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Shigeyuki Kondō

*K*3 Surfaces

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To Mizuho

Preface to the English translation

This book is an extended English version of the author's book "$K3$ surfaces" in Japanese, which forms volume 5 of the series "Suugaku no Kagayaki", published in 2015 by Kyoritsu Shuppan, Japan. Chapters 0–10 are an English translation of the above book by the author himself. Chapters 11 and 12 are new and added to this English version by the author.

In Chapter 11 we discuss finite groups of symplectic automorphisms of $K3$ surfaces. In particular we study a relation, discovered by S. Mukai, between these groups and the Mathieu group, a finite sporadic simple group. We give a lattice-theoretic proof of Mukai's result by using the classification of even definite unimodular lattices of rank 24, called Niemeier lattices.

In Chapter 12, we study the Kummer surface associated with the Jacobian of a curve of genus 2. It is known that the automorphism group of the Kummer surface is discrete and infinite. During the 19th century and at the beginning of the 20th century, geometers constructed many automorphisms of the Kummer surface. We show that these generate the automorphism group of a generic Kummer surface. Roughly speaking, we realize 32 non-singular rational curves on the Kummer surface forming a beautiful configuration, called the (16_6)-configuration, in terms of 32 octads appearing in the Steiner system. Here we use essentially a description of the fundamental domain of the reflection group of the even unimodular lattice of signature $(1, 25)$ discovered by J. H. Conway.

In this book we restrict ourselves to complex $K3$ and Enriques surfaces. Moreover, the last three chapters are based on the author's work. References are not complete and are kept to a necessary minimum. We refer the reader to the book on $K3$ surfaces by Huybrechts [Huy1] and a survey paper on Enriques surfaces by Dolgachev [Do3] for the case of positive characteristic and for recent progress.

Finally, Gerard van der Geer kindly encouraged the author to publish an English version and, moreover, he read the manuscript and pointed out many misprints. Matthias Schütt also read the manuscript and suggested several improvements to the author. The editors of the series "Suugaku no Kagayaki" and Kyoritsu Shuppan willingly accepted the English translation. In taking this opportunity the author would like to thank them heartily.

Preface

The main theme of this book is the Torelli-type theorem for $K3$ surfaces. A $K3$ surface is a connected compact 2-dimensional complex manifold that is simply connected and whose canonical line bundle is trivial. It is difficult to relate the name "$K3$ surface" to its definition, but A. Weil invented the name, with $K3$ resulting from the initials of Kummer, Kähler, Kodaira, as well as from a mountain in Karakoram called K2, which was unclimbed and mysterious at that time. The most famous example is the Kummer surface, discovered in the 19th century. In the case of elliptic curves, that is, 1-dimensional compact complex tori, the period of an elliptic curve determines its isomorphism class. For $K3$ surfaces one can define the notion of periods, and the claim that the isomorphism class of a $K3$ surface is determined by its period is the Torelli-type theorem for $K3$ surfaces. In the 1970s, the Torelli-type theorem was proved and then many results were established using this theorem. Since the 1990s, $K3$ surfaces have become of interest in mathematical physics. Nowadays, $K3$ surfaces are still mysterious: for example, a few years ago physicists discovered Mathieu Moonshine which claims a relation between the elliptic genus of $K3$ surfaces and one of the sporadic finite simple groups called the Mathieu group. The theory of lattices and their reflection groups is necessary to study $K3$ surfaces. In this book we start to explain these notions and give a proof of the Torelli-type theorem and its applications. We hope that this book shows the interplay between several sorts of mathematics. In particular, the author would be happy if this book were to prove helpful to the research of young people.

Finally, Hisanori Ohashi and the referee pointed out many misprints in the manuscript and gave the author useful comments. The author would like to take this opportunity to thank them.

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Introduction

In the following we give a brief outline of the book. For simplicity, we call a 1-dimensional compact connected complex manifold a curve. Curves are classified by their genus, and a curve of genus 0 is a projective line $\mathbb{P}^1$, and a curve of genus 1 is an elliptic curve. There exist g linearly independent holomorphic 1-forms on any curve of genus g. By taking period integrals of them we associate a g-dimensional abelian variety (a projective g-dimensional complex torus) called the Jacobian variety, and the Torelli theorem for curves claims that if their Jacobian varieties are isomorphic then the original curves are isomorphic. There exists a unique non-zero holomorphic 1-form on an elliptic curve up to a constant; on the other hand, any $K3$ surface has a unique non-zero holomorphic 2-form up to a constant. In this sense, $K3$ surfaces can be seen as a 2-dimensional generalization of elliptic curves. An elliptic curve can be realized as a cubic curve in a projective plane $\mathbb{P}^2$ by Weierstrass's $\wp$-function. On the other hand, a non-singular quartic surface in $\mathbb{P}^3$ is an example of a $K3$ surface. In the 19th century, E. Kummer discovered a $K3$ surface called the Kummer quartic surface. A Kummer quartic surface is realized as the quotient surface of the Jacobian of a curve of genus 2 and has 16 rational double points of type A_1. They form a beautiful microcosm with a line geometry in $\mathbb{P}^3$, but also are important in a proof of the Torelli-type theorem. At the present time a Kummer surface means the minimal model of the quotient surface of a 2-dimensional complex torus by the (-1)-multiplication. The set of isomorphism classes of Kummer surfaces has 4-dimensional parameters, but that of Kummer quartic surfaces has only 3-dimensional parameters. A difference from the case of curves is the existence of non-projective surfaces. For example, the existence of $K3$ surfaces not realized as quartic surfaces results from the following argument. Let V be the vector space of homogeneous polynomials of degree 4 in 4 variables. By counting monomials we know that V has dimension 35. Each point in the projective space $\mathbb{P}(V)$ defines a quartic surface and the set of isomorphism classes of quartic surfaces has $34 - \dim \mathrm{PGL}(4, \mathbb{C}) = 19$ parameters by considering the action of projective transformations. On the other hand, the isomorphism classes of all $K3$ surfaces have 20-dimensional parameters by deformation theory. Roughly speaking, the set of isomorphism classes of $K3$ surfaces is a 20-dimensional connected complex manifold in which there are countably many 19-dimensional submanifolds, each of which is the set of polarized $K3$ surfaces parametrized by an even positive integer called the degree of polarization. For example, a non-singular quartic surface has a polarization of degree 4. In the case of complex tori, they can be constructed concretely as the quotient of a complex vector space by a discrete subgroup, but it is difficult to construct a general projective $K3$ surface. This causes a difficulty in studying $K3$ surfaces uniformly.

Now we briefly recall the theory of periods of elliptic curves to understand the case of $K3$ surfaces. We denote by $\mathrm{Im}(z)$ the imaginary part of a complex number z. To each τ in the upper half-plane $H^+ = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$, we associate the subgroup $\mathbb{Z}+\mathbb{Z}\tau$ of the additive group $\mathbb{C}$ generated by $\{1, \tau\}$. The quotient group $E = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ naturally has the structure of a 1-dimensional compact complex manifold, which is called an elliptic curve. A holomorphic 1-form dz on $\mathbb{C}$ is invariant under translation and hence induces a nowhere-vanishing holomorphic 1-form ω_E on E. We remark that ω_E is unique up to a constant. On the other hand, E is a 2-dimensional real torus $S^1 \times S^1$ and hence $H_1(E, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Now let us fix a basis $\{\gamma_1, \gamma_2\}$ of $H_1(E, \mathbb{Z})$. Then the integrals

$$\int_{\gamma_1} \omega_E, \quad \int_{\gamma_2} \omega_E$$

are linearly independent over $\mathbb{R}$ and therefore, if necessary by changing γ_1 and γ_2, we may assume

$$\mathrm{Im}\left(\int_{\gamma_1} \omega_E \Big/ \int_{\gamma_2} \omega_E\right) > 0.$$

Then by defining

$$\tau_E = \left(\int_{\gamma_1} \omega_E\right) \Big/ \left(\int_{\gamma_2} \omega_E\right)$$

we have a point τ_E in H^+. Here we remark that τ_E is independent of the choice of ω_E, that is, the constant multiplication, because we take the ratio of two integrals. On the other hand, τ_E depends on the choice of a basis $\{\gamma_1, \gamma_2\}$. In fact, for another basis $\{\gamma_1', \gamma_2'\}$, let

$$\tau_E' = \left(\int_{\gamma_1'} \omega_E\right) \Big/ \left(\int_{\gamma_2'} \omega_E\right) \in H^+$$

and let

$$\gamma_1' = a\gamma_1 + b\gamma_2, \quad \gamma_2' = c\gamma_1 + d\gamma_2 \quad (a, b, c, d \in \mathbb{Z})$$

be the change of basis; then we have

$$\tau_E' = \frac{a\tau_E + b}{c\tau_E + d}.$$

The matrix of a base change is contained in $\mathrm{GL}(2, \mathbb{Z})$, and the conditions $\mathrm{Im}(\tau_E) > 0$ and $\mathrm{Im}(\tau_E') > 0$ imply that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$. Thus the changing of a basis corresponds to the action of an element of $\mathrm{SL}(2, \mathbb{Z})$ on the upper half-plane H^+

by a linear fractional transformation. After all, the point τ_E in the quotient space $H^+/\mathrm{SL}(2,\mathbb{Z})$ is independent of the choice of holomorphic 1-forms and a basis of the homology group, and depends only on the isomorphism class of E. We call τ_E the period of the elliptic curve E and the upper half-plane the period domain. Thus the set of isomorphism classes of elliptic curves (called the moduli space of elliptic curves) bijectively corresponds to $H^+/\mathrm{SL}(2,\mathbb{Z})$ by sending an elliptic curve to its period. This is an outline of the period theory of elliptic curves.

Now we return to the case of $K3$ surfaces. Let X be a $K3$ surface on which there exists a unique nowhere-vanishing holomorphic 2-form ω_X up to a constant. By integrating it over the second homology group $H_2(X,\mathbb{Z})$,

$$\omega_X\colon H_2(X,\mathbb{Z}) \to \mathbb{C}, \quad \gamma \to \int_\gamma \omega_X,$$

ω_X can be considered an element in $H^2(X,\mathbb{C})$, which is the period of the $K3$ surface X. The second cohomology group $H^2(X,\mathbb{Z})$ is a free abelian group of rank 22, and together with the cup

$$\langle\ ,\ \rangle\colon H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z}) \cong \mathbb{Z},$$

$H^2(X,\mathbb{Z})$ has the structure of a lattice. In this book a lattice means a pair of a free abelian group of finite rank and an integral-valued non-degenerate symmetric bilinear form on it. The period satisfies

$$\langle \omega_X, \omega_X\rangle = \int_X \omega_X \wedge \omega_X = 0, \quad \langle \omega_X, \bar{\omega}_X\rangle = \int_X \omega_X \wedge \bar{\omega}_X > 0,$$

which is called the Riemann condition. The topology of $K3$ surfaces is unique and is independent on complex structures. In particular, the isomorphism class of the lattice $H^2(X,\mathbb{Z})$ is independent on X and hence is denoted by L. Now we define

$$\Omega = \{\omega \in \mathbb{P}(L\otimes\mathbb{C}) : \langle\omega,\omega\rangle = 0,\ \langle\omega,\bar{\omega}\rangle > 0\},$$

which is called the period domain of $K3$ surfaces and corresponds to the upper half-plane of elliptic curves (here, for simplicity, we use the same symbol ω for a point in $L\otimes\mathbb{C}$ and its image in $\mathbb{P}(L\otimes\mathbb{C})$). Since L has rank 22, Ω is a 20-dimensional complex manifold. An isomorphism

$$\alpha_X\colon H^2(X,\mathbb{Z}) \to L$$

of lattices is called a marking for X and the pair (X,α_X) a marked $K3$ surface. To a marked $K3$ surface we associate a point $\alpha_X(\omega_X) \in \Omega$. By considering the projective space, this is independent of the choice of holomorphic 2-forms.

As in the case of elliptic curves, to get the period independent of the choice of α_X we need to take the quotient of Ω by the automorphism group $\mathrm{O}(L)$ of L, but the quotient space $\Omega/\mathrm{O}(L)$ has no complex structure. Therefore, we define the period only for marked $K3$ surfaces. And we can also define the period of a family of complex analytic surfaces $\pi\colon \mathcal{X} \to B$ which is a smooth deformation of a $K3$ surface. Here $\mathcal{X}$, B are complex manifolds, the fibers of π are $K3$ surfaces, and the fiber over the base point $t_0 \in B$ is the given $K3$ surface X. We may assume that B is a neighborhood or a germ at t_0. Moreover, we assume that B is contractible. Then a marking α_X of X induces a marking of every fiber simultaneously, and hence gives an associated holomorphic map

$$\lambda\colon B \to \Omega.$$

The map λ is called the period map for a family π. We have a map from the set of isomorphism classes of marked $K3$ surfaces to Ω by associating their periods, which is called the period map too. When we discuss the local isomorphism of the period map we use the former sense, and when discussing the surjectivity of the period map we use the period map in the latter sense.

Now consider two marked $K3$ surfaces whose periods coincide. Then the Torelli-type theorem for $K3$ surfaces answers the question of when the isomorphism

$$(\alpha_{X'})^{-1} \circ \alpha_X\colon H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z}) \tag{0.1}$$

of lattices preserving the classes of their holomorphic 2-forms is induced from an isomorphism between X and X'. If an isomorphism is induced from an isomorphism between complex manifolds, then it preserves the classes of Kähler forms. The Torelli-type theorem claims that the converse, that is, "an isomorphism of lattices preserving holomorphic 2-forms is induced from an isomorphism of complex manifolds if and only if it preserves the classes of Kähler forms", is true. In this book we assume the fact, proved by Siu, that every $K3$ surface is Kähler. We remark that all Kähler forms form a subset of $H^2(X,\mathbb{R})$, called the Kähler cone, which is a fundamental domain for an action of some reflection group on a cone, called the positive cone of the $K3$ surface. Preserving Kähler classes is nothing but preserving the Kähler cone.

Next we discuss the periods of projective $K3$ surfaces. The pair (X,H) of a projective $K3$ surface X and a primitive ample divisor H with $H^2 = 2d$ is called a polarized $K3$ surface of degree $2d$. Here H is called primitive if the quotient module $H^2(X,\mathbb{Z})/\mathbb{Z}H$ has no torsion. It follows from lattice theory that a primitive element of L with norm $2d$ is unique up to the action of the automorphism group $\mathrm{O}(L)$ of L. Therefore, for a fixed primitive element $h \in L$ with $\langle h,h\rangle = 2d$, we can take an isomorphism $\alpha_X\colon H^2(X,\mathbb{Z}) \to L$ satisfying $\alpha_X(H) = h$. On the other hand, ω_X is perpendicular to any classes represented by curves. In particular $\langle \omega_X, H\rangle = 0$. Thus we define

$$L_{2d} = \{x \in L : \langle x, h\rangle = 0\},$$

$$\Omega_{2d} = \{\omega \in \mathbb{P}(L_{2d} \otimes \mathbb{C}) : \langle \omega, \omega\rangle = 0,\ \langle \omega, \bar{\omega}\rangle > 0\},$$

and then associate a pair (X, H, α_X) to $\alpha_X(\omega_X) \in \Omega_{2d}$. Since L_{2d} has rank 21, the set Ω_{2d} is a 19-dimensional complex manifold. The group Γ_{2d} of isomorphisms of the lattice L fixing h acts on Ω_{2d} properly discontinuously and hence the quotient Ω_{2d}/Γ_{2d} has the structure of a complex analytic space. This follows from the fact that the lattice has the signature $(2, 19)$ and hence the associated Ω_{2d} has the structure of a bounded symmetric domain (more precisely, a disjoint union of two bounded symmetric domains). We note that the upper half-plane H^+ is the simplest example of a bounded symmetric domain. We may conclude that we can define the map from the set of isomorphism classes of polarized $K3$ surfaces of degree $2d$ to Ω_{2d}/Γ_{2d}, called the period map for polarized $K3$ surfaces, and the Torelli-type theorem for polarized $K3$ surfaces claims the injectivity of this map. In this case, if the images of two polarized $K3$ surfaces under the period map coincide, then there exists an isomorphism (0.1) of lattices preserving their periods and ample classes, and in particular preserving Kähler classes, and hence the proof of the Torelli-type theorem is reduced to the case of Kähler $K3$ surfaces.

The proof of the Torelli-type theorem consists of special and peculiar arguments. First, the local isomorphism of the period map is proved by deformation theory of complex structures. On the other hand, any Kummer surface is the quotient of a complex torus, and the complex torus can be reconstructed from the period of the Kummer surface. Then the Torelli-type theorem for Kummer surfaces follows from the Torelli theorem for complex tori. Moreover, it is proved that the period points of Kummer surfaces are dense in the period domain Ω. Finally, one can prove the Torelli-type theorem for the general case by using a density argument and the Torelli-type theorem for Kummer surfaces. This is an outline of the proof.

On the other hand, the proof of the surjectivity of the period map depends on a result of the Calabi conjecture. In the case of projective $K3$ surfaces there is another proof that uses degeneration. In this book we give only a brief outline of the surjectivity of the period map.

As we will mention in some history in Remark 0.1, the Torelli-type theorem for projective $K3$ surfaces was established first, then the one for Kähler $K3$ surfaces, and it was later that the surjectivity of the period map and finally the Kählerness of $K3$ surfaces were proved. In this book we will carry out the argument under the assumption that any $K3$ surface is Kähler.

The above is the main theme of this book, but concrete geometric examples are only Kummer surfaces because we treat analytic $K3$ surfaces mainly. Therefore we will consider Enriques surfaces and plane quartic curves in the final two chapters. An Enriques surface is a non-rational algebraic surface with vanishing geometric and arithmetic genus, discovered by F. Enriques, a member of the Italian school of algebraic geometry. Any Enriques surface is algebraic and its Picard number is 10, and hence it contains many curves, and various constructions by a projective geometry are known. A $K3$ surface appears as the universal covering (the covering degree is 2) of an Enriques surface. In other words, any Enriques surface can be defined as the quotient surface of a $K3$ surface by a fixed-point-free automorphism of order 2. In the case of polarized $K3$ surfaces we fix a sublattice $\mathbb{Z}H$ of rank 1 in $H^2(X,\mathbb{Z})$, and in the case of Enriques surfaces we will fix a sublattice of rank 10, which might be a typical example of a lattice polarized $K3$ surface. In this book, as applications of the Torelli-type theorem for $K3$ surfaces, we prove the Torelli-type theorem for Enriques surfaces, and mention the automorphism groups of Enriques surfaces and various concrete constructions of Enriques surfaces. In Chapter 9 we consider, as a topic, Reye congruence associated with a line geometry which was studied in the later half of the 19th century and the beginning of the 20th century.

In Chapter 10 we give an application to non-singular plane quartic curves (quartic curves in $\mathbb{P}^2$). Plane quartics are non-hyperelliptic curves of genus 3 and their Jacobian varieties are 3-dimensional principally polarized abelian varieties. As a higher-dimensional analogue of the quotient space $H^+/\mathrm{SL}(2,\mathbb{Z})$ in the theory of elliptic curves, the quotient space $\mathfrak{H}_3/\mathrm{Sp}_6(\mathbb{Z})$ of the 3-dimensional Siegel upper half-space $\mathfrak{H}_3$ by the symplectic group $\mathrm{Sp}_6(\mathbb{Z})$ is the set of isomorphism classes of 3-dimensional principally polarized abelian varieties (called the moduli space). Since the Torelli theorem for curves implies the injectivity of the map that associates to a curve its Jacobian, and both the moduli space of plane quartic curves and $\mathfrak{H}_3/\mathrm{Sp}_6(\mathbb{Z})$ have the same dimension 6, the moduli space of plane quartics and $\mathfrak{H}_3/\mathrm{Sp}_6(\mathbb{Z})$ are birational. In this book we associate a $K3$ surface, instead of the Jacobian, with a plane quartic. To the defining equation $f(x,y,z) = 0$ of a plane quartic where f is a homogeneous polynomial of degree 4, we associate the quartic surface in $\mathbb{P}^3$ defined by $t^4 = f(x,y,z)$ where t is a new variable. The main topic in the Chapter 10 is, by using the Torelli-type theorem for $K3$ surfaces, to show that the moduli space of plane quartics is birationally isomorphic to the quotient space of a 6-dimensional complex ball by a discrete group. Moreover, we will discuss del Pezzo surfaces of degree 2 and a root system of type E_7 which are deeply related to plane quartics.

