0: Introduction

- **Main Theorem A:** For E_2, KSBA = semitoroidal with explicit fans/chambers.
- **Main Theorem B**: Cusp correspondence involving E_0,E_2, andF_{4, h.e.}$
- **Main Theorem C:** Sterk diagrams are obtained by folding of K3 diagrams
- **Main Theorem D:** IAS for E_2 are obtained from IAS^2s for K3s which are invariant under imposed folding symmetries, so IAS^2s with two commuting involutions.
- **Main Theorem E:** The KSBA stable degenerations at each 0-cusp of E_2 admit explicit descriptions: the irreducible components are ADE surfaces which are determined by the IASs in Theorem D.

1: E_2 setup

- *Definition*: Enriques surfaces, numerical polarizations
- *Proposition*: there is *some* coarse space parameterizing num pol deg 2 Enriques
- *Proposition*: Gritsenko-Hulek = U/G via Horikawa.
- *Proposition*: the KSBA compactification exists and satisfies standard (?) nice properties
- *Remark*: what is known about the relevant stacks?

2: $F_{4,h.e.}$ theory from AE, AET, AEH, but only the KSBA aspects

- *Definition*: K3 surfaces and h.e. K3s
- *Construction*: of $F_{(2,2,0)}$ that uses new symplectic involution machinery
- *Definition*: semitoroidal compactifications
- **Theorem (Cite)**: for $F_{(2,2,0)}$, KSBA = semitoroidal with explicit fans
- **Theorem (Cite):** the maximal KSBA stable degeneration for $F_{(2,2,0)}$ admits an explicit description, the irreducible components are ADE surfaces which are encoded by subdiagrams of Coxeter diagrams at the cusps.

- 3: BB compactifications,
    - *Remark*: basic BB theory, correspondence between isotropic dudes and boundary components, why do we consider BB when we want KSBA
    - *Definition*: cusp diagram, how to read the geometry off of it.
    - *Proposition*. BB for $F_{4,h.e.}$ and its cusp diagram
        - **This is where I believe we need the Coxeter diagrams for the first time!!**

- *Proposition*. BB for E_2 and cusp its diagram
- *Remark/Definition*. introduce unpolarized Enriques surfaces E_0, explain why we need this
- Proposition: BB for E_0 and its cusp diagram
- **Main Theorem B**: Cusp correspondence involving E_0, E_2, and $F_{4,h.e.}$. Which degenerations are disc type vs $\mathbb{RP}^2$ type.
  - *Proof sketch*: the main preserved invariant is divisibility, the result follows from a computation.

- 4: Folding theory
  - *Definition*: Coxeter diagrams, node/edge conventions and how to read the diagram
  - *Definition*. general folding of diagrams
  - *Example*. A_7 folds into B_4
  - **Main Theorem C:** Sterk diagrams are obtained by folding of K3 diagrams
    - *Proof sketch*: we describe the symmetries $\mathbf{r}, \mathbf{h}, \mathbf{d}$ of the Coxeter diagrams. Compute the automorphism group of the diagram. Explicitly carry out folding procedure by finding some invariant roots, e.g. sums of exchanged roots, and then searching for enough remaining roots to complete the Coxeter diagram.

- 5: IAS for each of the 5 Sterk cusps of E_2
  - **Main Theorem D:** IAS for E_2 are obtained from IAS^2s for K3s which are invariant under imposed folding symmetries, so IAS^2s with two commuting involutions.
    - *Proof*: hopefully this follows from the lemma below.
  - *Definition (cite)*: basic review of IAS theory. Or just punt to appendix?
  - **Lemma**: a degeneration in E_2 of $\mathbb{RP}^2$ type is described by an integral affine structure on $\mathbb{RP}^2$ with charge 12.
    - *Proof sketch*: by Phil's thesis, an IAS^2 gives a Kulikov surface. Take a double cover of the SNC surface Y_0 from this result to get an X_0 with a fixed point free involution. Now prove that we can choose a smoothing that preserves the involution.
  - *Construction*: Describe the parameter choices ℓ_i giving 10-dim families of IAS^2s
  - *Construction*: Describe the toric varieties (maybe hard, skip?)
  - *Construction*: Describe the Symington polytopes P (with nontoric blowups) and verify the IAS^2s have charge 24

- 6: KSBA Stable Models for E_2
  - **Main Theorem E:** The maximal KSBA stable degenerations at each 0-cusp of E_2 admit an explicit description: the irreducible components are ADE surfaces determined by the IASs in **Theorem D**. Which ones are pumpkin vs smashed pumpkin type. Enumerated tables of all ADE surface possibilities.
    - *Proof*: follows from an analysis/description of the KSBA stable model gotten by collapsing the hemispheres in the IAS^2
  - *Lemma:* explicitly describe the equatorial behavior of each IAS, including the ramification divisor

    - **Main Theorem A:** For E_2, KSBA = semitoroidal with explicit fans/chambers (e.g. a subdivision of the Coxeter chambers into N subchambers)
        - *Proof sketch*: fan vs semifan follows from a finiteness condition on a subdiagram of the Coxeter diagram. What the actual fan/semifan is: an analysis that I do not know how to carry out yet.

- 7: Appendix
    - *Background*: semitoroidal compactifications
    - *Background*: Kulikov models
    - *Background*: IAS theory
    - *Background*: reflection groups as a theoretical background to Coxeter-Vinberg diagrams
    - *Background:* coming off tilings of hyperbolic spaces (coming from translating a hyperbolic polytope around by a Weyl group) as a way to canonically produce semifans

Big theoretical questions to address somewhere in remarks:

- What is this pseudo-algorithm we use on a Coxeter diagram at a 0-cusp to cook up an IAS^2?
- How does an IAS^2 explicitly encode a Kulikov model?
- How do we read the maximal KSBA stable degeneration from the Coxeter diagram?
- How do we read the fan data for a semitoroidal compactification off of the IAS data?