{(a,b)} \cdot \alpha_{(c,d)} = -1 + (a - c)^2 + (b - d)^2.$$

This implies that, given any $(a,b) \in \mathbb{Z}^2$, then

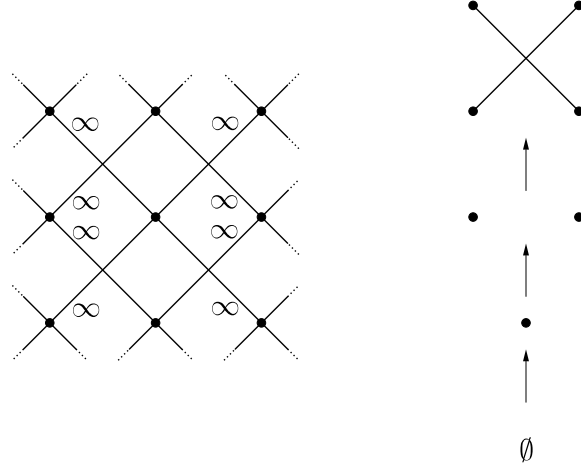- There is no edge between (a,b) and $(a \pm 1, b), (a, b \pm 1)$;

Figure 8.9: On the left, we show part of the Coxeter diagram for Γ_I where we omit the dotted edges. Here I corresponds to the odd 0-cusp of type 1. On the right are its elliptic and maximal parabolic subdiagrams, up to isometries of $\mathbb{Z}^{1,1} \oplus \mathbb{Z}^2(-2)$

- There is an edge labeled by ∞ between (a, b) and $(a \pm 1, b \pm 1), (a \pm 1, b \mp 1)$;

- (a, b) is connected by a dotted edge to any other vertex different from $(a \pm 1, b), (a, b \pm 1), (a \pm 1, b \pm 1), (a \pm 1, b \mp 1)$.

Part of the infinite Coxeter diagram is shown on the left in Figure 8.9. (The computation we just carried out shares some similarities with [Con83].)

Observation 8.62. Also in this case there is a bijection between the strata of the boundary of $\overline{M}^\nu_{D_{1,6}}$ containing the odd 0-dimensional stratum of type 1 (see Figure 8.6) and the elliptic and maximal parabolic subdiagrams of the Coxeter diagram up to isometries of $\mathbb{Z}^{1,1} \oplus \mathbb{Z}^2(-2)$ (on the right in Figure 8.9).

Remark 8.63. Let J be a $\mathbb{Q}$-isotropic plane containing I and consider $C_{J,+}$. Up to isometry there are two possibilities for $C_{J,+}$. Assume that $C_{J,+}$ is the ray generated by $(2, 0, 1, 1)$ in $C_{I,+} \subset (I^\perp/I)(\mathbb{R})$. One can easily see that the support space of $C_{J,+}$ is J (this is similar to what we did in Remark 8.61). If $C_{J,+}$ is the ray generated by $(1, -1, 0, 0)$, then $C_{J,+}$ is not orthogonal to any (-1)-vector in $(I^\perp/I)(\mathbb{Z})$, hence its support space is $J^\perp$. By [Oud10, Lemma 7.2], we have that J corresponds to the even 1-cusp in the first case, and the odd 1-cusp in the second case.

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Odd 0-cusp of type 2

Let U be the unique even unimodular lattice of signature $(1,1)$ (see Examples 2.9 (b)). Then for I corresponding to the odd 0-cusp of type 2, we have that $(I^\perp/I)(\mathbb{Z}) \cong U \oplus \mathbb{Z}^2(-2)$, which is an even lattice. In particular, $(I^\perp/I)(\mathbb{Z})$ does not contain (-1)-vectors. Therefore, the decomposition of $C_{I,+}$ induced by Σ is given by $C_{I,+}$ itself.

Remark 8.64. If J is a $\mathbb{Q}$-isotropic plane containing I, then the support space of $C_{J,+}$ is $J^\perp$.

8.6.3 Conclusions

Corollary 8.65. *The decomposition Σ in Definition 8.59 is an admissible decomposition of $C(\mathcal{D})$, and hence it gives rise to a semitoric compactification $\overline{\mathcal{D}/\Gamma}^\Sigma$.*

Proof. We have that Σ is admissible because Remarks 8.61, 8.63, and 8.64 imply that, for any $\mathbb{Q}$-isotropic plane J, the support space of $C_{J,+}$ is independent from the choice of $\mathbb{Q}$-isotropic line $I \subset J$. $\qquad\square$

Observation 8.66. The Baily-Borel compactification $\overline{\mathcal{D}/\Gamma}^{BB}$ can be thought of as the semitoric compactification of $\mathcal{D}/\Gamma$ associated to the trivial admissible decomposition of $C(\mathcal{D})$ (see [Loo03, Example 6.2]). In particular, by [Loo03, Lemma 6.6] we have a birational morphism $\overline{\mathcal{D}/\Gamma}^\Sigma \to \overline{\mathcal{D}/\Gamma}^{BB}$ which is an isomorphism on $\mathcal{D}/\Gamma$.