Lattice theory is necessary to discuss the Torelli-type theorem for $K3$ surfaces. First of all, we give preliminaries from lattice theory in Chapter 1. In Chapter 2 we study reflection groups and their fundamental domains. In Chapter 3 we introduce the classification of complex analytic surfaces and also the classification of singular fibers of elliptic surfaces. We give fundamental properties of $K3$ surfaces and examples (such as Kummer surfaces) of $K3$ surfaces in Chapter 4, and the Torelli theorem for 2-dimensional complex tori is proved. It will be used to prove the Torelli-type theorem for Kummer surfaces. Chapter 5 is devoted to introducing bounded symmetric domains of type IV, a higher-dimensional generalization of the upper half-plane, and then to introducing deformation theory of compact complex manifolds. This theory will be necessary for discussing the local isomorphicity of the period map of $K3$ surfaces. In Chapter 6 we give an explicit formulation of the Torelli-type theorem and its proof, and in Chapter 7 we explain the surjectivity of the period map. In Chapter 8 we give a couple of applications of the Torelli-type theorem to automorphisms of $K3$ surfaces. In Chapter 9 we introduce periods of Enriques surfaces, automorphism groups, and concrete examples. Chapter 10 is devoted to introducing plane quartic curves and related del Pezzo surfaces, and then giving a description of the moduli space of plane quartics as a complex ball quotient.

For the Torelli-type theorem for $K3$ surfaces, in addition to the original papers due to Piatetskii-Shapiro, Shafarevich [PS] and Burns, Rapoport [BR], we refer mostly to two books: the seminar note in French edited by Beauville [Be3] and the book by Barth, Hulek, Peters, Van de Ven [BHPV]. The references for algebraic and complex analytic surfaces are the articles Shafarevich [Sh], Kodaira [Kod1], [Kod2], Morrow, Kodaira [MK] and Beauville [Be1], and for the Torelli-type theorem for Enriques surfaces Namikawa [Na2]. The references are not complete and are kept to a necessary minimum. Of course this book does not cover all research on $K3$ surfaces. Topics not mentioned in this book include moduli spaces of vector bundles on a $K3$ surface and the Fourier–Mukai transform, Kähler symplectic manifolds which are higher-dimensional analogues of $K3$ surfaces, the case of positive characteristic and application to complex dynamical systems.

Remark 0.1. We summarize some history concerning the Torelli-type theorem for $K3$ surfaces. Weil [We] invented the name $K3$ surface, and thus $K3$ resulted from the initials of the three mathematicians Kummer, mentioned above, E. Kähler, and K. Kodaira, as well as from the mountain K2 located in Karakoram range, the second-highest mountain in the world (8611 m), which was unclimbed at that time (Weil's original is "ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cashemire"). Weil, together with A. Andreotti, proposed periods of $K3$ surfaces. Kodaira extended the classification of algebraic surfaces due to the Italian school to the case of complex analytic surfaces, and then established the local Torelli theorem for $K3$ surfaces (however, in [Kod2] giving the proof of this theorem, Kodaira mentioned that the local Torelli theorem is due to Andreotti and Weil). Moreover, Kodaira showed the density of periods of $K3$ surfaces with the structure of elliptic fibration in the period domain of $K3$ surfaces, and, as its application, he proved that any $K3$ surfaces are deformation equivalent and, in particular, all $K3$ surfaces are diffeomorphic.

Under the situation above, Piatetskii-Shapiro, Shafarevich [PS] had succeeded in proving the Torelli-type theorem for projective $K3$ surfaces. This was around 1970. Right after that, Burns, Rapoport [BR] succeeded in proving the Torelli-type theorem for Kähler $K3$ surfaces, not just projective ones. However, it remained open whether all $K3$ surfaces are Kähler or not. On the other hand, the surjectivity of the period map was a big remaining problem. In the middle of 1970, Horikawa [Ho1] and Shah [Sha] proved independently the surjectivity of the period map for polarized $K3$ surfaces of degree 2 by using geometric invariant theory. Right after that, Kulikov [Ku1], [Ku2], a member of the Shafarevich school, proved the surjectivity of the period map for projective $K3$ surfaces by classifying degenerations of $K3$ surfaces (right after that, Persson, Pinkham [PP] re-proved Kulikov's theorem). On the other hand, at that time Horikawa [Ho2] gave a proof of the Torelli-type theorem for Enriques surfaces. The proof of the surjectivity of the period map for the general case, not just for projective $K3$ surfaces, was given by Todorov [To] around 1980. Thus the Kählerness of $K3$ surfaces remained open and was finally solved by Siu [Si] in the first half of the 1980s.

1

Lattice theory

In this chapter we summarize the lattice theory used in the theory of $K3$ surfaces. First, we start with definitions and give, as an example, a root lattice and then introduce the discriminant quadratic form which is an invariant of even lattices and will be used to construct an overlattice of a given lattice. Finally, we give a classification of indefinite unimodular lattices and a theory of primitive embeddings of a lattice into a unimodular lattice which will be used in this book.

1.1 Basic properties

1.1.1 Definitions and examples. Let L ($\cong \mathbb{Z}^r$) be a free abelian group of rank r. Let

$$\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$$

be an integral-valued *symmetric bilinear form*, that is, the map satisfying

$$\langle x, y\rangle = \langle y, x\rangle, \quad \langle mx + ny, z\rangle = m\langle x, z\rangle + n\langle y, z\rangle$$

for any $x, y, z \in L$, $m, n \in \mathbb{Z}$. A symmetric bilinear form is called *non-degenerate* if $\langle x, y\rangle = 0$ for any $y \in L$ implies $x = 0$. Denote by L^* the dual $\mathrm{Hom}(L, \mathbb{Z})$ of L. For $x \in L$, we define $f_x \in L^*$ by $f_x(y) = \langle x, y\rangle$. Then non-degeneracy of L means that the natural map

$$L \to L^*, \quad x \to f_x \tag{1.1}$$

is injective.

We call the pair $(L, \langle\ ,\ \rangle)$ of L and a non-degenerate symmetric bilinear form a *lattice* of rank r. For simplicity we denote by L the lattice $(L, \langle\ ,\ \rangle)$. Lattices L_1 and L_2 are *isomorphic* if there is an isomorphism between free abelian groups L_1 and L_2 preserving the bilinear forms. An isomorphism of L to itself is called an *automorphism*. We denote by $\mathrm{O}(L)$ the group of automorphisms of L and call it the *orthogonal group*.

We fix a basis $\{e_i\}_{i=1}^r$ of L as a free abelian group. For each $x \in L$, write $x = \sum_i x_i e_i$, $x_i \in \mathbb{Z}$ and also define $a_{ij} = \langle e_i, e_j\rangle \in \mathbb{Z}$. Then $f(x) = \langle x, x\rangle = \sum_{i,j} a_{ij} x_i x_j$

is a quadratic form. It follows from Sylvester's theorem about the signature of a quadratic form that

$$f(x) = t_1^2 + \cdots + t_p^2 - t_{p+1}^2 - \cdots - t_{p+q}^2, \quad p + q = r, \tag{1.2}$$

where $t_1, \ldots, t_{p+q}$ are variables defined over $\mathbb{R}$. The pair (p, q) of integers in equation (1.2) is called the signature of L. And the difference $p - q$ is denoted by $\mathrm{sign}(L)$ which is sometimes called the signature too. The absolute value of the determinant of the matrix $A = (a_{ij})$ is denoted by $d(L)$. We denote $\langle x, x\rangle$ by x^2 and call it the norm of $x \in L$.

Exercise 1.1. Show that $d(L)$ is independent of the choice of a basis of L.

A lattice L is called *positive definite* or *negative definite* if $p = r$ or $q = r$ in equation (1.2), respectively. We call L *definite* if it is positive or negative definite. A lattice L is called *indefinite* if $p > 0$ and $q > 0$, and L is called *unimodular* if $d(L) = 1$.

Exercise 1.2. Show that a lattice L is unimodular if and only if the natural map (1.1) is an isomorphism.

A lattice L is *even* if all diagonal elements of the matrix A are even integers, that is, $\langle x, x\rangle$ is an even integer for any $x \in L$. If L is not even, then L is called an *odd lattice*.

For lattices L, M, we denote by $L \oplus M$ the orthogonal direct sum of L and M, and by $L^{\oplus m}$ the orthogonal direct sum of m-copies of L. For a lattice $(L, \langle\ ,\ \rangle)$ and an integer $m \neq 0$, we denote by $L(m)$ the lattice $(L, m\langle\ ,\ \rangle)$.

A subgroup S of a lattice L is called a *sublattice* of L if S with the restriction of the bilinear form is a lattice. For a sublattice S of L, we define $S^\perp$ by

$$S^\perp = \{x \in L : \langle x, y\rangle = 0\ \forall\, y \in S\}$$

and call it the *orthogonal complement* of S. Since S is non-degenerate, $S \cap S^\perp = \{0\}$ and $S \oplus S^\perp$ is a sublattice of L of finite index.

Example 1.3. Denote by $I_\pm$ the lattice of rank 1 with the quadratic form $f(x) = \pm x^2$. Then $I_+^{\oplus p} \oplus I_-^{\oplus q}$ is an odd unimodular lattice of signature (p, q). And we denote by $\langle m\rangle$ the lattice generated by an element of norm m. Then $\langle \pm 1\rangle = I_\pm$.

Example 1.4. We denote by U the lattice of rank 2 defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is an even unimodular lattice of signature $(1, 1)$. And $U(m)$ is a lattice of rank 2 defined by the matrix $\begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$.

A negative definite lattice generated by elements of norm -2 is called a *root lattice*. Usually, a root lattice is defined as a positive definite lattice; however, for convenience of application to algebraic geometry, we take a negative definite one for a root lattice.

Example 1.5. We consider $\mathbb{Z}^{m+1}$ a negative definite lattice $I_-^{\oplus(m+1)}$ given in Example 1.3, and define its sublattice A_m by

$$A_m = \left\{(x_1, \dots, x_{m+1}) \in \mathbb{Z}^{m+1} : \textstyle\sum_{i=1}^{m+1} x_i = 0\right\}.$$

Consider a basis

$$\{e_1 = (1,0,\dots,0),\ e_2 = (0,1,0\dots,0),\ \dots,\ e_{m+1} = (0,\dots,0,1)\}$$

of $\mathbb{Z}^{m+1}$. It is easy to see that A_m is of rank m and $r_i = e_i - e_{i+1}$, $i = 1,\dots,m$ is its basis. Since $r_i^2 = -2$, A_m is a root lattice.

To describe root lattices it is convenient to use *Dynkin diagrams*. In Example 1.5, we represent a vertex as $\circ$ for each r_i, and join two vertices corresponding to r_i and r_j by $\langle r_i, r_j\rangle$-tuple edges. Thus we have the diagram in Figure 1.1, called the Dynkin

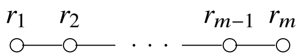


Figure 1.1. Dynkin diagram of type A_m.

diagram of the root lattice A_m. For any root lattice L, we can define the Dynkin diagram similarly. It is known that there exists a basis $r_1,\dots,r_n$ of L satisfying

$$r_i^2 = -2, \quad \langle r_i, r_j\rangle \geq 0,\ \ i \neq j.$$

The elements $\{r_1,\dots,r_n\}$ are called simple roots of L. The basis of A_m given as above is nothing but a basis of simple roots. In the same way as in the case of A_m, the Dynkin diagram can be defined for a basis of simple roots. Simple roots can be described in terms of a fundamental domain (see Remark 2.17). Let Δ be the set of all elements in L of norm -2. Since L is definite, Δ is a finite set (see Exercise 1.6).

For each $r \in \Delta$ we denote by $r^\perp$ the hyperplane in $L \otimes \mathbb{R}$ perpendicular to r. Then each connected component of

$$L \otimes \mathbb{R} \setminus \bigcup_{r\in\Delta} r^\perp$$

is a polyhedron with finitely many hyperplanes (or parts of them) as the boundary. Two connected components adjacent along $r^\perp$ can be interchanged by an isomorphism s_r of L defined by

$$s_r : x \to x + \langle x, r\rangle r,$$

which is called a *reflection*. Thus any connected components are mapped to each other under the action of the group $W(\Delta)$ generated by reflections $\{s_r : r \in \Delta\}$, and in particular isomorphic to each other as polyhedrons. Each connected component is called a fundamental domain of the action of $W(\Delta)$ on $L \otimes \mathbb{R}$. Elements of norm -2 defining hyperplanes of a fundamental domain are simple roots.

Exercise 1.6. Show that Δ is a finite set.

A root lattice is called *irreducible* if its Dynkin diagram is connected. Any root lattice is the orthogonal direct sum of irreducible root lattices, and the Dynkin diagram of an irreducible root lattice is of type A_m given above, or one of the diagrams in Figure 1.2 of type D_n ($n \geq 4$), or of type E_k ($k = 6, 7, 8$) (Proposition 1.12).

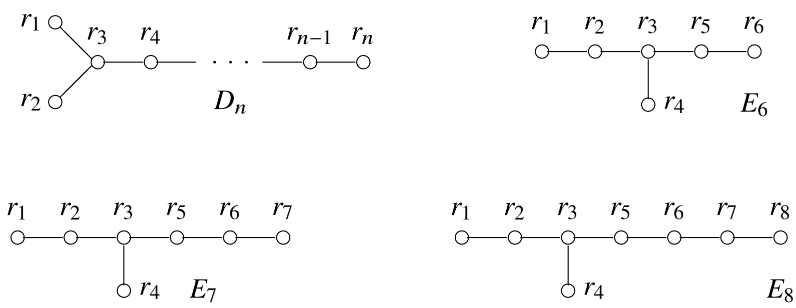


Figure 1.2. Dynkin diagrams.

A direct calculation shows that

$$d(A_m) = m + 1, \quad d(D_n) = 4, \quad d(E_6) = 3, \quad d(E_7) = 2, \quad d(E_8) = 1.$$

In particular, E_8 is unimodular.

Exercise 1.7. Consider $\mathbb{Z}^n$ as the lattice $I_-^{\oplus n}$. Then show that

$$D_n \cong \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 + \cdots + x_n \equiv 0 \bmod 2\}.$$

In the following we show that any connected Dynkin diagram D is one of the Dynkin diagrams from Figures 1.1, 1.2. Let $r_1, \ldots, r_n$ be simple roots corresponding to the vertices of D.

Lemma 1.8. *We have* $\langle r_i, r_j \rangle \leq 1$.

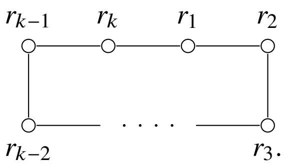
Proof. Assume $\langle r_i, r_j \rangle \geq 2$. Since a root lattice is negative definite and $r_i + r_j \neq 0$, we have $(r_i + r_j)^2 < 0$. On the other hand, by $r_i^2 = -2$, we have

$$(r_i + r_j)^2 = -4 + 2\langle r_i, r_j \rangle \geq 0$$

which is a contradiction. □

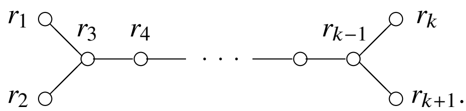
A similar argument shows the following.

Lemma 1.9. *The following diagram does not appear as a subdiagram of* D $(k \geq 3)$:



Exercise 1.10. Prove Lemma 1.9.

Lemma 1.11. *The following diagram does not appear as a subdiagram of* D $(k \geq 4)$:

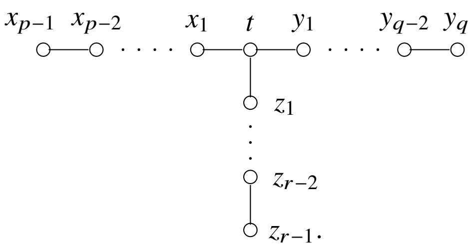


Proof. Assume the existence of this diagram. By considering a non-zero element

$$r_1 + r_2 + r_k + r_{k+1} + 2(r_3 + \cdots + r_{k-1}),$$

we have a contradiction as in the case of Lemma 1.8. □

It follows from Lemmas 1.8, 1.9, 1.11 that a connected Dynkin diagram is as follows:



Now we consider the vector

$$w = t + \frac{1}{p}\sum_{i=1}^{p-1}(p-i)x_i + \frac{1}{q}\sum_{i=1}^{q-1}(q-i)y_i + \frac{1}{r}\sum_{i=1}^{r-1}(r-i)z_i.$$

Since $\langle w, x_i\rangle = \langle w, y_j\rangle = \langle w, z_k\rangle = 0$, we have

$$w^2 = \langle w, t\rangle = -2 + \frac{p-1}{p} + \frac{q-1}{q} + \frac{r-1}{r} = 1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}.$$

Since a root lattice is negative definite and $w \neq 0$, we have the inequality

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1. \tag{1.3}$$

Here we may assume that $p \le q \le r$. If $(p,q,r) = (1,q,r)$, then D is of type A_{q+r-1}. If $(p,q,r) = (2,2,r)$, then D is of type D_{r+2}. Finally, if $(p,q,r) = (2,3,r)$, then it follows from inequality (1.3) that $r = 3,4,5$, and if $(p,q,r) = (2,3,3), (2,3,4), (2,3,5)$, then D is of type E_6, E_7, E_8, respectively. By inequality (1.3), we see that these are the only possible values of (p,q,r). Therefore we have the following proposition.

Proposition 1.12. *A connected Dynkin diagram is of type A_m, D_n, $n \geq 4$ or E_k, $k = 6,7,8$.*

Remark 1.13. The proof of Proposition 1.12 is the one given in Ebeling [E].

1.1.2 Discriminant quadratic forms.

Definition 1.14. Let L be an even lattice. We consider L a subgroup of the dual L^* under the injection (1.1), and denote by A_L the quotient L^*/L. The map

$$q_L\colon A_L \to \mathbb{Q}/2\mathbb{Z}, \quad q_L(x+L) = \langle x, x\rangle \bmod 2\mathbb{Z} \tag{1.4}$$

is called the *discriminant quadratic form* of L. Here $\langle\ ,\ \rangle$ is the bilinear form extended to $L\otimes\mathbb{Q}$. Moreover, by defining

$$b_L\colon A_L \times A_L \to \mathbb{Q}/\mathbb{Z}, \quad b_L(x+L, y+L) = \langle x, y\rangle \bmod \mathbb{Z}, \tag{1.5}$$

we have $q_L(x+y) - q_L(x) - q_L(y) \equiv 2b_L(x,y) \bmod 2\mathbb{Z}$.

We remark here that b_L is determined by q_L. If L is unimodular, then $A_L = 0$ and $q_L = 0$. However, for general even lattices, q_L is, besides the signature, one of the most important invariants.

Example 1.15. Consider the case $L = A_m$. We use the same notation as in Example 1.5. Then

$$\delta = \frac{1}{m+1}\sum_{k=1}^{m} k r_k$$

is contained in L^* and $\delta \bmod L$ is a generator of A_L by the fact $d(A_m) = m+1$. Thus we have $A_L \cong \mathbb{Z}/(m+1)\mathbb{Z}$, $q_L(\delta) = -\frac{m}{m+1}$.

Definition 1.16. Recall that $\mathrm{O}(L)$ is the group of automorphisms of a lattice L which is called the orthogonal group of L. We denote by $\mathrm{O}(q_L)$ the group of automorphisms of the abelian group A_L preserving q_L. Any element in $\mathrm{O}(L)$ induces an automorphism of L^* and hence an automorphism of A_L preserving q_L. In other words, we have a homomorphism of groups $\mathrm{O}(L) \to \mathrm{O}(q_L)$. We denote by $\widetilde{\mathrm{O}}(L)$ the kernel of this homomorphism.

Example 1.17. We use the same notation as in Example 1.5. Vectors in $I_-^{\oplus(m+1)}$ with norm -1 are exactly $\pm e_i$, $i = 1, \dots, m+1$. Since any automorphism preserves the norm of vectors, we see that $\mathrm{O}(I_-^{\oplus(m+1)}) \cong (\mathbb{Z}/2\mathbb{Z})^{m+1} \cdot \mathfrak{S}_{m+1}$. Here the symmetric group $\mathfrak{S}_{m+1}$ of degree $m+1$ acts as permutations of the coordinates, and the group $(\mathbb{Z}/2\mathbb{Z})^{m+1}$ is generated by multiplications of each coordinate by -1.

Exercise 1.18. Calculate the subgroup G of $\mathrm{O}(I_-^{\oplus(m+1)})$ preserving A_m. Moreover, calculate the image of G under the map $\mathrm{O}(A_m) \to \mathrm{O}(q_{A_m})$.

1.1.3 Overlattices.

We introduce a method to construct an even lattice, called an *overlattice*, from a given even lattice. Let L be an even lattice. A subgroup H of A_L is called *isotropic* if $q_L|H = 0$. For an isotropic subgroup H, define

$$L_H = \{x \in L^* : x \bmod L \in H\}.$$

Then $(L_H, \langle\,,\,\rangle)$ is an even lattice because H is isotropic. It follows from the definition that

$$L \subset L_H \subset L_H^* \subset L^*, \quad d(L) = d(L_H)\cdot [L_H : L]^2.$$

Moreover, since any element in L_H^* has integral values with any element in L_H with respect to $\langle\,,\,\rangle$, we have $A_{L_H} \cong H^\perp/H$ and $q_{L_H} = q_L|H^\perp/H$. In general, an even lattice containing L as a sublattice of finite index is called an *overlattice* of L. The lattice L_H is an overlattice of L. Conversely, for any overlattice L' of L, L'/L is an isotropic subgroup of A_L. Thus we have the following.

Theorem 1.19. *The set of overlattices of L bijectively corresponds to the set of isotropic subgroups of A_L.*

Example 1.20. Let e, f be a basis of $L = U(2)$ satisfying $\langle e, e\rangle = \langle f, f\rangle = 0$, $\langle e, f\rangle = 2$. Then $e/2$, $f/2$ generate $A_{U(2)} \cong (\mathbb{Z}/2\mathbb{Z})^2$. The isotropic subgroups are exactly $\langle e/2\rangle$ and $\langle f/2\rangle$, and both the corresponding overlattices are isomorphic to U. Note that the lattice obtained from $U(2)$ by adding the vector $(e+f)/2$ with norm 1 is isomorphic to the odd lattice $I_+ \oplus I_-$.

Exercise 1.21. Show that the lattice E_8 is obtained as an overlattice of D_8.

1.2 Classification of indefinite unimodular lattices

In this section we give the classification of indefinite unimodular lattices. First, we discuss the case of odd unimodular lattices (Theorem 1.22), and then give a property of the signatures of unimodular lattices (Theorem 1.25), and finally classify even unimodular lattices (Theorem 1.27).

1.2.1 Classification of indefinite odd unimodular lattices.

Theorem 1.22. *Let L be an indefinite odd unimodular lattice of signature (p,q). Then*

$$L \cong I_+^{\oplus p} \oplus I_-^{\oplus q}.$$

In particular, the isomorphism class is determined by its signature.

Proof. An element x of L is called *isotropic* if $x^2 = 0$. We assume the following proposition.

Proposition 1.23. *Let L be an indefinite unimodular lattice. Then L contains a non-zero isotropic element.*

We give a related result which is not needed in the proof of Theorem 1.22. For the proofs of these propositions, we refer the reader to Serre [Se].

Proposition 1.24 (Meyer). *Let L be an indefinite lattice with* $\mathrm{rank}(L) \geq 5$ (*not necessarily unimodular*). *Then L has a non-zero isotropic element.*

First of all, there exists a non-zero isotropic element x in L by Proposition 1.23. If necessary by considering x/m $(m \in \mathbb{Z})$ instead of x, we may assume that x is primitive, where a non-zero element x in L is called *primitive* if $x/m \in L$ $(m \in \mathbb{Z})$ implies $m = \pm 1$.

Step (1) There exists $y \in L$ satisfying $\langle x, y\rangle = 1$.
Since L is non-degenerate, the image of the homomorphism $f_x\colon L \to \mathbb{Z}$ is isomorphic

to $m\mathbb{Z}$ ($m > 0$). If $m > 1$, then we have $x/m \in L^* = L$, which contradicts the primitivity of x. Hence $m = 1$, that is, the homomorphism is surjective, and the assertion follows.

Step (2) We may assume that $\langle y, y\rangle$ is odd.
Assume $\langle y, y\rangle$ is even. Since L is an odd lattice, there exists an element $t \in L$ such that $\langle t, t\rangle$ is odd. If we take $y' = t + (1 - \langle x, t\rangle)y$, then $\langle x, y'\rangle = 1$ and $\langle y', y'\rangle$ is odd.

By Step (2), we may assume that $\langle y, y\rangle = 2m + 1$. Now we put

$$e_1 = y - mx, \quad e_2 = y - (m + 1)x,$$

and then we have $\langle e_1, e_1\rangle = 1$, $\langle e_2, e_2\rangle = -1$, $\langle e_1, e_2\rangle = 0$. Therefore, if we denote by L_1 the sublattice of L generated by e_1, e_2, then the following holds:

Step (3) $L_1 \cong I_+ \oplus I_-$.

Step (4) Let $L_2 = L_1^{\perp}$. Then $L \cong L_1 \oplus L_2$.
Since L_1 is non-degenerate, we have $L_1 \cap L_2 = \{0\}$. For any $x \in L$, by considering f_x a function on L_1, the unimodularity $L_1^* \cong L_1$ implies that there exists $x_1 \in L_1$ satisfying

$$f_x(y) = \langle x_1, y\rangle$$

for any $y \in L_1$. Thus $x - x_1$ is an element of L_2, and hence $x \in L$ can be written as a sum $x = x_1 + (x - x_1)$ of elements in L_1 and L_2.

Thus we have a decomposition $L = I_+ \oplus I_- \oplus L_2$. If $L_2 = 0$, then we are done. If $L_2 \neq 0$, then $L_2 \oplus I_+$ or $L_2 \oplus I_-$ is an indefinite odd unimodular lattice and therefore we have finished the proof by induction on the rank of L. □

1.2.2 Signatures of unimodular lattices. Next we introduce a result on the signatures of unimodular lattices. Let L be a unimodular lattice of rank r which might be definite or indefinite. Then $\bar{L} = L/2L$ is an r-dimensional vector space over the finite field $\mathbb{F}_2$. Denote by $\bar{x} \in \bar{L}$ the element corresponding to $x \in L$. The map

$$\langle\ ,\ \rangle\colon \bar{L} \times \bar{L} \to \mathbb{F}_2, \quad \langle \bar{x}, \bar{y}\rangle = \langle x, y\rangle \bmod 2$$

is a non-degenerate quadratic form over $\mathbb{F}_2$. Define a map $f\colon \bar{L} \to \mathbb{F}_2$ by

$$f(\bar{x}) = \langle x, x\rangle \bmod 2.$$

Then we have $f(\bar{x} + \bar{y}) = f(\bar{x}) + f(\bar{y})$, that is, $f \in \mathrm{Hom}(\bar{L}, \mathbb{F}_2) \cong \bar{L}$. Therefore there exists $\bar{u} \in \bar{L}$ satisfying

$$\langle \bar{u}, \bar{x}\rangle = f(\bar{x})$$

for any $\bar{x} \in \bar{L}$. Now we take an element $u \in L$ satisfying $u \bmod 2L = \bar{u}$. Then $\langle u, u\rangle \bmod 8$ is independent of the choice of u. In fact, if $u' = u + 2x$, $x \in L$, then it follows from $\langle \bar{u}, \bar{x}\rangle = f(\bar{x})$ that $\langle u, x\rangle + \langle x, x\rangle$ is an even integer, and

$$\langle u', u'\rangle = \langle u + 2x, u + 2x\rangle = \langle u, u\rangle + 4(\langle u, x\rangle + \langle x, x\rangle) \equiv \langle u, u\rangle \bmod 8.$$

We call u a *characteristic element* of L.

Theorem 1.25. *Let L be a unimodular lattice of signature* (p, q) *and u its characteristic element. Then*

$$\langle u, u\rangle \equiv \operatorname{sign}(L) = p - q \mod 8. \tag{1.6}$$

Proof. Let L_1, L_2 be unimodular lattices with characteristic elements u_1, u_2 respectively. We remark that a characteristic element of the orthogonal direct sum $L_1 \oplus L_2$ is given by $u = u_1 + u_2$. In the case of a unimodular lattice $I_\pm$ of rank 1, its characteristic element u obviously satisfies

$$\langle u, u\rangle \equiv \pm 1 \bmod 8.$$

If L is an indefinite odd unimodular lattice, then we have $L \cong I_+^{\oplus p} \oplus I_-^{\oplus q}$ by Theorem 1.22 and hence assertion (1.6) follows from the above remark. For a general L, note that $L \oplus I_+ \oplus I_-$ is indefinite odd unimodular, and also the contribution of $I_+ \oplus I_-$ to the signature and that of its characteristic element to $\langle u, u\rangle$ in formula (1.6) are both zero. Thus we have proved the theorem. □

Corollary 1.26. *Let L be an even unimodular lattice of signature* (p, q). *Then* $p - q$ *is a multiple of* 8.

Proof. In the case that L is an even lattice, we can take 0 as a characteristic element by the fact $f = 0$, and hence $p - q$ is a multiple of 8 by Theorem 1.25. □

1.2.3 Classification of indefinite even unimodular lattices.

Theorem 1.27. *Let L be an indefinite even unimodular lattice. Then we have the following*:

(1) *In the case* $p \le q$, $L \cong U^{\oplus p} \oplus E_8^{\oplus (q-p)/8}$.

(2) *In the case* $p \ge q$, $L \cong U^{\oplus q} \oplus E_8(-1)^{\oplus (p-q)/8}$.

In particular, the isomorphism class of L is determined by its signature.

Proof. In this case, the arguments of Steps (1)–(4) in the proof of Theorem 1.22 hold too, except that we use U for L_1 instead of $I_+ \oplus I_-$. In fact, in Step (1) we have $\langle y, y\rangle = 2m$ because L is even, and defining $e_1 = x$, $e_2 = y - mx$, we get a lattice generated by e_1, e_2 isomorphic to U. Thus we have an orthogonal decomposition $L = U \oplus L_2$. However, in this case, we cannot apply the induction if L_2 is definite. Instead of induction, we use the following:

Step (5) Let F_1, F_2 be even unimodular lattices with $F_1 \oplus I_+ \oplus I_- \cong F_2 \oplus I_+ \oplus I_-$. Then $U \oplus F_1 \cong U \oplus F_2$.

We fix an isomorphism $f\colon F_1 \oplus I_+ \oplus I_- \to F_2 \oplus I_+ \oplus I_-$. Now consider

$$E_i = \{x \in F_i \oplus I_+ \oplus I_- : \langle x, x\rangle \equiv 0 \bmod 2\}.$$

Then E_i is an even sublattice of $F_i \oplus I_+ \oplus I_-$ of index 2. Since f preserves norm, it induces an isomorphism from E_1 to E_2. On the other hand, by Example 1.20 we see that $E_i \cong F_i \oplus U(2)$ and $E_i^*/E_i \cong (\mathbb{Z}/2\mathbb{Z})^2$. The group E_i^*/E_i has 3 elements of order 2, with one of them non-isotropic and the others isotropic with respect to q_{E_i}. The lattice $F_i \oplus I_+ \oplus I_-$ is obtained from E_i by adding the non-isotropic element, and the overlattice corresponding to a non-zero isotropic vector is isomorphic to $F_i \oplus U$. The isomorphism $f\colon E_1 \to E_2$ induces an isomorphism $f\colon E_1^* \to E_2^*$ which sends isotropic subgroups of E_1^*/E_1 to those of E_2^*/E_2. Therefore we conclude that it gives an isomorphism $f\colon F_1 \oplus U \to F_2 \oplus U$ (which is the end of the proof of Step (5)).

If the signature of $L = U \oplus L_2$ is (p, q), then that of L_2 is $(p-1, q-1)$. Since the isomorphism class of an indefinite odd lattice is determined by its signature (Theorem 1.22), Step (5) implies that the isomorphism class of any indefinite even unimodular lattice is determined by its signature too. On the other hand, $p - q$ is a multiple of 8 (Corollary 1.26), and $U^{\oplus p} \oplus E_8^{\oplus(q-p)/8}$ $(q \geq p)$ and $U^{\oplus q} \oplus E_8(-1)^{\oplus(p-q)/8}$ $(p \geq q)$ are both even unimodular lattices of signature (p, q). Therefore L is isomorphic to $U^{\oplus p} \oplus E_8^{\oplus(q-p)/8}$ or $U^{\oplus q} \oplus E_8(-1)^{\oplus(p-q)/8}$ according to the size of p, q. □

Remark 1.28. In the case of even unimodular definite lattices, the isomorphism class is not determined by its signature (its rank in this case). In the case of rank 8, there exists exactly one isomorphism class E_8, but in the case of rank 16, there are two isomorphism classes, $E_8 \oplus E_8$ and the overlattice of the root lattice D_{16}, and in the case of rank 24, there are 24 isomorphism classes.[1] When the rank is greater than or equal to 32, the classification of such lattices is not known. The contents of this section are the same as in Serre [Se].

[1]Added in English translation: See Theorem 11.2.

1.3 Embeddings of lattices

We introduce basic facts on embeddings of lattices and some results which will be used in later.

1.3.1 Primitive embeddings of even lattices into an even unimodular lattice.

Definition 1.29. Let L, S be lattices. A linear map from S to L preserving the bilinear forms is called an *embedding* of lattices. In this case, by identifying S with the image, S can be considered a sublattice of L. An embedding $S \subset L$ of lattices is called *primitive* if the quotient L/S is torsion-free.

In the following we assume that all lattices are even. Let L be an even unimodular lattice and let S be a primitive sublattice of L. The orthogonal complement $T = S^{\perp}$ of S is also a primitive sublattice of L. The quotient $H = L/(S \oplus T)$ is a finite abelian group, and H is an isotropic subgroup of $A_S \oplus A_T$, with A_S, A_T the discriminant groups, under the inclusion $S \oplus T \subset L \subset S^* \oplus T^*$.

Exercise 1.30. Show that $|H|^2 = d(L) \cdot |H|^2 = d(S) \cdot d(T)$.

Consider the projections

$$p_S \colon A_S \oplus A_T \to A_S, \quad p_T \colon A_S \oplus A_T \to A_T$$

and their restrictions $p_S|H$, $p_T|H$ to H.

Lemma 1.31. *Both maps $p_S|H \colon H \to A_S$, $p_T|H \colon H \to A_T$ are bijective.*

Proof. Assume that $(x \bmod S, y \bmod T) \in H$, $x \in S^*$, $y \in T^*$ is contained in the kernel of $p_S|H$. Then $x \in S$. Since $x + y \in L$, we have $y \in L \cap T^*$, and hence $y \in T$ by the primitivity of T in L. Therefore we have $(x \bmod S, y \bmod T) = 0$ and the injectivity of $p_S|H$. Similarly $p_T|T$ is injective. The assertion now follows from the fact $|A_S| \cdot |A_T| = |H|^2$. □

The map

$$\gamma_{ST} = p_T \circ (p_S|H)^{-1} \colon A_S \to A_T \tag{1.7}$$

is an isomorphism of abelian groups. For $(x \bmod S, y \bmod T) \in H$, the fact that $x + y$ is in L implies that $x^2 + y^2 \in 2\mathbb{Z}$, and hence $q_S(x \bmod S) + q_T(y \bmod T) = 0$. Thus we have the following theorem.

Theorem 1.32. *Let L be an even unimodular lattice, S a primitive sublattice of L, and let $T = S^{\perp}$. Then the following holds*:

$$A_S \cong A_T, \quad q_S(\alpha) = -q_T(\gamma_{ST}(\alpha)) \ \ (\forall\, \alpha \in A_S).$$

Corollary 1.33. *Let $S_1, S_2 \subset L$ be primitive sublattices and let $T_i = S_i^{\perp}$, $i = 1, 2$. Let $\varphi\colon S_1 \to S_2$ be an isomorphism of lattices. Then the following are equivalent:*

(1) *The map φ can be extended to an isomorphism of L, that is, there exists an automorphism $\tilde{\varphi}\colon L \to L$ of L with $\tilde{\varphi}|S_1 = \varphi$.*

(2) *There exists an isomorphism $\psi\colon T_1 \to T_2$ satisfying*

$$\bar{\psi} \circ \gamma_{S_1,T_1} = \gamma_{S_2,T_2} \circ \bar{\varphi}.$$

Here $\bar{\psi}\colon A_{T_1} \to A_{T_2}$, $\bar{\varphi}\colon A_{S_1} \to A_{S_2}$ are isomorphisms induced from ψ, φ respectively.

Proof. Assume that assertion (1) holds. Then $\psi = \tilde{\varphi}|T_1$ satisfies assertion (2). Conversely, assume (2). The map defined by $\tilde{\varphi} = (\varphi, \psi)\colon S_1 \oplus T_1 \to S_2 \oplus T_2$ induces a map $S_1^* \oplus T_1^* \to S_2^* \oplus T_2^*$. We denote this map by the same symbol $\tilde{\varphi}$. Then the condition in (2) implies that $\tilde{\varphi}(L/(S_1 \oplus T_1)) = L/(S_2 \oplus T_2)$, and hence $\tilde{\varphi}(L) = L$ as desired. □

Next we consider a condition under which an even lattice can be primitively embedded into an even unimodular lattice. Suppose that even lattices S, T and an isomorphism $\gamma\colon A_S \to A_T$ satisfying $q_S(\alpha) = -q_T(\gamma(\alpha))$ $(\alpha \in A_S)$ are given. Then

$$H = \{(\alpha, \gamma(\alpha)) : \alpha \in A_S\}$$

is an isotropic subgroup of $A_{S\oplus T}$ with respect to $q_{S\oplus T}$, and by Theorem 1.19 the corresponding overlattice L contains $S \oplus T$ as a sublattice of index $|H|$. It follows from the fact $d(L) \cdot |H|^2 = d(S) \cdot d(T)$ that $d(L) = 1$ and hence L is unimodular. By construction, $p_S|H$ is isomorphic and hence S and T are primitive sublattices of L. Thus we have the following.

Theorem 1.34. *Let S, T be even lattices and let $\gamma\colon A_S \to A_T$ be an isomorphism satisfying $q_S = -q_T \circ \gamma$. Then there exists an even unimodular lattice L such that S can be primitively embedded in L and T is the orthogonal complement of S in L.*

Example 1.35. The root lattice E_7 can be primitively embedded in the root lattice E_8 whose orthogonal complement is isomorphic to the root lattice A_1. To see this, we take $S = E_7$ and $T = A_1$. Since $d(E_7) = d(A_1) = 2$, $A_S \cong A_T \cong \mathbb{Z}/2\mathbb{Z}$. Let t be a basis of A_1, and let $r_1, \ldots, r_7$ be a basis given in the Dynkin diagram, Figure 1.2. Then we can take

$$\alpha = \frac{r_4 + r_5 + r_7}{2} \bmod S, \quad \beta = \frac{t}{2} \bmod T$$

as generators of A_S, A_T respectively. Thus we have $q_S(\alpha) = -3/2$, $q_T(\beta) = -1/2$ which satisfy the assumption of Theorem 1.34. Therefore S can be primitively embedded in an even unimodular lattice of rank 8 and its orthogonal complement is $T = A_1$. Finally, if we consider the element

$$r_8 = -r_1 - 2r_2 - 3r_3 - \tfrac{3}{2}r_4 - \tfrac{5}{2}r_5 - 2r_6 - \tfrac{3}{2}r_7 - \tfrac{1}{2}t,$$

we have that $r_8^2 = -2$, $\langle r_8, r_i\rangle = 0$, $i = 1, \ldots, 6$, $\langle r_8, r_7\rangle = 1$. Thus a basis $r_1, \ldots, r_7$ of E_7 together with r_8 gives a basis of the lattice with the Dynkin diagram of type E_8, and hence $L \cong E_8$.

Exercise 1.36. Show that the root lattice E_6 can be primitively embedded in the root lattice E_8 whose orthogonal complement is isomorphic to the root lattice A_2.

It is important to consider the problem of whether an even lattice S can be primitively embedded in an even unimodular lattice L or not, and if the answer is yes, then of the uniqueness of embeddings of S into L modulo $\mathrm{O}(L)$. For example, to apply Theorem 1.34 to this problem, we need to show the existence of an even lattice T of rank $(\mathrm{rank}(L) - \mathrm{rank}(S))$ and with the discriminant quadratic form $-q_S$ which is a difficult problem in general. In the following, we mention two propositions which will be used in later chapters (Lemmas 9.11, 9.14, Section 9.3, Lemma 10.12). These propositions are due to Nikulin [Ni4].