Theorem 8.67. *Consider the two birational modifications $\overline{M}^\nu_{D_{1,6}} \to \overline{\mathcal{D}/\Gamma}^{BB} \leftarrow \overline{\mathcal{D}/\Gamma}^\Sigma$ of the Baily-Borel compactification of $\mathcal{D}/\Gamma$. Then these are isomorphic in a neighborhood of the preimage of the odd 1-cusp of type 2. Moreover, there is an intersection-preserving bijection between the boundary strata of the two compactifications which also preserves the dimensions of the strata.*

Proof. The statement of the theorem summarizes Observations 8.55, 8.60, and 8.62. $\qquad\square$

Conjecture 8.68. *The KSBA compactification $\overline{M}^\nu_{D_{1,6}}$ and the semitoric compactification $\overline{\mathcal{D}/\Gamma}^\Sigma$ of $\mathcal{D}/\Gamma$ are isomorphic.*

Remark 8.69. To prove Conjecture 8.68 the first step would be to show that the birational map $\overline{M}^\nu_{D_{1,6}} \dashrightarrow \overline{\mathcal{D}/\Gamma}^\Sigma$ is an isomorphism in a neighborhood of the preimage of the even 0-cusp, which is where we have a toroidal behavior. To prove this, we plan to use the standard extension criterion

used for instance in [AB12, CMGHL15], but adapted to our specific situation. For this adaptation, the results in [Laz08] will be helpful to us.

8.7 Final remarks

Remark 8.70. In the case considered in this dissertation, it is remarkable that the theory of stable pairs produced a compactification $\overline{M}^{\nu}_{D_{1,6}}$ which has a toroidal behavior at the even 0-dimensional stratum, is isomorphic to the Baily-Borel compactification at the odd 0-dimensional stratum of type 2, and is a "mixture" of toroidal and Baily-Borel at the odd 0-dimensional stratum of type 1. In particular, this illustrates the behavior that should be expected in general when using stable pairs to compactify moduli spaces.

Remark 8.71. Associated to our 4-dimensional family of $D_{1,6}$-polarized Enriques surfaces there is a 4-dimensional family of K3 surfaces given by the $\mathbb{Z}_2^3$-covers $X \to \mathrm{Bl}_3 \mathbb{P}^2$ branched along three pairs of lines (see also Remark 8.15). One can also consider the minimal desingularization of the $\mathbb{Z}_2$-cover of $\mathbb{P}^2$ branched along six lines, which gives rise to a second 4-dimensional family of K3 surfaces. These two families are distinct and they are related as follows. We know from Remark 8.15 that X can be viewed as a $(2,2,2)$ hypersurface in $(\mathbb{P}^1)^3$. The group action $\mathbb{Z}_2^3 \curvearrowright X$ can be realized as the action of the restriction to X of the involutions of $(\mathbb{P}^1)^3$ given in an affine patch by $(x,y,z) \mapsto ((-1)^i x, (-1)^j y, (-1)^k z)$, $i,j,k \in \{0,1\}$. The subgroup $G < \mathbb{Z}_2^3$ corresponding to $i + j + k = 2$ is isomorphic to $\mathbb{Z}_2^2$ and acts symplectically on X. This means that the minimal resolution of X/G is again a K3 surface, which in this case belongs to the second 4-dimensional family of K3 surfaces above.

Remark 8.72. Degenerations of six undistinguished lines in $\mathbb{P}^2$ with weight $\frac{1}{2} + \epsilon$ were considered in [Ale15, Section 6.2.1]. However, our case is different for the following reasons:

- First of all, we already explained in Remark 8.12 that our situation cannot be reduced to considering line arrangements in $\mathbb{P}^2$;

- We keep track of three pairs of lines, and this is necessary to reconstruct the overlying degenerations of Enriques surfaces;

- Given a degeneration of $\mathbb{P}^2$ with the six lines, it is not clear how to immediately obtain the corresponding degeneration of $\mathrm{Bl}_3 \mathbb{P}^2$ and the divisor on it;

- The boundaries of the two compactifications parametrizing in the interior six lines in $\mathbb{P}^2$ and three pairs of lines in $\mathrm{Bl}_3 \mathbb{P}^2$ respectively are distinct, and the reason is the following. A $D_{1,6}$-polarized Enriques surface S can be realized as the minimal desingularization of the $\mathbb{Z}_2^2$-cover of $\mathbb{P}^2$ branched along three pairs of lines. In this way S comes with a natural polarization of degree 4, i.e. the pullback of a line in $\mathbb{P}^2$. However, the polarization coming from the $\mathbb{Z}_2^2$-cover of $\mathrm{Bl}_3 \mathbb{P}^2$ has degree 6 and is given by $E_1 + E_2 + E_3$ (see Observation 8.4).

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