Proposition 1.37. *Let T be an indefinite even lattice of signature (t_+, t_-) and with $q = q_T$. Suppose that*

$$\mathrm{rank}(T) \geq l(A_T) + 2.$$

Here $l(A_T)$ is the number of minimal generators of the finite abelian group A_T. Then an even lattice of signature (t_+, t_-) and with discriminant quadratic form q is unique up to isomorphisms, that is, it is isomorphic to T. Moreover, the natural map $\mathrm{O}(T) \to \mathrm{O}(q_T)$ is surjective.

A lattice L is called *2-elementary* if A_L is a 2-elementary abelian group, that is, $A_L \cong (\mathbb{Z}/2\mathbb{Z})^l$. We define an invariant δ of a 2-elementary lattice L by $\delta = 0$ if the image of the discriminant quadratic form $q_L\colon A_L \to \mathbb{Q}/2\mathbb{Z}$ is contained in $\mathbb{Z}/2\mathbb{Z}$, and otherwise $\delta = 1$.

Exercise 1.38. Show that root lattices A_1, D_4, E_7 are 2-elementary, and determine l and δ for these lattices. Determine whether D_5 is 2-elementary or not.

Proposition 1.39. *If an indefinite even 2-elementary lattice exists, then its isomorphism class is determined by the signature, l and δ. Moreover, the natural map $\mathrm{O}(L) \to \mathrm{O}(q_L)$ is surjective.*

In this book we do not go further into the general theory of embeddings of lattices, but in the next section we will introduce a sufficient condition that is elementary to prove. All of this section is treated in Nikulin [Ni4].

1.3.2 Elementary transformations and embeddings of lattices.

Definition 1.40. Let L be an even lattice. Let f, ξ be elements of L satisfying $f^2 = \langle f, \xi \rangle = 0$. For each $x \in L$, we define

$$\phi_{f,\xi}(x) = x + \langle x, \xi \rangle f - \tfrac{1}{2}\xi^2 \langle x, f \rangle f - \langle x, f \rangle \xi, \tag{1.8}$$

which is an automorphism of the lattice L. We call $\phi_{f,\xi}$ the *elementary transformation associated with* f, ξ.

Exercise 1.41. Show that $\phi_{f,\xi}$ preserves the bilinear form of L and satisfies $\phi_{f,\xi}(f) = f$. Moreover, prove that $\phi_{f,\xi} \in \widetilde{\mathrm{O}}(L)$.

In the following, we denote by $\{e, f\}$ a basis of U satisfying $e^2 = f^2 = 0$, $\langle e, f \rangle = 1$.

Lemma 1.42. *Any element $me + f + x$ of $L = U \oplus K$, $m \in \mathbb{Z}$, $x \in K$, can be sent to an element of the form $ne + f$, $n \in \mathbb{Z}$ by an automorphism of L.*

Proof. Apply the elementary transformation $\phi_{e,x}$. □

Exercise 1.43. Under the same notation as in Lemma 1.42, show that the map $K \to \mathrm{O}(L)$, $x \to \phi_{e,x}$ is a monomorphism.

Lemma 1.44. *Any non-zero element x of $U \oplus U$ can be sent to an element of the form $me + nf$, $m, n \in \mathbb{Z}$, $m|n$, by an automorphism. Here $\{e, f\}$ is a basis of the first factor of $U \oplus U$.*

Proof. Let $\{e', f'\}$ be a basis of the second factor of $U \oplus U$, and let

$$x = a_1 e + a_2 f + a_3 e' + a_4 f', \quad a_i \in \mathbb{Z}.$$

Denote by $M_2(\mathbb{Z})$ the additive abelian group consisting of 2×2 matrices with integer coefficients. For $A = \begin{pmatrix} a_1 & -a_3 \\ a_4 & a_2 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & -b_3 \\ b_4 & b_2 \end{pmatrix}$ in $M_2(\mathbb{Z})$, by defining a bilinear form by

$$\langle A, B \rangle = a_1 b_2 + a_2 b_1 + a_3 b_4 + a_4 b_3,$$

$M_2(\mathbb{Z})$ has the structure of a lattice. Its associated quadratic form is nothing but $2 \cdot \det$. Under the map

$$(a_1, a_2, a_3, a_4) \to \begin{pmatrix} a_1 & -a_3 \\ a_4 & a_2 \end{pmatrix},$$

the lattices $U \oplus U$ and $M_2(\mathbb{Z})$ are isomorphic. By the elementary divisor theorem, there exist elements $C, D \in \mathrm{GL}(2,\mathbb{Z})$ such that

$$C \cdot \begin{pmatrix} a_1 & -a_3 \\ a_4 & a_2 \end{pmatrix} \cdot D = \begin{pmatrix} m' & 0 \\ 0 & n' \end{pmatrix}. \tag{1.9}$$

Here m', n' are non-negative integers with $m'|n'$. The transformation (1.9) preserves the quadratic form up to ± 1, and hence there is an automorphism of $U \oplus U$ sending x to $me + nf$. □

Lemma 1.45. *Let L be an even unimodular lattice and assume that L has an orthogonal decomposition $L = U^{\oplus 2} \oplus K$. Then* $\mathrm{O}(L)$ *acts transitively on the set of primitive elements of L with the same norm.*

Proof. We denote by U_1, U_2 respectively the first and the second factor of $U \oplus U$, and let $\{e_i, f_i\}$ be a basis of U_i satisfying $\langle e_i, e_i\rangle = \langle f_i, f_i\rangle = 0$, $\langle e_i, f_i\rangle = 1$, $i = 1, 2$. We will show that any primitive element $y \in L$ with norm $2m$ can be sent to an element of the form $e_1 + mf_1$ by an automorphism of L. Let $y = y' + y''$, $y' \in U_1 \oplus U_2$, $y'' \in K$. First of all, we show that y' can be assumed to be primitive modulo the action of $\mathrm{O}(L)$. If $y' = 0$, then we choose an element $\xi \in K$ satisfying $\langle y, \xi\rangle \neq 0$, and then applying the elementary transformation $\phi_{f_1,\xi}$ to y, we may assume that $\langle y, e_1\rangle \neq 0$, that is, $y' \neq 0$. Moreover, we may assume that $\langle e_1, y\rangle | \langle f_1, y\rangle$ by Lemma 1.44. It follows from Step (1) in the proof of Theorem 1.22 that there exists an element $u \in L$ satisfying $\langle y, u\rangle = 1$. Let $\xi = u - \langle u, e_1\rangle f_1$. Since $\langle e_1, \xi\rangle = 0$, we can define the elementary transformation $\phi_{e_1,\xi}$. Put $y_0 = \phi_{e_1,\xi}(y)$. Then $\phi_{e_1,\xi}(e_1) = e_1$ implies that $\langle y_0, e_1\rangle = \langle y, e_1\rangle$. Note that $\langle e_1, y\rangle | \langle f_1, y\rangle$. Then we have

$$\begin{aligned} \langle y_0, f_1\rangle &= \langle y, f_1\rangle + \langle y, \xi\rangle\langle e_1, f_1\rangle - \tfrac{1}{2}\langle \xi, \xi\rangle\langle y, e_1\rangle\langle e_1, f_1\rangle - \langle y, e_1\rangle\langle \xi, f_1\rangle \\ &\equiv \langle y, \xi\rangle \bmod \langle y, e_1\rangle \equiv \langle y, u\rangle \bmod \langle y, e_1\rangle \equiv 1 \bmod \langle y_0, e_1\rangle. \end{aligned}$$

Thus, by considering y_0 instead of y, we may assume that y' is primitive. Again by Lemma 1.44, y can be sent to an element of the form $e_1 + nf_1 + v$, $v \in K$ under an automorphism of L. Finally, the assertion follows from Lemma 1.42. □

By using induction, we can generalize Lemma 1.45 as follows.

Proposition 1.46. *Let L be an even unimodular lattice and assume that L has an orthogonal decomposition $L = U^{\oplus k} \oplus K$. Then any even lattice of rank less than or equal to k can be primitively embedded into L . Moreover, a primitive embedding of an even lattice of rank less than or equal to $k - 1$ into L is unique modulo* $\mathrm{O}(L)$*. Here we do not assume that even lattices are non-degenerate.*

Proof. We first show the existence of a primitive embedding. Let $\{e_i, f_i\}$ be a basis of each direct summand of $U^{\oplus k}$ ($e_i^2 = f_i^2 = 0, \langle e_i, f_j\rangle = \delta_{ij}, i, j = 1, \dots, k$). Let F be an even lattice of rank $l \leq k$ and let $y_1, \dots, y_l$ be its basis. Then the map defined by

$$y_i \to x_i = e_i + \frac{1}{2}\langle y_i, y_i\rangle f_i + \sum_{j=1}^{i-1}\langle y_i, y_j\rangle f_j, \quad i = 1, \dots, l \tag{1.10}$$

from F to $U^{\oplus k}$ is an embedding. Since $\langle x_i, f_i\rangle = 1$, this embedding is primitive.

Next we will prove the uniqueness. Let F be a primitive sublattice of L of rank $k-1$ and let $\{y_1, \dots, y_{k-1}\}$ be a basis of F. We will show that there exists an element in $\mathrm{O}(L)$ which sends y_i to x_i, $1 \leq i \leq k-1$ given in (1.10) by induction on the rank of F.

In the case of rank 1, the assertion is nothing but Lemma 1.45. By the induction hypothesis, we may assume that $y_i = x_i$, $1 \leq i \leq k-2$. Now we will find an automorphism of L which fixes x_i, $1 \leq i \leq k-2$ and sends y_{k-1} to x_{k-1}. Let E be a sublattice generated by $e_{k-1}, f_{k-1}, e_k, f_k$. Then E is unimodular and hence is a direct summand of L. As in the proof of Lemma 1.45, we may assume that $\langle y_{k-1}, e_{k-1}\rangle \neq 0$ by applying the elementary transformation associated with ξ and f_{k-1} where ξ and f_{k-1} are perpendicular to $x_1, \dots, x_{k-2}$. Moreover, F is primitive and hence its basis can be extended to a basis of L. Therefore there exists an element f of the dual L^* of L satisfying

$$f(x_i) = 0, \;\; 1 \leq i \leq k-2, \quad f(y_{k-1}) = 1.$$

Since L is unimodular, there exists an element $u \in L$ satisfying $f(x) = \langle u, x\rangle$ ($x \in L$). Let $\xi = u - \langle u, e_{k-1}\rangle f_{k-1}$. Then $\langle e_{k-1}, \xi\rangle = 0$ and hence we can consider the elementary transformation $\phi_{e_{k-1},\xi}$. Since both e_{k-1}, ξ are perpendicular to $x_1, \dots, x_{k-2}$, we have $\phi_{e_{k-1},\xi}(x_i) = x_i$, $1 \leq i \leq k-2$. Again, by the same argument used in the proof of Lemma 1.45 concerning y', we may assume that the projection of y_{k-1} into E is primitive. Let M be a sublattice generated by $e_1, f_1, \dots, e_{k-2}, f_{k-2}$; then

$$L = M \oplus M^\perp, \quad M^\perp = E \oplus K.$$

Let $y_{k-1} = y'_{k-1} + y''_{k-1}$, $y'_{k-1} \in M$, $y''_{k-1} \in M^\perp$. Then it follows that y''_{k-1} is primitive. Again, by the same argument as in the proof of Lemma 1.44, we may assume that

$$y_{k-1} = y'_{k-1} + e_{k-1} + m f_{k-1}.$$

By the definition of x_i, $\{x_1, \dots, x_{k-2}, f_1, \dots, f_{k-2}\}$ is a basis of M. Therefore there exists $w \in M^* = M$ satisfying

$$\langle w, x_i\rangle = 0, \quad \langle w, f_i\rangle = \langle y_{k-1}, f_i\rangle, \quad 1 \leq i \leq k-2.$$

The elementary transformation $\phi_{f_{k-1},w}$ fixes all $x_1,\dots,x_{k-2}$. Let $y = \phi_{f_{k-1},w}(y_{k-1})$. Then we have

$$\begin{aligned} \langle y, f_i\rangle &= \langle y_{k-1}, f_i\rangle - \langle y_{k-1}, f_{k-1}\rangle\langle w, f_i\rangle = 0, && i = 1,\dots,k-2,\\ \langle y, x_i\rangle &= \langle y_{k-1}, x_i\rangle, && i = 1,\dots,k-2. \end{aligned}$$

On the other hand, by definition of x_i, we have

$$\langle x_{k-1}, f_i\rangle = 0, \quad \langle x_{k-1}, x_i\rangle = \langle y_{k-1}, x_i\rangle, \quad i = 1,\dots,k-2.$$

Since $\{x_1,\dots,x_{k-2}, f_1,\dots,f_{k-2}\}$ is a basis of M, the projections of x_{k-1} and y into M coincide. Moreover, x_{k-1} and y have the same norm, and their projections into $M^\perp$ have the same form $e_{k-1} + n f_{k-1}$, and therefore we conclude $x_{k-1} = y$. □

Remark 1.47. The references for this section are the articles Piatetskii-Shapiro, Shafarevich [PS] and Looijenga, Peters [LP].

2

Reflection groups and their fundamental domains

We introduce a fundamental domain of a reflection group associated to a real space with a quadratic form of signature $(1, r)$. In the theory of $K3$ surfaces, the Kähler cone appears as a fundamental domain of a reflection group acting on a cone, called the positive cone.

2.1 Reflection groups and fundamental domains

In this section we consider an n-dimensional real vector space V with a quadratic form of signature $(1, n-1)$. This case is important for applications to $K3$ surfaces.

Definition 2.1. Let V be an n-dimensional real vector space and let

$$\langle\ ,\ \rangle \colon V \times V \to \mathbb{R}$$

be a symmetric bilinear form of signature $(1, n-1)$. Denote by $\mathrm{O}(V)$ the group of all automorphisms of the vector space V preserving the bilinear form. It is called the orthogonal group of V. An element δ of V with $\delta^2 = \langle \delta, \delta \rangle = -2$ is called a *root* of V. For a root δ of V, we define a map $s_\delta \colon V \to V$ by

$$s_\delta(x) = x + \langle x, \delta \rangle \delta.$$

A simple calculation shows that s_δ preserves the bilinear form and satisfies $s_\delta^2 = 1$. By definition, s_δ is the identity map on the hyperplane

$$H_\delta = \{x \in V : \langle x, \delta \rangle = 0\}$$

perpendicular to δ. We call $s_\delta \in \mathrm{O}(V)$ the *reflection* with respect to H_δ. Suppose that Δ_0 is a set of roots of V that might be a finite or infinite set. Let W be the subgroup of $\mathrm{O}(V)$ generated by reflections associated with roots in Δ_0. It is called the *reflection group* associated with Δ_0. We define

$$\Delta = W(\Delta_0), \quad \mathfrak{H} = \{H_\delta : \delta \in \Delta\}.$$

Then Δ is the set of all roots that induce reflections in W, and $\mathfrak{H}$ is the set of hyperplanes defined by roots in Δ. For any $H \in \mathfrak{H}$ and $w \in W$, we have $w(H) \in \mathfrak{H}$.

Exercise 2.2. Show that s_δ preserves the bilinear form and satisfies $s_\delta^2 = 1$.

Next we introduce the space on which W acts. Consider the set

$$P(V) = \{x \in V : x^2 > 0\}.$$

There exists a basis of V over $\mathbb{R}$ such that the quadratic form is given by

$$x^2 = x_1^2 - x_2^2 - \cdots - x_n^2,$$

and hence $P(V)$ has two connected components according to whether $x_1 > 0$ or $x_1 < 0$. We fix one of them, denote it by $P^+(V)$, and call it a *positive cone* (see Figure 2.1).

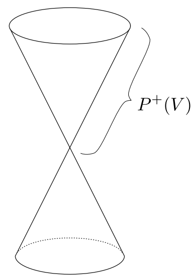


Figure 2.1. Positive cone.

Note that the subspace H_δ has signature $(1, n-2)$ for each root $\delta \in \Delta$. This implies that the intersection of H_δ and $P^+(V)$ is non-empty. Since the reflection s_δ fixes each point of H_δ, we conclude that s_δ preserves $P^+(V)$ and the reflection group W acts on $P^+(V)$. We denote by $\overline{P^+(V)}$ the closure of $P^+(V)$ in $V - \{0\}$.

Lemma 2.3. *Let $x \in P^+(V)$, $y \in \overline{P^+(V)}$. Then $\langle x, y\rangle > 0$.*

Proof. By the assumption on x, y, we have

$$x_1^2 > \sum_{i=2}^n x_i^2, \quad y_1^2 \geq \sum_{i=2}^n y_i^2.$$

By combining this with the Cauchy–Schwarz inequality, we can prove the assertion. □

Definition 2.4. Suppose that a topological group G acts on a topological space continuously, that is, the map

$$G \times M \to M, \quad (g, x) \to g \cdot x$$

is continuous. We assume that both G, M are Hausdorff, and M is locally compact. An action of G on M is called *proper* if the map

$$G \times M \to M \times M, \quad (g, x) \to (g \cdot x, x) \tag{2.1}$$

is proper, that is, the inverse image of any compact set is compact. An action of G on M is called *properly discontinuous* if the number of $g \in G$ such that $g(K) \cap K \neq \emptyset$ for any compact set $K \subset M$ is finite. If G acts on M properly discontinuously, then any orbit of G is a discrete set in M and the stabilizer subgroup of G is finite. When G acts on M properly, then Γ is a discrete subgroup of G if and only if Γ acts on M properly discontinuously. In fact, for a compact set $K \subset M$, let $K_1 \times K$ be the inverse image of $K \times K$ by the map (2.1) which is compact. If $g(K) \cap K \neq \emptyset$, then $g \in K_1$. Since $K_1 \subset G$ is compact and Γ is discrete, the number of $g \in \Gamma$ satisfying $g(K) \cap K \neq \emptyset$ is finite. Conversely, if Γ is not discrete in G, then there exists a convergent sequence $\{\gamma_n\}$, $\gamma_n \in \Gamma$. Then $\{\gamma_n(x)\}$ $(x \in M)$ converges to a point in M by continuity of the action. This contradicts the discreteness of the orbits of Γ.

Now we return to the reflection groups. The reflection group is a subgroup of the topological group $\mathrm{O}(V)$.

Assumption: We assume that W acts on $P^+(V)$ properly discontinuously.

The set of hyperplanes $\mathfrak{H}$ is called *locally finite* if for any point $x \in P^+(V)$, there exists a neighborhood U of x in $P^+(V)$ such that the number of $H \in \mathfrak{H}$ with $H \cap U \neq \emptyset$ is finite.

Lemma 2.5. *$\mathfrak{H}$ is locally finite.*

Proof. By the assumption, W acts on $P^+(V)$ properly discontinuously. Therefore for any point in $P^+(V)$, there exists a neighborhood U such that the number of $w \in W$ with $w(U) \cap U \neq \emptyset$ is finite. In particular, the number of $\delta \in \Delta$ satisfying $s_\delta(U) \cap U \neq \emptyset$ is finite. This implies that the number of $H \in \mathfrak{H}$ intersecting with U is finite. □

Corollary 2.6. *The union $\bigcup_{H \in \mathfrak{H}} H$ is a closed set in $P^+(V)$.*

Proof. Let $x \in P^+(V) \setminus \bigcup_{H \in \mathfrak{H}} H$. Then for any $\delta \in \Delta$, s_δ does not fix x. It follows from Lemma 2.5 that there exists a neighborhood U with $s_\delta(U) \cap U = \emptyset$ $(\forall\, \delta \in \Delta)$. In particular, we have $U \subset P^+(V) \setminus \bigcup_{H \in \mathfrak{H}} H$ and hence we have proved the assertion. □

Definition 2.7. A *chamber* is a connected component of the complement of the union of hyperplanes in $\mathfrak{H}$:

$$P^+(V) \setminus \bigcup_{H \in \mathfrak{H}} H.$$

Let C be a chamber and denote by $\bar{C}$ the closure of C in $P^+(V)$. A hyperplane $H \in \mathfrak{H}$ is called a *face* if $H \cap \bar{C}$ contains an open set of H. A *simple root* with respect to C is a root that defines a face of C. For an interior point $x_0 \in C$, by defining

$$\Delta^+ = \{\delta \in \Delta : \langle \delta, x_0 \rangle > 0\}, \quad \Delta^- = \{-\delta : \delta \in \Delta^+\},$$

we have a decomposition

$$\Delta = \Delta^+ \cup \Delta^-, \tag{2.2}$$

which is independent of the choice of x_0 and depends only on C.

Lemma 2.8. *Let δ, δ' be simple roots with respect to a chamber C. Then*

$$\langle \delta, \delta' \rangle \geq 0.$$

Proof. Let $x_0 \in C$. Then we have $\langle x_0, \delta \rangle > 0$ and $\langle x_0, \delta' \rangle > 0$. The root $s_\delta(\delta') = \delta' + \langle \delta, \delta' \rangle \delta$ is perpendicular to x_0 if and only if

$$\langle x_0, \delta' \rangle + \langle \delta, \delta' \rangle \langle x_0, \delta \rangle = 0.$$

If $\langle \delta, \delta' \rangle < 0$, then we can move x_0 in C continuously to a point on the hyperplane $H_{s_\delta(\delta')}$. This means that the hyperplane $H_{s_\delta(\delta')}$ cuts C, which contradicts the fact that C is a chamber. □

Theorem 2.9. *Let C be a chamber. Then C is a* fundamental domain *of W with respect to the action on $P^+(V)$. That is, the following two conditions hold*:

(i) $P^+(V) = \bigcup_{w \in W} w(\bar{C})$.

(ii) *If $w \in W$, $w(C) \cap C \neq \emptyset$, then $w = 1$.*

Proof. Let S be the set of simple roots with respect to C, and let W_S be a subgroup of W generated by reflections s_δ ($\delta \in S$). It is enough to prove the following three properties:

(1) Let C' be a chamber. Then there exists a $w \in W_S$ satisfying $w(C') = C$.

(2) $W_S = W$.

(3) For $\delta \in \Delta$, let P_δ be the set of $w \in W$ such that C and $w(C)$ belong to the same half-space with respect the hyperplane H_δ. Then

$$\bigcap_{\delta \in S} P_\delta = \{1\}.$$

In the following we prove (1), (2), (3) in order.

Proof of (1): First, we fix $x \in C'$ and $a \in C$. Consider the orbit $W_S \cdot x$ of x under the action of W_S. It follows from Lemma 2.3 that $\langle y, a \rangle > 0$ for any $y \in W_S \cdot x$. Since W (and hence W_S) acts on $P^+(V)$ properly discontinuously, there exists $y_0 \in W_S \cdot x$ satisfying

$$\langle y_0, a \rangle \leq \langle y, a \rangle \quad (\forall\, y \in W_S \cdot x).$$

Then, for any $\delta \in S$, we have

$$\langle a, y_0 \rangle \leq \langle a, s_\delta(y_0) \rangle = \langle a, y_0 \rangle + \langle a, \delta \rangle \langle \delta, y_0 \rangle,$$

and hence $\langle \delta, y_0 \rangle \geq 0$ because $\langle a, \delta \rangle > 0$. Since $y_0 \notin H_\delta$, we have $\langle \delta, y_0 \rangle > 0$ and hence $y_0 \in C$. Therefore, for $w \in W_S$ with $y_0 = w(x)$, we have $w(C') = C$.

Proof of (2): For any $\delta \in \Delta$, there exists a chamber C' which has a face H_δ defined by δ. By claim (1), there exists a $w \in W_S$ with $w(C') = C$. In this case, the hyperplane $w(H_\delta)$ is a face of C, and if we denote by $\delta' \in S$ the root defining this face, then $s_\delta = w^{-1} s_{\delta'} w \in W_S$. Thus we have $W = W_S$.

Proof of (3): We first prepare two lemmas.

Lemma 2.10. *For $\delta, \delta' \in S$ and $w \in P_\delta$, if $w s_{\delta'} \notin P_\delta$, then $w s_{\delta'} = s_\delta w$.*

Proof. By the assumption, C and $w(C)$ sit in the same half-space with respect to the hyperplane H_δ, and C and $w(s_{\delta'}(C))$ sit on different sides, and hence $w(C)$ and $w(s_{\delta'}(C))$ sit on different sides. Therefore C and $s_{\delta'}(C)$ sit on different sides with respect to $w^{-1}(H_\delta)$. Since C and $s_{\delta'}(C)$ touch along $H_{\delta'}$, we have $H_{\delta'} = w^{-1}(H_\delta)$ and $s_{\delta'} = w^{-1} s_\delta w$. □

Lemma 2.11. *Any $w \in W = W_S$ can be represented as a product of reflections $w = s_{\delta_1} \cdots s_{\delta_l}$ $(\delta_1, \ldots, \delta_l \in S)$. We denote by $l(w)$ the minimum of the numbers l among such representations of w. Then*

$$P_\delta = \{w \in W : l(s_\delta w) > l(w)\}.$$

Proof. We divide the proof into two cases:

(a) The case $w \notin P_\delta$. For simplicity, we set $q = l(w)$ and $w = s_{\delta_1} \cdots s_{\delta_q}$ $(\delta_1, \ldots, \delta_q \in S)$. For each $1 \leq j \leq q$, we define $w_j = s_{\delta_1} \cdots s_{\delta_j}$ and $w_0 = 1$. By the assumption, $w_0 = 1 \in P_\delta$, $w \notin P_\delta$ and hence there exists a j such that $w_{j-1} \in P_\delta$, $w_j \notin P_\delta$. It follows from Lemma 2.10 that $w_{j-1} s_{\delta_j} = s_\delta w_{j-1}$, that is, $s_{\delta_1} \cdots s_{\delta_{j-1}} s_{\delta_j} = s_\delta s_{\delta_1} \cdots s_{\delta_{j-1}}$. Therefore we have

$$s_\delta w = s_\delta s_{\delta_1} \cdots s_{\delta_{j-1}} s_{\delta_j} \cdots s_{\delta_q} = s_{\delta_1} \cdots s_{\delta_{j-1}} s_{\delta_{j+1}} \cdots s_{\delta_q},$$

and $l(s_\delta w) < l(w)$.

(b) The case $w \in P_\delta$. Put $w' = s_\delta w$. Then $w' \notin P_\delta$. By case (a), we have $l(s_\delta w) = l(w') > l(s_\delta w') = l(w)$ and thus have proved Lemma 2.11. □

Now consider $w \in W$, $w \neq 1$. Then $q = l(w) \geq 1$. Let $w = s_{\delta_1} \cdots s_{\delta_q}$ $(\delta_1, \dots, \delta_q \in S)$. Since $s_{\delta_1} w = s_{\delta_2} \cdots s_{\delta_q}$, we have $l(s_{\delta_1} w) < l(w)$ and then, by Lemma 2.11, we have $w \notin P_{\delta_1}$. Thus we have finished the proofs of claim (3) and Theorem 2.9. □

2.2 Reflection groups associated with lattices

Finally, in this chapter we deal with the reflection groups appearing in the case of $K3$ surfaces. Let L be an even lattice of signature $(3, n)$. Consider a 2-dimensional positive definite subspace E of $L \otimes \mathbb{R}$. Let V be the orthogonal complement of E which has a quadratic form of signature $(1, n)$. We consider the following set as Δ_0:

$$\Delta_0 = \{\delta \in L \cap V : \delta^2 = -2\}.$$

By definition of Δ_0, $\Delta = \Delta_0$. Let $S = L \cap V$, then S is a primitive subgroup of L; however, it might be degenerate. And it might be happen that $\Delta_0 = \emptyset$. Since any s_δ associated with $\delta \in \Delta_0$ preserves L, it is contained in the orthogonal group $\mathrm{O}(L)$ of L, and in particular $W \subset \mathrm{O}(L) \cap \mathrm{O}(V)$.

Lemma 2.12. *The action of W on $P^+(V)$ is properly discontinuous.*

Proof. We denote by $P^+(V)^{(1)}$ the subset of $P^+(V)$ consisting of elements of norm 1. Then we have

$$P^+(V) \cong P^+(V)^{(1)} \times \mathbb{R}_{>0},$$

and the action of W on $P^+(V)$ preserves this decomposition. Since W acts on $\mathbb{R}_{>0}$ trivially, it is enough to consider the action of W on $P^+(V)^{(1)}$. The orthogonal group $G = \mathrm{O}(V)$ acts on $P^+(V)^{(1)}$ transitively and the stabilizer subgroup K of $x \in P^+(V)^{(1)}$ is compact. This follows from the fact that $\langle x, x \rangle > 0$ and hence the orthogonal complement of x is negative definite. Thus the action of G on $P^+(V)^{(1)} \cong G/K$ is proper. On the other hand, W $(\subset \mathrm{O}(L) \subset \mathrm{O}(L \otimes \mathbb{R}))$ is a discrete subgroup, and hence is also discrete in G. Therefore W acts on $P^+(V)$ properly discontinuously. □

By this lemma, Theorem 2.9 holds in this case.

Finally, we consider the case that S is an even lattice of signature $(1, r)$. This happens in the case of projective $K3$ surfaces. In this case, we set

$$V = S \otimes \mathbb{R}, \quad \Delta_0 = \Delta = \{\delta \in S : \delta^2 = -2\}.$$

Remark 2.13. The reflection s_δ acts on $A_S = S^*/S$ trivially. Hence W acts on A_S trivially too.

Exercise 2.14. Prove Remark 2.13.

Exercise 2.15. Show that W is a normal subgroup of the orthogonal group $\mathrm{O}(S)$ of S.

For a chamber $C \subset S \otimes \mathbb{R}$ with respect to W, define

$$\mathrm{Aut}(C) = \{\varphi \in \mathrm{O}(S) : \varphi(C) = C\}.$$

Then the following holds.

Corollary 2.16. $\mathrm{O}(S)/\{\pm 1\} \cdot W \cong \mathrm{Aut}(C)$.

Proof. Let $\varphi \in \mathrm{O}(S)$. If necessary by considering $-\varphi$, we may assume that φ preserves $P^+(S \otimes \mathbb{R})$. Theorem 2.9 implies that there exists a $w \in W$ satisfying $w \circ \varphi(C) = C$. Thus we have finished the proof. □

Remark 2.17. In the above we have considered only even lattices, but we do not need this assumption. For odd lattices we should consider reflections associated with elements of norm -1. We restrict to the case of even lattices to avoid this complexity. E. B. Vinberg studied reflections of the space of signature $(1, r-1)$ deeply. We refer the reader interested in this subject to Vinberg [V1]. On the other hand, in the case that V is definite, considering the action of W on V itself, all results in this section hold after small modifications. The main reference for this chapter is Bourbaki [Bou].

3

Complex analytic surfaces

In this chapter we first recall fundamental tools for complex analytic surfaces. Then we mention the classification of complex analytic surfaces. Finally, we will introduce the classification of singular fibers of elliptic surfaces. In particular we show that the dual graphs of reducible singular fibers coincide with the extended Dynkin diagrams of type $\tilde{A}_m$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$.

3.1 Basics of complex analytic surfaces

Let X be an n-dimensional compact complex manifold. We assume that X is Hausdorff. In the case $n = 1$ we call X a *non-singular curve*, and in the case $n = 2$ a *non-singular surface*. For simplicity, we call these a curve and a surface if there is no confusion. Let $H_i(X, \mathbb{Z})$, $H^i(X, \mathbb{Z})$ be the singular homology group, the singular cohomology group of X, respectively, and $\pi_i(X)$ the ith homotopy group. The first homotopy group $\pi_1(X)$ is nothing but the fundamental group. Let $\mathcal{O}_X$ be the structure sheaf of X, $\mathcal{O}_X^*$ the sheaf of germs of holomorphic functions without zeros, and Ω_X^k the sheaf of germs of holomorphic k-forms. We denote by T_X the holomorphic tangent bundle of X, by T_X^* its dual, and by K_X the canonical line bundle of X, that is, $K_X = \wedge^n T_X^*$. We also denote by $c_i(X)$ $(= c_i(T_X))$ the ith Chern class and by $e(X)$ the *Euler number* $c_n(X)$ of X. For a non-singular curve C, the genus of C is denoted by $g(C)$ $(= \dim H^1(C, \mathcal{O}_C))$. The cohomology group $H^q(X, \Omega_X^p)$ is a finite-dimensional complex vector space whose dimension is denoted by $h^{p,q}(X)$. In particular, $h^{0,n}(X)$, $h^{0,1}(X)$ are denoted by $p_g(X)$, $q(X)$, respectively and called the *geometric genus*, *irregularity*.

The set of isomorphism classes of line bundles on X is naturally identified with $H^1(X, \mathcal{O}_X^*)$ which has the structure of an abelian group. The group multiplication corresponds to the tensor product of line bundles and the inverse of L to the dual L^* of L. We denote $H^1(X, \mathcal{O}_X^*)$ by $\mathrm{Pic}(X)$ and call it the *Picard group*. The exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0 \tag{3.1}$$

induces the exact sequence of cohomology groups

$$\cdots \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \cdots \tag{3.2}$$

and δ is the map sending L to its Chern class $c_1(L)$. The image of δ is denoted by $\mathrm{NS}(X)$ and is called the *Néron–Severi group*. The kernel of δ is denoted by $\mathrm{Pic}^0(X)$.

In the following we assume that X is a surface. The cup product

$$\langle\ ,\ \rangle\colon H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}$$

is a non-degenerate symmetric bilinear form modulo torsion and the quotient group $H^2(X, \mathbb{Z})/\{\text{torsion}\}$ has the structure of a lattice. A *divisor* is a formal sum $D = \sum_{i=1}^n m_i C_i$ ($m_i \in \mathbb{Z}$) of a finite number of irreducible curves on X. A divisor D is called *effective* if all coefficients are non-negative. A non-zero effective divisor may be called a *positive divisor*. The line bundle associated with a divisor D is denoted by $[D]$ or $\mathcal{O}_X(D)$. For any irreducible curves C, C' on X, their intersection number $C \cdot C'$ is defined and coincides with the cup product of their cohomology classes. For any divisors, the same property holds. For line bundles L, L', we denote $\langle c_i(L), c_1(L')\rangle$ by $c_1(L) \cdot c_1(L')$ for simplicity. A divisor D is called *nef* if its intersection number with any irreducible curve is non-negative. Let L be a line bundle on X, and denote by $\chi(L)$ the alternating sum $\sum_{i=0}^2 (-1)^i \dim H^i(X, L)$. We denote by $h^i(X, L)$ the dimension of $H^i(X, L)$. The next result is called the *Riemann–Roch* theorem, which is fundamental.

Theorem 3.1 (Riemann–Roch theorem for surfaces).

$$\chi(L) = \frac{1}{2}(c_1(L)^2 + c_1(L) \cdot c_1(X)) + \chi(\mathcal{O}_X).$$

By definition we have $\chi(\mathcal{O}_X) = p_g(X) - q(X) + 1$. Moreover, by using the *Serre duality*

$$H^i(X, L) \cong H^{2-i}(X, K_X \otimes L^*)^*,$$

the left-hand side in the Riemann–Roch theorem is given by

$$h^0(X, L) - h^1(X, L) + h^0(X, K_X \otimes L^*).$$

Theorem 3.2 (Noether's formula).

$$p_g(X) - q(X) + 1 = \frac{1}{12}(c_1(X)^2 + c_2(X)).$$

Let X be a surface and $C \subset X$ a non-singular curve. Then the following theorem, called the *adjunction formula*, is useful.

Theorem 3.3 (Adjunction formula).

$$K_C = (K_X + C)|C, \quad 2g(C) - 2 = K_X \cdot C + C^2.$$

For any irreducible curve, not necessarily non-singular, on X, we define $p_a(C)$ by

$$p_a(C) = \tfrac{1}{2}(K_X \cdot C + C^2) + 1,$$

which is called the *arithmetic genus* or the *virtual genus*. Let $\nu\colon \tilde{C} \to C$ be the normalization of C. Then we have

$$p_a(C) = g(\tilde{C}) + \sum_{x \in C} \dim(\nu_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C)_x. \tag{3.3}$$

In particular, $p_a(C) \geq 0$, and $p_a(C) = 0$ implies that C is a non-singular rational curve.

We denote by $(b^+(X), b^-(X))$ the signature of the cup product

$$H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \to \mathbb{R},$$

and the difference $b^+(X) - b^-(X)$ is called the *index* of the surface.

Theorem 3.4 (Hirzebruch's index theorem).

$$b^+(X) - b^-(X) = \tfrac{1}{3}(c_1(X)^2 - 2c_2(X)).$$

If a surface is Kähler, then the Hodge decomposition is a powerful method; however, for a general complex analytic surface the following holds.

Theorem 3.5. (1) *In the case* $b_1(X) \equiv 0 \bmod 2$*,*

$$2p_g(X) = b^+(X) - 1, \quad 2q(X) = b_1(X), \quad h^{1,0}(X) = q(X).$$

(2) *In the case* $b_1(X) \equiv 1 \bmod 2$*,*

$$2p_g(X) = b^+(X), \quad 2q(X) = b_1(X) + 1, \quad h^{1,0}(X) = q(X) - 1.$$

A *complete linear system*, denoted by $|D|$, is the set of all effective divisors linearly equivalent to a divisor D. The zero divisor of any non-zero section of $H^0(X, \mathcal{O}_X(D))$ gives an element in $|D|$, and this correspondence identifies $|D|$ and $\mathbb{P}(H^0(X, \mathcal{O}_X(D)))$. A subspace of $|D|$ is also called a *linear system*. The *dimension of a linear system* is the dimension of the associated projective space. Let P be a linear system. The *fixed component* of P is the maximum effective divisor F such that $D - F$ is effective for any $D \in P$. Since the linear system P is isomorphic to $P - F$, by replacing P

by $P - F$, we may assume that P has no fixed component. We call the intersection of all divisors in $P - F$ *base points*. Now assume that P has no fixed component. Associating $x \in X$ to the hyperplane consisting of divisors in P passing through x, we have a meromorphic map

$$\Phi_P : X \to P^*. \tag{3.4}$$

Note that Φ_P is not defined at the base points of P. Conversely, for a meromorphic map

$$\varphi : X \to \mathbb{P}^n,$$

we can define the pullback $\varphi^* H$ of a hyperplane $H \subset \mathbb{P}^n$, and thus we have an n-dimensional linear system. A divisor D is called *very ample* if $|D|$ has no fixed component and base points, and $\Phi_{|D|}$ gives an embedding into a projective space. A divisor D is called *ample* if mD ($m > 0$) is very ample. Let X be a projective manifold, that is, it is embedded in $\mathbb{P}^N$; then the restriction $H|X$ of a hyperplane $H \subset \mathbb{P}^N$ is called a *hyperplane section*. The next result is useful as a criterion of ampleness of a divisor D.

Theorem 3.6 (Nakai's criterion). *A divisor D is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for any irreducible curve C.*

Theorem 3.7 (Hodge index theorem). *Let D, C be divisors with $D^2 > 0$, $D \cdot C = 0$. Then $C^2 \leq 0$, and the equality holds only if the class of C in $H^2(X, \mathbb{Q})$ is 0.*

The next theorem is useful to construct examples of $K3$ surfaces.

Theorem 3.8 (Lefschetz hyperplane theorem). *Let $X \subset \mathbb{P}^N$ be an $n(\geq 2)$-dimensional non-singular closed manifold. Let H be a hyperplane such that $H \cap X$ is non-singular. Then the natural maps*

$$H_i(H \cap X, \mathbb{Z}) \to H_i(X, \mathbb{Z}), \quad \pi_i(H \cap X) \to \pi_i(X)$$

are isomorphic for $0 \leq i \leq n - 2$.

Let X be a compact complex manifold and let K_X be the canonical line bundle. We denote by $\kappa(X)$, called the *Kodaira dimension*, the maximum dimension of the image under the meromorphic map $\Phi_{|mK_X|}$, where m runs over non-negative integers. In other words, if we set $p_m(X) = \dim H^0(X, mK_X)$, then we can define it as the asymptotic behavior of $p_m(X)$, $m \to \infty$ coinciding with that of $m^{\kappa(X)}$. Also, it is known that $\kappa(X) + 1$ is equal to the transcendental degree of the graded ring $\bigoplus_{m\geq 0} H^0(X, mK_X)$ over $\mathbb{C}$. Here, if $H^0(X, mK_X) = \{0\}$ for any natural number m, that is, $p_m(X) = 0$, then we set $\kappa(X) = -\infty$. Thus the Kodaira dimension takes values $-\infty, 0, 1, \ldots, n = \dim(X)$.

Later we construct a $K3$ surface as a double covering branched along a divisor. Then the following result will be used.

Proposition 3.9. *Let M be a compact complex manifold and let D be a non-singular effective divisor. Then the following are equivalent*:

(1) *There exists a double covering $\pi\colon \widetilde{M} \to M$ branched along D.*

(2) $\frac{1}{2}D \in \mathrm{Pic}(M)$.

Moreover, if $D' = \frac{1}{2}D$, then $K_{\widetilde{M}} = \pi^*(K_M \otimes \mathcal{O}(D'))$.

Proof. Assume that there exists a line bundle $p\colon \mathcal{L} \to M$ satisfying $\mathcal{L}^{\otimes 2} = [D]$. Let s be a section of $\mathcal{L}^{\otimes 2}$ with $(s) = D$. Then

$$\widetilde{M} = \left\{ y \in \mathcal{L} : y^{\otimes 2} = s(p(y)) \right\}$$

is the desired double covering.

Next we show the latter half. Let $(x_1, \dots, x_n)$ be local coordinates and assume that $x_1 = 0$ is the local defining equation of D. We can take local coordinates $(y_1, \dots y_n)$ of $\widetilde{M}$ such that $\pi(y_1, y_2, \dots, y_n) = (y_1^2, y_2, \dots, y_n) = (x_1, x_2, \dots, x_n)$. Then

$$\pi^*(dx_1 \wedge dy_2 \wedge \cdots \wedge dx_n) = y_1\, dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n,$$

and hence we have the relation between canonical line bundles. □

We will discuss a theory of periods of $K3$ surfaces later. Here we consider this from a general viewpoint.

Definition 3.10. Let L be a free abelian group of finite rank. A *Hodge structure* on L of weight m or a *Hodge decomposition* is a direct decomposition of $L \otimes \mathbb{C}$ into subspaces $H^{p,q}$ $(p, q \geq 0)$,

$$L \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{p,q},$$

such that $H^{q,p}$ is the complex conjugate of $H^{p,q}$. We define $h^{p,q} = \dim H^{p,q}$ which is called the *Hodge number*.

Let X be a compact complex manifold. A hermitian metric on X is called a *Kähler metric* if the associated $(1,1)$-form is d-closed. If X has a Kähler metric, X is called a *Kähler manifold*. The cohomology class of a Kähler form is called the *Kähler class*. In the case that X is a compact Kähler manifold, it follows from Hodge theory that there exists a direct decomposition

$$H^m(X, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(X),$$

that is, the free part of $H^m(X,\mathbb{Z})$ has a Hodge structure of weight m. Here $H^{p,q}(X)$ is the Dolbeault cohomology group. Moreover, there exists an isomorphism

$$H^q(X,\Omega_X^p) \cong H^{p,q}(X)$$

of cohomology groups. In particular, we will identify $H^{p,0}(X)$ and $H^0(X,\Omega_X^p)$ frequently.

Definition 3.11. A *polarized* Hodge structure of weight m is a Hodge structure $L \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{p,q}$ of weight m with a bilinear form

$$Q\colon L \otimes \mathbb{Q} \times L \otimes \mathbb{Q} \to \mathbb{Q}$$

satisfying the following conditions:

(1) Q is symmetric if m is even and alternating if m is odd.

(2) If $p \neq s$, then $Q(H^{p,q}, H^{r,s}) = 0$.

(3) If $\omega \neq 0 \in H^{p,q}$, then
$$\sqrt{-1}^{\,p-q} Q(\omega, \bar{\omega}) > 0.$$

Example 3.12. Let X be an n-dimensional projective manifold and $h \in H^2(X,\mathbb{Z})$ the hyperplane section. Defining

$$\begin{aligned} P^{n-k}(X) &= \{x \in H^{n-k}(X,\mathbb{C}) : \langle x, h^{k+1}\rangle = 0\}, \\ H^{p,q} &= P^{n-k}(X) \cap H^{p,q}(X) \quad (p+q = n-k), \end{aligned}$$

and

$$Q(x,y) = (-1)^{(n-k)(n-k-1)/2} \int_X h^k \wedge x \wedge y \quad (x,y \in P^{n-k}(X)),$$

we have a polarized Hodge structure $(H^{p,q}, Q)$ on $P^{n-k}(X)$.

Example 3.13. Let C be a compact Riemann surface. Then the Hodge decomposition $H^1(C,\mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C)$ with the cup product is a polarized Hodge structure. It is known that $H^{1,0}(C) \cong H^0(C,\Omega_C^1)$. For $\gamma \in H_1(C,\mathbb{Z})$, defining

$$\gamma\colon H^0(C,\Omega_C^1) \to \mathbb{C}, \quad \omega \to \int_\gamma \omega,$$

we have an injection $H_1(C,\mathbb{Z}) \to H^0(C,\Omega_C^1)^*$, and the quotient $H^0(C,\Omega_C^1)^*/H_1(C,\mathbb{Z})$ is the *Jacobian* $J(C)$ of C.

Example 3.14. In the case of $K3$ surfaces, the theory of periods is nothing but the Hodge structure on $H^2(X,\mathbb{Z})$ of weight 2,

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X),$$

where $H^{0,2}$ and $H^{2,0}$ are complex conjugations of each other and $H^{1,1}$ is defined over $\mathbb{R}$. If X is projective, then $P^2(X)$ has a polarized Hodge structure of weight 2 as mentioned in Example 3.12

For more details we refer the reader to Barth, Hulek, Peters, Van de Ven [BHPV], Beauville [Be1], Griffiths, Harris [GH].

3.2 Classification of complex analytic surfaces

Let X be a connected compact complex manifold. The transcendental degree of the meromorphic function field of X over $\mathbb{C}$ is called the algebraic dimension of X and denoted by $a(X)$. The value $a(X)$ can be $0, 1, \ldots, \dim(X)$. A surface with algebraic dimension 2 is called an *algebraic surface*.

A non-singular rational curve C on a surface with self-intersection number $C^2 = -1$ is called an *exceptional curve*. A surface is called *minimal* if it contains no exceptional curves. If a surface X contains an exceptional curve, then X is obtained from a surface by blowing up at a point p and C is the inverse image of p. In other words, we can blow down C to a point and obtain a new non-singular surface Y. Since the 2nd Betti number of Y is equal to that of X minus 1, by repeating this process we get a surface without exceptional curves, that is, a minimal surface. We first give a classification table (Table 3.1) of minimal complex analytic surfaces in terms of meromorphic function fields. We will give a definition of each surface later.

Table 3.1. Classification of surfaces by algebraic dimension

$a(X)$	Class of X
2	(Projective) algebraic surfaces
1	Elliptic surfaces
0	Complex tori, $K3$ surfaces, surfaces with $p_g = 0$, $b_1 = q = 1$

Remark 3.15. A complex torus and a $K3$ surface exist in any of the cases $a(X) = 0, 1, 2$.

In the first half of the 20th century, the classification of algebraic surfaces was made by the Italian school. Later, the classification of complex analytic surfaces was given by Kodaira, and Table 3.2 is the classification table in terms of Kodaira dimensions of minimal surfaces.

Table 3.2. Classification of surfaces by Kodaira dimension

$\kappa(X)$	Class of X
$-\infty$	Ruled surfaces, surfaces of type VII_0
0	Complex tori, bielliptic surfaces, $K3$ surfaces, Enriques surfaces, Kodaira surfaces
1	Elliptic surfaces
2	Surfaces of general type

A ruled surface is a 2-dimensional analogue of the projective line $\mathbb{P}^1$, a surface with $\kappa(X) = 0$ that of an elliptic curve, and a surface of general type that of a curve of genus greater than or equal to 2.

In the following we give the definition of surfaces in Tables 3.1, 3.2:

(1) A *ruled surface* X is an analytic fiber bundle $\pi\colon X \to C$ over a curve C such that each fiber $\pi^{-1}(x)$ $(x \in C)$ is isomorphic to $\mathbb{P}^1$ and the structure group is $\mathrm{PGL}(2, \mathbb{C})$. Only if C is a rational curve, is the surface X a *rational surface*, that is, X is obtained from the projective plane $\mathbb{P}^2$ by blowing up and blowing down successively.

(2) Let v_1, v_2, v_3, v_4 be elements in a 2-dimensional complex vector space linearly independent over $\mathbb{R}$ and let Γ be a free abelian group of rank 4 generated by them. The group Γ is a discrete subgroup of V and acts on V as translations, and the quotient $A = V/\Gamma$ has the structure of an abelian group and also the structure of a complex manifold. As a topological space, A is homeomorphic to $(S^1)^4$ and in particular compact. We call A a *complex torus*. The canonical line bundle is trivial and $p_g = 1$, $q = 2$ hold. In this book, we consider only 2-dimensional complex tori and hence we omit the dimension. If a complex torus is projective, it is called an *abelian surface*. The Jacobian of a curve of genus 2 (Example 3.13) is a typical example of an abelian surface.

(3) Let E, F be elliptic curves, and suppose that a finite group G acts on E as translations and on F such that $F/G \cong \mathbb{P}^1$. Then the quotient surface $(E \times F)/G$ is called a *bielliptic surface*. This name comes from the fact that

the projections $E \times F \to F$, $E \times F \to E$ induce two structures of elliptic fibrations on $(E \times F)/G$. A bielliptic surface is algebraic and $p_g = 0$, $q = 1$.

(4) A surface X is called a $K3$ *surface* if the canonical line bundle is trivial and $q(X) = 0$. Since K_X is trivial, by the adjunction formula it contains no exceptional curve and hence is minimal. We will discuss the details of $K3$ surfaces in Chapter 4.

(5) A surface X is called an *Enriques surface* if $p_g(X) = q(X) = 0$ and $K_X^{\otimes 2}$ is trivial. This surface was discovered by Enriques who was a central person in the Italian school. An Enriques surface has an unramified double covering which is a $K3$ surface and, conversely, if a $K3$ surface has a fixed-point-free automorphism of order 2, then the quotient by this automorphism is an Enriques surface. We will discuss Enriques surfaces in Chapter 9.

(6) An *elliptic surface* is a holomorphic map $\pi\colon X \to C$ from a surface X to a curve C with connected fibers such that any fiber except over finitely many points of C is an elliptic curve. When π is given by $\Phi_{|mK_X|}$, it is an elliptic surface with $\kappa(X) = 1$. There are surfaces with Kodaira dimension $-\infty, 0$ which have the structure of an elliptic surface. One such example is the surface obtained from a projective plane $\mathbb{P}^2$ by blowing up the 9 intersection points of two cubic curves. Later we introduce $K3$ and Enriques surfaces with the structure of an elliptic fibration.

(7) We call a surface with Kodaira dimension 2 a *surface of general type*. Among surfaces in $\mathbb{P}^3$ defined by a homogeneous polynomial in 4 variables of degree m, it is of general type if $m \geq 5$. On the other hand, it is rational if $m = 1, 2, 3$ and a $K3$ surface if $m = 4$ by the adjunction formula and the Lefschetz hyperplane theorem.

The above surfaces appear only in the classification of algebraic surfaces, except for complex tori, $K3$ surfaces, and elliptic surfaces which also appear in the non-algebraic case. On the other hand, the following surfaces do not appear in the classification of algebraic surfaces:

(8) A surface with $\kappa(X) = -\infty$, $b_1(X) = 1$ is called a *surface of class* VII_0. It is classically known that a surface, called a *Hopf surface*, whose universal covering is $\mathbb{C}^2 \setminus \{0\}$ is an example of such a surface. It was conjectured that the class of VII_0 surfaces consisted only of Hopf surfaces, but in 1972 Inoue discovered a different surface of class VII_0. It is now called an *Inoue surface*. Unfortunately a complete classification is not known. The name VII_0 comes from the numbering in the classification table given by Kodaira [Kod2]. The

suffix 0 in VII_0 means a minimal surface. (However, the definition of VII_0 here differs from that given by Kodaira. We follow the one given in Barth, Hulek, Peters, Van de Ven[BHPV].)

(9) A *primary Kodaira surface* is a surface with $b_1 = 3$ which has the structure of a locally trivial elliptic surface over an elliptic curve, and a surface with a primary Kodaira surface as its unramified covering is called a *secondary Kodaira surface*. The latter has $b_1 = 1$ and the structure of a locally trivial elliptic surface over a rational curve.

3.3 Elliptic surfaces and their singular fibers

An elliptic surface $\pi\colon X \to C$ is called *relatively minimal* if no fiber contains an exceptional curve. If a fiber contains an exceptional curve, then by blowing down it, we can reduce to a relatively minimal one.

Example 3.16. Let C_1, C_2 be two non-singular cubic curves meeting at 9 distinct points. Let F_1, F_2 be their respective defining polynomials, and let $C_{(t:s)}$ be the linear system of cubic curves defined by $tF_1 + sF_2 = 0$ for $(t : s) \in \mathbb{P}^1$. A general element in the linear system is non-singular and after blowing up the 9 base points of the linear system we obtain a non-minimal surface, but it is relatively minimal as an elliptic surface.

Example 3.17. Consider the complex manifolds

$$W_0 = \mathbb{P}^2 \times \mathbb{C}_0, \quad W_1 = \mathbb{P}^2 \times \mathbb{C}_1.$$

Here $\mathbb{C}_0$, $\mathbb{C}_1$ are complex planes $\mathbb{C}$. We identify a point $((x : y : z), u)$ of W_0 and a point $((x_1 : y_1 : z_1), u_1)$ of W_1 by the equations

$$uu_1 = 1, \quad x = u^4 x_1, \quad y = u^6 y_1, \quad z = z_1$$

and denote by W the obtained manifold. The projection

$$\pi\colon ((x : y : z), u) \to u$$

gives the structure of a fiber bundle on W over $\mathbb{P}^1$ with fibers $\mathbb{P}^2$.

For $\tau = (\tau_0, \tau_1, \dots, \tau_8, \sigma_1, \dots, \sigma_{12}) \in \mathbb{C}^{21}$, we put

$$g(u) = \tau_0 \prod_{\nu=1}^{8} (u - \tau_\nu), \quad h(u) = \prod_{\nu=1}^{12} (u - \sigma_\nu),$$

and define a submanifold Y_τ of W by the equations

$$\begin{cases} y^2z - 4x^3 + g(u)xz^2 + h(u)z^3 = 0, \\ y_1^2z_1 - 4x_1^3 + u_1^8 g(1/u_1)x_1z_1^2 + u_1^{12}h(1/u_1)z_1^3 = 0. \end{cases} \tag{3.5}$$

For a fixed τ, this is a cubic curve in $\mathbb{P}^2$. Moreover, we define a rational function $\mathcal{J}_\tau(u)$ by

$$\mathcal{J}_\tau(u) = \frac{g(u)^3}{g(u)^3 - 27h(u)^2}.$$

Now we assume that $\tau \in \mathbb{C}^{21}$ satisfies the following three conditions:

$$\tau_0 \neq 0, \quad \tau_0^3 \neq 27; \tag{3.6}$$

$$\text{if } g(\sigma_\lambda) = 0, \text{ then } \sigma_\nu \neq \sigma_\lambda \ \ (\nu \neq \lambda); \tag{3.7}$$

$$\text{any pole of } \mathcal{J}_\tau(u) \text{ has multiplicity } 1. \tag{3.8}$$

Exercise 3.18. Assume that τ satisfies the three conditions (3.6)–(3.8). Then show that Y_τ is non-singular.

Denote by π_τ the restriction of the projection π to Y_τ. Then

$$\pi_\tau : Y_\tau \longrightarrow \mathbb{P}^1$$

is an elliptic surface. Let F_u be the fiber $\pi_\tau^{-1}(u)$ over u. Now we assume that $\sigma_\nu = \tau_\nu$ $(1 \leq \nu \leq r)$ and σ_ν is different to any τ_λ for $r+1 \leq \nu \leq 12$. Let $a_1, \dots, a_j$ be poles of $\mathcal{J}_\tau(u)$. Then the multiplicity of each pole is 1 by condition (3.8). Thus we have

$$g(u)^3 - 27h(u)^2 = (\tau_0^3 - 27)\prod_{\nu=1}^{r}(u-\tau_\nu)^2\prod_{\rho=1}^{j}(u-a_\rho).$$

We remark here that $j + 2r = 24$. Condition (3.7) implies that $\tau_\nu \neq \tau_\lambda$ for $1 \leq \nu < \lambda \leq r$.

Exercise 3.19. Show that F_u is a non-singular elliptic curve if $u \neq \tau_1, \dots, \tau_r, a_1, \dots, a_j$, F_ρ $(\rho = a_1, \dots, a_j)$ is a rational curve with a node, and F_ν $(1 \leq \nu \leq r)$ is a cubic curve with a *cusp* defined by $y^2z - 4x^3 = 0$. Here a *node* is a singularity locally isomorphic to the singular point $(0,0)$ of the curve defined by $x^2 + y^2 = 0$.

Thus the elliptic surface Y_τ has j cubic curves with a node, r cubic curves with a cusp as fibers, and the others are non-singular. Moreover, we have $j + 2r = 24$. This equation means that the Euler number $e(Y_\tau) = c_2(Y_\tau)$ of Y_τ is the sum of those of the singular fibers. In general, the Euler number of an elliptic surface coincides with the sum of those of the singular fibers. As in Table 3.3, the Euler number of a cubic curve with a node (type I_1) is 1 and that of a cubic curve with a cusp (type II) is 2.

A fiber of an elliptic surface is called *singular* if it is not non-singular. The singular fibers of relatively minimal elliptic surfaces are classified as in Table 3.3.

Table 3.3.

Singular fiber	Extended Dynkin diagram	Euler number
$m\mathrm{I}_0$ $(m \geq 2)$	—	0
$m\mathrm{I}_1$ $(m \geq 1)$	—	1
$m\mathrm{I}_n$ $(m \geq 1, n \geq 2)$	$\tilde{A}_{n-1}$	n
II	—	2
III	$\tilde{A}_1$	3
IV	$\tilde{A}_2$	4
I_n^* $(n \geq 1)$	$\tilde{D}_{n+4}$	$n+6$
II^*	$\tilde{E}_8$	10
III^*	$\tilde{E}_7$	9
IV^*	$\tilde{E}_6$	8

In the following, we explain the notation in Table 3.3. First of all, let

$$F = \sum_i m_i C_i$$

be the irreducible decomposition of a singular fiber F. Here C_i is an irreducible curve and m_i is a positive integer. A fiber F is called a *multiple fiber* if the greatest common divisor of the m_i is greater than or equal to 2. In Table 3.3, the symbol $m\mathrm{I}_n$ means a singular fiber of type I_n with multiplicity m. The irreducible decomposition of each fiber is as follows:

(1) $m\mathrm{I}_0$: $F = mC$, where C is a non-singular elliptic curve.

(2) $m\mathrm{I}_1$: $F = mC$, where C is a rational curve with a node.

(3) $m\mathrm{I}_{n+1}$ $(n \geq 1)$: $F = m(C_1 + \cdots + C_{n+1})$, where each component C_i is a non-singular rational curve, and the non-zero intersection numbers between them are given by $C_1 \cdot C_2 = C_2 \cdot C_3 = \cdots = C_n \cdot C_{n+1} = C_{n+1} \cdot C_1 = 1$ and others are zero except for the cases $n = 1, 2$. In the case $n = 1$, C_1 and C_2 meet transversally at two distinct points and in the case $n = 2$, C_1, C_2, C_3 do not meet at one point.

(4) II: $F = C$, where C is a rational curve with a cusp.

(5) III: $F = C_1 + C_2$, where C_1, C_2 are both non-singular rational curves and meet at one point with multiplicity 2.

(6) IV: $F = C_1 + C_2 + C_3$, where C_1, C_2, C_3 are non-singular rational curves and 3 curves meet at one point transversally with each other.

(7) I^*_{n-4} $(n \geq 4)$: $F = C_1 + C_2 + 2(C_3 + \cdots + C_{n-1}) + C_n + C_{n+1}$, where each C_i is a non-singular rational curve and their intersection is described by the following dual graph of them. In the following cases (8)–(10), irreducible components of singular fibers are all non-singular rational curves and their intersections are similarly described.

(8) II^*: $F = 2C_1 + 4C_2 + 6C_3 + 3C_4 + 5C_5 + 4C_6 + 3C_7 + 2C_8 + C_9$.

(9) III^*: $F = C_1 + 2C_2 + 3C_3 + 4C_4 + 3C_5 + 2C_6 + C_7 + 2C_8$.

(10) IV^*: $F = C_1 + 2C_2 + 3C_3 + 2C_4 + C_5 + 2C_6 + C_7$.

In the case that the number of irreducible components of a singular fiber is greater than or equal to 2, all irreducible components are non-singular rational curves. In this case we define the *dual graph* of a singular fiber as follows. We represent a vertex $\circ$ for each irreducible component and join two vertices corresponding to C_i, C_j by $C_i \cdot C_j$-tuple edges. The obtained dual graph coincides with one of the diagrams $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, called *extended Dynkin diagrams*, given in Figure 3.1. The dual graph of a singular fiber of type I^*_{n-4}, II^*, III^*, or IV^* mentioned above corresponds to $\tilde{D}_n, \tilde{E}_8, \tilde{E}_7$, or $\tilde{E}_6$, respectively. The dual graph of a singular fiber of type I_{n+1}, III, or IV is given by $\tilde{A}_n, \tilde{A}_1$, or $\tilde{A}_2$, respectively.

Remark 3.20. It is known that there exist invariants called the functional invariant and the topological invariant for an elliptic surface which determine the type of singular fibers (Kodaira [Kod1]).

Remark 3.21. A singular fiber F with multiplicity greater than or equal to 2 occurs only when F is not simply connected.

Finally, we give a proof of the classification table, Table 3.3, of singular fibers following Kodaira [Kod1] (it might be a little longer). For simplicity, we assume that all singular fibers are reduced (i.e., with multiplicity 1).

First, consider the case that a singular fiber F is irreducible. In this case, it follows from formula (3.3) that F is a rational curve with a node or a cusp, and hence is of type I_1 or II.

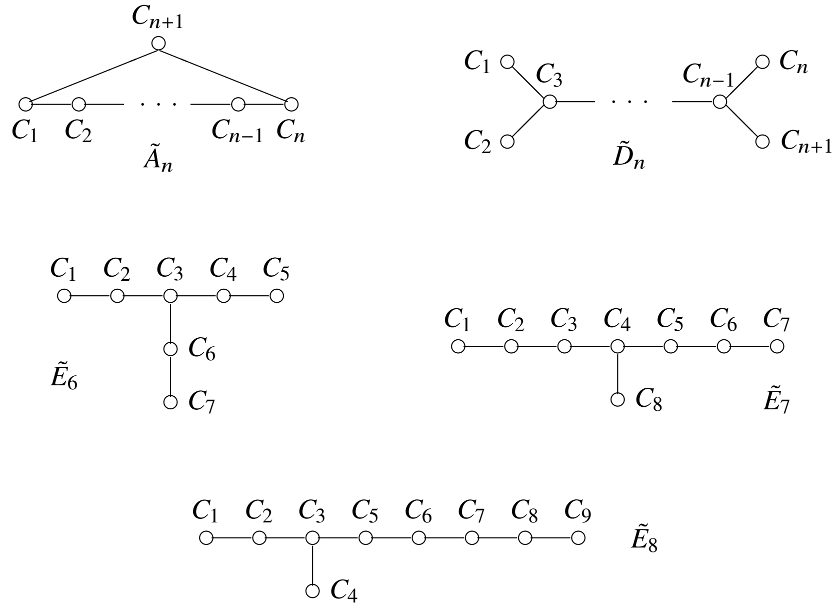


Figure 3.1. Extended Dynkin diagrams.

Lemma 3.22. *Assume the number of irreducible components of* $F = \sum_i m_i C_i$ *is greater than or equal to* 2*. Then any irreducible component* C_i *is a non-singular rational curve and*

$$C_i^2 = -2, \quad 2m_i = \sum_{j \neq i} m_j C_j \cdot C_i. \tag{3.9}$$

Proof. First, the cohomology classes of the singular fiber $F = \sum_i m_i C_i$ and a general fiber F' coincide, and hence $F \cdot C_i = F' \cdot C_i = 0$, that is, we have

$$m_i C_i^2 + \sum_{j \neq i} m_j C_j \cdot C_i = 0. \tag{3.10}$$

On the other hand, by the adjunction formula $K_X \cdot F' + F'^2 = 2p_a(F') - 2 = 0$ and $F'^2 = 0$, we have $K_X \cdot F' = 0$. Combining these, we have $\sum_i m_i C_i \cdot K_X = F \cdot K_X = F' \cdot K_X = 0$, and then together with $K_X \cdot C_i = 2p_a(C_i) - 2 - C_i^2$, we have

$$\sum_i m_i \bigl(2p_a(C_i) - 2 - C_i^2\bigr) = 0. \tag{3.11}$$

Since F is connected, it follows that

$$\sum_{j \neq i} m_j C_j \cdot C_i \geq \sum_{j \neq i} C_j \cdot C_i \geq 1.$$

Equation (3.10) implies that $C_i^2 \le -1$. By the assumption that the surface is relatively minimal, F contains no exceptional curves and hence $C_i^2 \le -2$ if $p_a(C_i) = 0$. Therefore, for any C_i, it follows that $2p_a(C_i) - 2 - C_i^2 \ge 0$. By considering equation (3.11), we have $2p_a(C_i) - 2 - C_i^2 = 0$, that is, $p_a(C_i) = 0$, $C_i^2 = -2$. Thus any irreducible component C_i is a non-singular rational curve and satisfies

$$C_i^2 = -2, \quad 2m_i = \sum_{j \neq i} m_j C_j \cdot C_i. \qquad \square$$

By using Lemma 3.22, we will give the classification of singular fibers with at least 2 irreducible components. We assume that singular fibers are reduced (multiplicity 1). We determine the singular fibers in the following order:

(i) the case that there exist irreducible components C_i, C_j with $C_i \cdot C_j \ge 2$;

(ii) the case that at least 3 irreducible components meet at one point;

(iii) the case that the dual graph of a singular fiber contains a cycle;

(iv) the case that the dual graph of a singular fiber contains no cycles.

Case (i) Assume that there exist irreducible components C_i, C_j with $C_i \cdot C_j \ge 2$. We may assume $m_i \le m_j$. Since

$$2m_i = m_j C_j \cdot C_i + \sum_{k \neq i,j} m_k C_k \cdot C_i \ge 2m_i,$$

we have $m_i = m_j$, $C_j \cdot C_i = 2$, $C_k \cdot C_i = 0$ $(k \neq i, j)$. The same argument shows that $C_j \cdot C_k = 0$ $(k \neq i, j)$, and then the connectedness of F implies that $F = C_i + C_j$, and thus F is of type III or I_2.

In the following, it suffices to consider the case that for any irreducible components C_i, C_j $(i \neq j)$,

$$C_i \cdot C_j \le 1. \tag{3.12}$$

Case (ii) Assume that 3 irreducible components C_1, C_2, C_3 meet at one point. We may assume that $m_1 \le m_2 \le m_3$. By equation (3.9), we have

$$2m_1 = m_2 + m_3 + \sum_{i \neq 1,2,3} C_i \cdot C_1 \ge 2m_1.$$

It follows that $m_1 = m_2 = m_3 = m$ and, moreover, $C_i \cdot C_1 = 0$ $(i \neq 1, 2, 3)$. The connectedness of F implies that $F = C_1 + C_2 + C_3$ and F is of type IV.

Case (iii) Assume that the dual graph of irreducible components $C_1, \dots, C_n$ of F is of type $\tilde{A}_{n-1}$ ($n \geq 2$). Here assume that $C_1 \cdot C_2 = \cdots = C_{n-1} \cdot C_n = C_n \cdot C_1 = 1$ and $m_1 \leq m_j$ ($j \neq 1$). By equation (3.9), we have

$$2m_1 = m_2 + m_n + \sum_{j \neq 1,2,n} m_j C_j \cdot C_1 \geq 2m_1.$$

It follows that $m_1 = m_2 = m_n$ and C_1 does not meet any irreducible components except for C_2, C_n. The same argument shows that $m_2 = m_3$ and C_2 does not meet any irreducible components except for C_1, C_3. By applying this argument successively and by using the connectedness of F, finally we have $F = C_1 + \cdots + C_n$ and F is of type I_n.

Case (iv) We may assume that m_1 is the minimum among the coefficients m_i of the irreducible decomposition $F = \sum_{i=1}^n m_i C_i$. Suppose that C_1 meets at least two other irreducible components, for example, C_2, C_3. Then it follows from

$$2m_1 = m_2 + m_3 + \sum_{j \neq 1,2,3} m_j C_j \cdot C_1 \geq 2m_1$$

that $m_1 = m_2 = m_3$ and C_1 does not meet any components except for C_2, C_3. If C_2 meets C_3, then the dual graph contains a cycle, which is a contradiction. Next we consider C_2 instead of C_1. Equation (3.9) implies that C_2 meets an irreducible component other than C_1. By repeating this argument, finally the dual graph contains a cycle, which is a contradiction. Therefore C_1 meets only one irreducible component. We denote this component by C_2. Then it follows from equation (3.9) that

$$2m_1 = m_2. \tag{3.13}$$

In the following, we divide the proof according to how C_2 meets components other than C_1.

Case (iv-1) Assume that C_2 meets at least 3 components C_3, C_4, C_5 other than C_1. Then it follows from equations (3.9) and (3.13) that

$$m_3 + m_4 + m_5 \geq 3m_1 = m_3 + m_4 + m_5 + \sum_{j \geq 6} m_j C_j \cdot C_2.$$

Therefore $m_1 = m_3 = m_4 = m_5$. By this and equations (3.9), (3.13), we can see that C_3, C_4, C_5 intersect only C_2. The connectedness of F implies that $n = 5$ and $F = C_1 + 2C_2 + C_3 + C_4 + C_5$, that is, F is of type I_0^*.

Case (iv-2) Assume that C_2 meets exactly two components C_3, C_4 other than C_1. Since there are no cycles in the dual graph, we obtain $C_3 \cdot C_4 = 0$, $3m_1 = m_3 + m_4$, and

$$2m_3 = 2m_1 + \sum_{j \geq 5} m_j C_j \cdot C_3, \quad 2m_4 = 2m_1 + \sum_{j \geq 5} m_j C_j \cdot C_4.$$

We may assume $m_3 \leq m_4$. If C_3 meets a component other than C_2, for example C_5, then it follows from

$$3m_1 \geq 2m_3 = 2m_1 + m_5 + \cdots \geq 3m_1$$

that $m_1 = m_5$, $3m_1 = 2m_3$. By combining this with $2m_5 = m_3 + \sum_{j \geq 6} m_j C_j \cdot C_5$, we have

$$\frac{1}{2} m_1 = \sum_{j \geq 6} m_j C_j \cdot C_5,$$

which contradicts the minimality of m_1. Therefore C_3 meets only C_2 and

$$m_3 = m_1, \quad m_4 = 2m_1, \quad 2m_1 = \sum_{j \geq 5} m_j C_j \cdot C_4.$$

By the last equation, we can see that C_4 meets at most two components among C_j, $j \geq 5$. If C_4 meets C_5, C_6, then $m_5 = m_6 = m_1$ and

$$F = C_1 + 2C_2 + C_3 + 2C_4 + C_5 + C_6,$$

and hence F is of type I_1^*. If C_4 meets only component C_5, then we have

$$m_5 = 2m_1, \quad 2m_1 = \sum_{j \geq 6} m_j C_j \cdot C_5.$$

By repeating this process, we see that F is of type I_n^*.

Case (iv-3) Assume that C_2 meets exactly one component C_3 other than C_1. In this case it follows from $2m_2 = m_1 + m_3$ and $m_2 = 2m_1$ that

$$3m_1 = m_3. \tag{3.14}$$

Therefore it suffices to consider the following case: there are irreducible components $C_1, \ldots, C_h$ ($h \geq 3$) such that each C_i ($2 \leq i \leq h-1$) meets exactly two components C_{i-1}, C_{i+1}, and C_h meets C_{h-1} and at least two components C_{h+1}, C_{h+2}. Since there are no cycles, we have $C_{h+1} \cdot C_{h+2} = 0$. Then it follows from equation (3.9) that

$$m_i = im_1 \quad (i = 2, \dots, h), \tag{3.15}$$

$$(h+1)m_1 = m_{h+1} + m_{h+2} + \sum_{j \geq h+3} m_j C_j \cdot C_h, \tag{3.16}$$

$$2m_{h+1} = hm_1 + \sum_{j \geq h+3} m_j C_j \cdot C_{h+1}, \tag{3.17}$$

$$2m_{h+2} = hm_1 + \sum_{j \geq h+3} m_j C_j \cdot C_{h+2}. \tag{3.18}$$

Note that C_h meets only $C_{h-1}, C_{h+1}, C_{h+2}$. In fact, if C_h meets C_{h+3}, then by equation (3.9), we have

$$2m_{h+3} = m_h + \cdots = hm_1 + \cdots \geq hm_1,$$

and hence by equations (3.16)–(3.18), we obtain

$$2(h+1)m_1 \geq 2m_{h+1} + 2m_{h+2} + 2m_{h+3} \geq 3hm_1,$$

which contradicts $h \geq 3$. Thus C_h meets only $C_{h-1}, C_{h+1}, C_{h+2}$.

Next, by equation (3.16) we have

$$(h+1)m_1 = m_{h+1} + m_{h+2} \tag{3.19}$$

and, by equations (3.17), (3.18),

$$2m_1 = \sum_{j \geq h+3} m_j C_j \cdot C_{h+1} + \sum_{j \geq h+3} m_j C_j \cdot C_{h+2}. \tag{3.20}$$

Here we may assume that $m_{h+1} \geq m_{h+2}$. Then we have

$$2m_{h+1} \geq (h+1)m_1 = hm_1 + m_1$$

by equation (3.19), and hence by equation (3.17) we can prove that C_{h+1} meets at least one of C_j $(j \geq h+3)$. We denote this component by C_{h+3}. We claim that C_{h+1} does not meet other components. If C_{h+1} meets C_{h+4}, then it follows from equation (3.17) and the minimality of m_1 that

$$2m_{h+1} \geq hm_1 + m_{h+3} + m_{h+4} \geq (h+2)m_1. \tag{3.21}$$

By equation (3.20) we obtain $2m_1 \geq m_{h+3} + m_{h+4} \geq 2m_1$, and in particular $m_1 = m_{h+3} = m_{h+4}$. By this, equation (3.9), and inequality (3.21), we have

$$2m_1 = 2m_{h+4} = m_{h+1} + \cdots \geq \frac{h+2}{2} m_1,$$

which contradicts $h \geq 3$. Thus C_{h+1} meets only C_h, C_{h+3}.

In the following, we divide the proof according to how C_{h+2} meets other components.

Case (iv-3-α) Assume that C_{h+2} meets the component C_{h+4}, as well as C_h. Then by equation (3.20) we have

$$2m_1 \geq m_{h+3} + m_{h+4} \geq 2m_1,$$

and hence $m_1 = m_{h+3} = m_{h+4}$. Moreover, we obtain

$$2m_1 = 2m_{h+3} \geq m_{h+1} = \frac{h}{2}m_1 + \frac{1}{2}m_{h+3} = \frac{h+1}{2}m_1$$

by equations (3.9), (3.17). Hence it follows that $h = 3$ and C_{h+3} meets only C_{h+1}. Similarly C_{h+4} meets only C_{h+2} and equation (3.19) implies that $m_{h+1} = m_{h+2} = 2m_1$. Since F is assumed to be reduced, we have $m_1 = 1$ and

$$F = C_1 + 2C_2 + 3C_3 + 2C_4 + 2C_5 + C_6 + C_7,$$

that is, F is a singular fiber of type IV*.

Case (iv-3-β) Assume that C_{h+2} meets only C_h. In this case, it follows from equations (3.9), (3.15) that $2m_{h+2} = m_h = hm_1$. Therefore, by equation (3.19), we obtain $m_{h+1} = (\frac{h}{2} + 1)m_1$. On the other hand, it follows from equation (3.20) that $2m_1 = m_{h+3}$. Moreover, by

$$2m_{h+3} = m_{h+1} + \sum_{j \geq h+4} m_j C_j \cdot C_{h+3},$$

we obtain the equation

$$3m_1 = \frac{h}{2}m_1 + \sum_{j \geq h+4} m_j C_j \cdot C_{h+3}. \tag{3.22}$$

Hence it follows from the minimality of m_1 and $h \geq 3$ that $h = 3, 4$, or 6.

In the case $h = 6$, equation (3.22) implies that C_{h+3} does not meet C_j ($j \geq h+4 = 10$). By the connectedness of F, we obtain

$$F = C_1 + 2C_2 + 3C_3 + 4C_4 + 5C_5 + 6C_6 + 4C_7 + 3C_8 + 2C_9$$

which is a singular fiber of type II*.

In the case $h = 4$, it follows from equation (3.22) that C_{h+3} meets another component C_{h+4}, as well as C_{h+1}, and hence $m_{h+4} = m_1 = \frac{1}{2}m_{h+3}$. Then equation (3.9) implies that C_{h+4} meets only C_{h+3}. We conclude that

$$F = C_1 + 2C_2 + 3C_3 + 4C_4 + 3C_5 + 2C_6 + 2C_7 + C_8$$

which is a singular fiber of type III^*.

Finally, we show that $h = 3$ does not occur. Assume $h = 3$. Then C_{h+3} meets exactly one component C_{h+4} other than C_{h+1} and $m_{h+4} = \frac{3}{2}m_1$ because of equation (3.22) and the minimality of m_1. By $m_{h+3} = 2m_1$, we have

$$3m_1 = 2m_{h+4} = 2m_1 + \sum_{j \geq h+5} m_j C_j \cdot C_{h+4}.$$

This implies that C_{h+4} meets another component C_{h+5}, $m_{h+5} = m_1$, and

$$2m_1 = 2m_{h+5} = \frac{3}{2}m_1 + \sum_{j \geq h+6} m_j C_j \cdot C_{h+5}.$$

This contradicts the minimality of m_1.

Remark 3.23. The classification of singular fibers of elliptic surfaces and their concrete construction are due to Kodaira [Kod1]. We have also referred to Kodaira [Kod2].

4

$K3$ surfaces and examples

First, we study fundamental properties of a $K3$ surface. As an example, we introduce the Kummer surface associated with a curve of genus 2 discovered in the 19th century. Finally, we state the Torelli theorem for 2-dimensional complex tori which will be used in the proof of the Torelli-type theorem for Kummer surfaces.

4.1 Definition and examples of $K3$ surfaces

Let X be a $K3$ surface. Recall that X is a connected compact 2-dimensional complex manifold (surface) satisfying the conditions $K_X = 0$, $q(X) = 0$ (Section 3.2). By the adjunction formula (Theorem 3.3), X is a minimal surface.

The transition function of the canonical bundle is defined by the Jacobian $J_{ji} = \det\left(\frac{\partial(x_i,y_i)}{\partial(x_j,y_j)}\right)$ which is nowhere zero on $U_i \cap U_j$ where $\{(U_i,(x_i,y_i))\}_i$ is an open covering. Let $\psi = \{\psi_i\}$ be a section of K_X. Then we have $\psi_j = J_{ji}\psi_i$ on $U_i \cap U_j$, that is, $\psi_j\, dx_j \wedge dy_j = \psi_i\, dx_i \wedge dy_i$. Thus we have an isomorphism $\mathcal{O}(K_X) \cong \Omega_X^2$ of sheaves. In particular, $K_X = 0$ is equivalent to the existence of a nowhere-vanishing holomorphic 2-form on X.

On the other hand, by the condition $q(X) = 0$ and the exact sequence of cohomology (3.2), we obtain the injection

$$H^1(X,\mathcal{O}_X^*) \xrightarrow{\delta} H^2(X,\mathbb{Z}).$$

By definition, $c_1(X) = \delta(-K_X)$, and thus the conditions $c_1(X) = 0$ and $K_X = 0$ are equivalent.

Example 4.1. A non-singular quartic surface X_4 in a projective space $\mathbb{P}^3$ is a $K3$ surface. It follows from the adjunction formula (Theorem 3.3) and the Lefschetz hyperplane theorem (Theorem 3.8) respectively that the canonical line bundle is trivial and the irregularity is zero. Similarly, the intersection $X_6 = Q_2 \cap Y_3$ of a quadric Q_2 and a cubic Y_3 in $\mathbb{P}^4$ and the intersection $X_8 = Q_1 \cap Q_2 \cap Q_3$ of 3 quadrics Q_i in $\mathbb{P}^5$ are $K3$ surfaces if they are non-singular. The hyperplane sections $H|X$ of these $K3$ surfaces have the self-intersection numbers 4, 6, 8 and they are called $K3$ surfaces of degrees 4, 6, 8, respectively. A $K3$ surface of degree 2 is obtained as a double cover of the projective plane $\mathbb{P}^2$. The branch divisor is a non-singular plane sextic (see Proposition 3.9).

Exercise 4.2. Show that the elliptic surface Y_τ given in Example 3.17 is a $K3$ surface. Here we can use the fact $c_2(Y_\tau) = 24$.

It follows from Theorem 3.5 that $b_1(X) = 0$. Moreover, we have the following.

Lemma 4.3. $H_1(X, \mathbb{Z}) = 0$.

Proof. Since $H_1(X, \mathbb{R}) = 0$, $H_1(X, \mathbb{Z})$ is a finite abelian group. Consider an element of $H_1(X, \mathbb{Z})$ of order n. We have an unramified covering of X of degree n,

$$\pi\colon \tilde{X} \to X,$$

associated with this element. Then $\tilde{X}$ is also a connected compact analytic surface with $e(\tilde{X}) = ne(X) = 24n$. There exists a nowhere-vanishing holomorphic 2-form ω_X on X, and its pullback $\pi^*(\omega_X)$ by π is the one on $\tilde{X}$, and hence $K_{\tilde{X}} = 0$. It follows from Noether's formula (Theorem 3.2) that

$$24n = e(\tilde{X}) = c_2(\tilde{X}) = 12(p_g(\tilde{X}) - q(\tilde{X}) + 1) = 12(2 - q(\tilde{X})).$$

This implies $n = 1$. □

Remark 4.4. A $K3$ surface is simply connected (see Corollary 6.40).

By Lemma 4.3, we have $H^1(X, \mathbb{Z}) = H^3(X, \mathbb{Z}) = 0$. Moreover, by the universal coefficient theorem

$$0 \to \mathrm{Ext}(H_1(X, \mathbb{Z}), \mathbb{Z}) \to H^2(X, \mathbb{Z}) \to \mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}) \to 0,$$

$H^2(X, \mathbb{Z})$ is a finitely generated free abelian group. On the other hand, by putting $c_1(X) = 0$ in Noether's formula (Theorem 3.2),

$$c_1(X)^2 + c_2(X) = 12(p_g(X) - q(X) + 1) = 24,$$

we obtain that the Euler number $e(X) = c_2(X)$ of X is 24. Thus we have proved $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{22}$. Next we study the structure of the lattice $H^2(X, \mathbb{Z})$ with the cup product $\langle\ ,\ \rangle$.

Theorem 4.5. *The lattice* $(H^2(X, \mathbb{Z}), \langle\ ,\ \rangle)$ *is isomorphic to* $U^{\oplus 3} \oplus E_8^{\oplus 2}$.

Proof. The lattice $(H^2(X, \mathbb{Z}), \langle\ ,\ \rangle)$ is unimodular by Poincaré duality, and has signature $(3, 19)$ by Hirzebruch's index theorem (Theorem 3.4). We use the Wu formula in topology to prove that it is an even lattice. First, recall that there exists a homomorphism called the Steenrod operator,

$$\mathrm{Sq}^i : H^n(X, \mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(X, \mathbb{Z}/2\mathbb{Z}) \quad (n, i \geq 0),$$

satisfying

$$\mathrm{Sq}^0(a) = a, \quad \mathrm{Sq}^n(a) = \langle a, a \rangle, \quad \mathrm{Sq}^i(a) = 0 \quad (i > n), \quad a \in H^n(X, \mathbb{Z}/2\mathbb{Z})$$

(Milnor, Stasheff [MS, §8]), where $\langle\ ,\ \rangle$ is the cup product. By the duality, there exists a $v_k \in H^k(X, \mathbb{Z}/2\mathbb{Z})$ satisfying

$$(\langle a, v_k \rangle, \mu) = (\mathrm{Sq}^k(a), \mu)$$

corresponding to a homomorphism

$$H^{4-k}(X, \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}, \quad a \to (\mathrm{Sq}^k(a), \mu).$$

Here $(\ ,\ )$ is the Kronecker index and μ is a generator of $H^4(X, \mathbb{Z}/2\mathbb{Z})$. Wu's formula (Milnor, Stasheff [MS, Thm. 11.14]) claims that the second Stiefel–Whitney class $w_2 \in H^2(X, \mathbb{Z}/2\mathbb{Z})$ coincides with $\sum_{i+j=2} \mathrm{Sq}^i(v_j) = v_2$. Therefore we have

$$(\langle x, x \rangle, \mu) = (\mathrm{Sq}^2(x), \mu) = (\langle x, w_2 \rangle, \mu)$$

for $x \in H^2(X, \mathbb{Z}/2\mathbb{Z})$. On the other hand, w_2 is the modulo 2 reduction of $c_1(X)$ in $H^2(X, \mathbb{Z}/2\mathbb{Z})$ (Hirzebruch [Hi, p. 73]), and hence $\langle x, x \rangle$ is an even integer. Now the assertion follows from Theorem 1.27. □

By definition of a $K3$ surface, $H^0(X, \Omega^2_X) \cong \mathbb{C}$, that is, there exists a unique nowhere-vanishing holomorphic 2-form ω_X on X up to a constant. The equations

$$\langle \omega_X, \omega_X \rangle = \int_X \omega_X \wedge \omega_X, \quad \langle \omega_X, \bar{\omega}_X \rangle = \int_X \omega_X \wedge \bar{\omega}_X$$

imply that ω_X satisfies a condition called the *Riemann condition*,

$$\langle \omega_X, \omega_X \rangle = 0, \quad \langle \omega_X, \bar{\omega}_X \rangle > 0, \tag{4.1}$$

where the cup product is the one extended to $H^2(X, \mathbb{C})$. By an elementary calculation, the Riemann condition is equivalent to

$$\langle \mathrm{Re}(\omega_X), \mathrm{Re}(\omega_X) \rangle = \langle \mathrm{Im}(\omega_X), \mathrm{Im}(\omega_X) \rangle > 0, \quad \langle \mathrm{Re}(\omega_X), \mathrm{Im}(\omega_X) \rangle = 0. \tag{4.2}$$

Therefore the subspace $E(\omega_X)$ in $H^2(X, \mathbb{R})$ generated by $\mathrm{Re}(\omega_X)$, $\mathrm{Im}(\omega_X)$ is a 2-dimensional positive definite subspace.

Exercise 4.6. Show that conditions (4.1) and (4.2) are equivalent.

Definition 4.7. We denote by $H^{1,1}(X,\mathbb{R})$ the orthogonal complement of $E(\omega_X)$ in $H^2(X,\mathbb{R})$.

Since $H^2(X,\mathbb{R})$ has signature $(3,19)$, the signature of $H^{1,1}(X,\mathbb{R})$ is $(1,19)$.

Lemma 4.8. *Let X be a $K3$ surface and let $c \in H^2(X,\mathbb{Z})$. Then the following are equivalent*:

(1) *There exists a line bundle L on X with $c = c_1(L)$.*

(2) $c \in H^{1,1}(X,\mathbb{R})$.

(3) $\langle c, \omega_X \rangle = 0$.

Proof. By $H^1(X,\mathcal{O}_X) = 0$ and the exact sequence (3.2), we have

$$0 \to H^1(X,\mathcal{O}_X^*) \xrightarrow{\delta} H^2(X,\mathbb{Z}) \xrightarrow{i} H^2(X,\mathcal{O}_X).$$

The map δ sends a line bundle L to the first Chern class $c_1(L)$ and the map i coincides with the projection

$$H^2(X,\mathbb{C}) \to H^{0,2}(X),$$

and hence the lemma follows. □

In view of Lemma 4.8, the following sublattice of $H^2(X,\mathbb{Z})$ is important (see Remark 4.10).

Definition 4.9. For a $K3$ surface X, we define

$$S_X = \{x \in H^2(X,\mathbb{Z}) : \langle x, \omega_X \rangle = 0\} = H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{R})$$

and call it the *Néron–Severi lattice*. Note that S_X is nothing but the Néron–Severi group of X, and hence it is frequently denoted by $\mathrm{NS}(X)$. Since the Picard group and the Néron–Severi group are isomorphic under the injection

$$H^1(X,\mathcal{O}_X^*) \xrightarrow{\delta} H^2(X,\mathbb{Z}),$$

we also call S_X the *Picard lattice*. If $x \in S_X$ is represented by a divisor D, we denote $x = [D]$ or simply $x = D$. Since the rank of $H^{1,1}(X,\mathbb{R})$ is 20, we have

$$0 \le r = \operatorname{rank} S_X \le 20.$$

The structure of the lattice S_X coincides with the one defined by the intersection form. The rank of the Néron–Severi lattice is called the *Picard number* and is denoted by $\rho(X)$. The orthogonal complement

$$T_X = \{x \in H^2(X,\mathbb{Z}) : \langle x, y\rangle = 0 \; \forall \, y \in S_X\}$$

of S_X is called the *transcendental lattice*.

Remark 4.10. In this book, we assume that a lattice is non-degenerate; however, Néron–Severi lattices and transcendental lattices might be degenerate (see Proposition 4.11). To avoid complication, we call the degenerate ones "lattices" too.

Note that the signature of $H^{1,1}(X,\mathbb{R})$ is $(1, 19)$ and hence S_X has at most one positive eigenvalue.

Proposition 4.11. *Let $a(X)$ be the algebraic dimension of a $K3$ surface X and let r be the rank of S_X. Then the following hold*:

(1) *If $a(X) = 2$ then S_X is non-degenerate and has the signature $(1, r - 1)$.*

(2) *If $a(X) = 1$ then S_X has a 1-dimensional kernel and the quotient by the kernel is negative definite.*

(3) *If $a(X) = 0$ then S_X is negative definite.*

Remark 4.12. We will present an example of a $K3$ surface of case (2) from Proposition 4.11 in Remark 6.41.

Lemma 4.13. *Let C be an irreducible curve on a $K3$ surface X. Then the following hold*:

(1) *If $C^2 = -2$ then C is a non-singular rational curve and $h^0(\mathcal{O}_X(C)) = 1$.*

(2) *If $C^2 = 0$ then $p_a(C) = 1$ and $h^0(\mathcal{O}_X(C)) = 2$.*

(3) *If $C^2 \geq 2$ then $p_a(C) = \frac{1}{2}C^2 + 1$ and $h^0(\mathcal{O}_X(C)) = p_a(C) + 1$. In this case X is algebraic.*

Proof. By definition of the arithmetic genus, we have $C^2 = 2p_a(C) - 2 \geq -2$. In the exact sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(C)) \to H^0(C, \mathcal{O}_C(C)) \to H^1(X, \mathcal{O}_X) = 0$$

associated with the exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0$, by observing that $\mathcal{O}_C(C) = K_C$ by the adjunction formula, we obtain

$$h^0(\mathcal{O}_X(C)) = h^0(K_C) + 1 = p_a(C) + 1.$$

Since an irreducible curve with arithmetic genus 0 is a non-singular rational curve, we have assertion (1). Similarly we can prove assertions (2), (3). □

Finally, we state a result due to Siu [Si].

Theorem 4.14 (Siu). *Every $K3$ surface is Kähler.*

As mentioned in the introduction, this theorem was proved after the establishment of the Torelli-type theorem for Kähler $K3$ surfaces and the surjectivity of the period map. It will be repeated that the main theme in this book is an introduction of the proof of the Torelli-type theorem for Kähler $K3$ surfaces assuming Siu's theorem. The Kählerness will be used essentially in the proof of Lemma 6.52.

4.2 Reflection group associated with non-singular rational curves and the Kähler cone

Definition 4.15. For a $K3$ surface X, we define

$$\Delta(X) = \{\delta \in S_X : \langle \delta, \delta \rangle = -2\}.$$

For an element δ in $\Delta(X)$, we define a reflection s_δ of $H^{1,1}(X, \mathbb{R})$ by

$$s_\delta(x) = x + \langle x, \delta \rangle \delta \quad (x \in H^{1,1}(X, \mathbb{R})).$$

We denote by $W(X)$ a subgroup of $\mathrm{O}(H^{1,1}(X, \mathbb{R}))$ generated by all reflections $\{s_\delta : \delta \in \Delta(X)\}$. The cone $P(X) = \{x \in H^{1,1}(X, \mathbb{R}) : \langle x, x \rangle > 0\}$ has two connected components and the one containing a Kähler class is denoted by $P^+(X)$ and called the *positive cone* (see Figure 2.1). For each $\delta \in \Delta(X)$, define

$$H_\delta = \{x \in P^+(X) : \langle x, \delta \rangle = 0\}.$$

As mentioned in Section 2.1, $W(X)$ acts on $P(X)^+$.

Lemma 4.16. *Let $\delta \in S_X$ with $\delta^2 \geq -2$. Then δ or $-\delta$ is represented by an effective divisor.*

Proof. Let L be a line bundle representing δ. Then by the Riemann–Roch theorem for surfaces and Serre duality, we have

$$\begin{aligned}\dim H^0(X, \mathcal{O}(L)) + \dim H^2(X, \mathcal{O}(L)) &= \dim H^0(X, \mathcal{O}(L)) + \dim H^0(X, \mathcal{O}(-L)) \\ &\geq 2 + \delta^2/2 \geq 1,\end{aligned}$$

and hence $\dim H^0(X, \mathcal{O}(L)) > 0$ or $\dim H^0(X, \mathcal{O}(-L)) > 0$. Thus δ or $-\delta$ is represented by an effective divisor. □

We set

$$\Delta(X)^+ = \{\delta \in S_X : \delta \text{ is an effective divisor with } \langle \delta, \delta \rangle = -2\},$$
$$\Delta(X)^- = \{-\delta : \delta \in \Delta(X)^+\}.$$

Then by Lemma 4.16, we have a decomposition

$$\Delta(X) = \Delta(X)^+ \cup \Delta(X)^-. \tag{4.3}$$

It follows from Lemma 2.12 that results stated in Section 2.1, in particular Theorem 2.9, hold. As mentioned in Section 2.1, each chamber determines a decomposition (2.2), but in the above case, the decomposition (4.3) is determined geometrically. This decomposition of $\Delta(X)$ determines a fundamental domain

$$D(X) = \{x \in P^+(X) : \langle x, \delta \rangle > 0 \,\forall\, \delta \in \Delta(X)^+\}.$$

A geometric meaning of $D(X)$ is as follows. Let $\kappa \in P^+(X)$. For an irreducible curve C, it follows from Lemma 4.13 that $p_a(C) \geq 1$ if and only if $C^2 \geq 0$, and $C^2 = -2$ if and only if C is a non-singular rational curve. By Lemma 2.3, the intersection number of C and κ is positive where C is an irreducible curve with $C^2 \geq 0$. Thus $\kappa \in D(X)$ if and only if the intersection number of κ and any curve is positive.

Definition 4.17. We call $D(X)$ the *Kähler cone* of X. It is known that any element in $D(X)$ is a Kähler class although it is non-trivial (see Theorem 7.5).

Remark 4.18. For $\delta \in \Delta(X)$, the reflection s_δ is defined by

$$s_\delta(x) = x + \langle x, \delta \rangle \delta,$$

and hence it can be considered an isomorphism of the lattice $H^2(X, \mathbb{Z})$. Since $\langle \delta, \omega_X \rangle = 0$, s_δ fixes ω_X. Therefore $W(X)$ fixes ω_X.

Remark 4.19. Consider the case that X is projective. It follows from Nakai's criterion (Theorem 3.6) that a divisor H with $H^2 > 0$ is ample if and only if the intersection number of H and any curve is positive. Hence $D(X) \cap S_X$ is nothing but the set of ample classes. In this case we consider $S_X \otimes \mathbb{R}$, instead of $H^{1,1}(X, \mathbb{R})$, and a fundamental domain

$$A(X) = \{x \in S_X \otimes \mathbb{R} \cap P^+(X) : \langle x, \delta \rangle > 0 \,\forall\, \delta \in \Delta(X)^+\}$$

of $W(X)$ with respect to its action on $S_X \otimes \mathbb{R} \cap P^+(X)$. The cone $A(X)$ is called the *ample cone*.

4.3 Kummer surfaces[1]

Definition 4.20 (Kummer surface). Let $A = \mathbb{C}^2/\Gamma$ be a complex torus and let -1_A be an automorphism of A of order 2 induced by the multiplication of $\mathbb{C}^2$ by -1 (Section 3.2). The fixed points of -1_A are the points $\frac{1}{2}\Gamma/\Gamma$ of order 2 in A, and the quotient surface $A/\{\pm 1_A\}$ has 16 rational double points of type A_1 (see Remark 4.22). We denote by $\mathrm{Km}(A)$ the minimal resolution of $A/\{\pm 1_A\}$ and call it a *Kummer surface*.

In the following, we study Kummer surfaces in detail. A fixed point of -1_A is isolated and -1_A can be given by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ for suitable local coordinates $\{U, (x, y)\}$ around the fixed point. Let

$$\tilde{\sigma}\colon \tilde{A} \to A$$

be the blowing up 16 points of order 2. We may assume that there exist local coordinates $\{U_1, (x_1, y_1)\}$, $\{U_2, (x_2, y_2)\}$ covering $\tilde{\sigma}^{-1}(U)$ such that U_1 and U_2 are identified by $y_1 y_2 = 1$, $x_1 = x_2 y_2$, and $\tilde{\sigma}$ is given by

$$\tilde{\sigma}(x_1, y_1) = (x_1, x_1 y_1), \quad \tilde{\sigma}(x_2, y_2) = (x_2 y_2, x_2).$$

The inversion -1_A induces an automorphism

$$\iota\colon (x_1, y_1) \to (-x_1, y_1), \quad (x_2, y_2) \to (-x_2, y_2)$$

of $\tilde{A}$ of order 2. The set of fixed points of ι is 16 exceptional curves obtained by the blowing up. Hence (x_1^2, y_1), (x_2^2, y_2) are local coordinates of the quotient surface $\tilde{A}/\iota$ and $\tilde{A}/\iota$ is non-singular.

Consider the natural map $\tilde{\pi}\colon \tilde{A} \to \tilde{A}/\iota$. If we denote by C the image of an exceptional curve E under $\tilde{\pi}$, then C is the ramification curve of the double covering $\tilde{\pi}$, and hence

$$2C^2 = (\tilde{\pi}^*(C))^2 = (2E)^2 = -4.$$

Therefore the image of each exceptional curve on $\tilde{A}/\iota$ is a non-singular rational curve with the self-intersection number -2. Thus $\tilde{A}/\iota$ is the minimal non-singular model $\mathrm{Km}(A)$ of $A/\{\pm 1_A\}$.

Theorem 4.21. $\mathrm{Km}(A)$ *is a* $K3$ *surface.*

Proof. For simplicity, we denote $\mathrm{Km}(A)$ by X. First, we show that there exists a nowhere-vanishing holomorphic 2-form on X. Since A is a 2-dimensional complex torus, it has a nowhere-vanishing holomorphic 2-form invariant under the action of -1_A. If a 2-form is given by $dx \wedge dy$ on U, then we have $x_1\, dx_1 \wedge dy_1 = -x_2\, dx_2 \wedge dy_2$

[1] Added in English translation: For a history of the research of Kummer surfaces we refer the reader to [Do4].

on U_1, U_2. It induces a holomorphic 2-form $d(x_1^2) \wedge dy_1 = -d(x_2^2) \wedge dy_2$ on X. Thus there exists a nowhere-vanishing holomorphic 2-form on X.

Next we show that the Euler number $e(X)$ of X is 24. The complex torus A has $e(A) = 0$, and hence $\tilde{A}$ has $e(\tilde{A}) = 16$ because it is obtained from A by blowing up 16 points. Since the map $\tilde{\pi}\colon \tilde{A} \to X$ is a double covering branched along 16 non-singular rational curves, we have

$$e(\tilde{A}) = 2e(X) - 16e(\mathbb{P}^1),$$

and thus $e(X) = 24$. Finally, it follows from the Noether formula that

$$c_1(X)^2 + e(X) = 12 \sum (-1)^i \dim H^i(X, \mathcal{O}_X),$$

and hence $\dim H^1(X, \mathcal{O}_X) = 0$. Thus X is a $K3$ surface. □

Remark 4.22. A singularity appearing on the quotient surface $A/\{\pm 1\}$ is called a *rational double point of type* A_1. The defining equations of rational double points are given as follows (e.g., Barth, Hulek, Peters, Van de Ven [BHPV, Chap. III, §7]). Here a singularity appears at the origin of $\mathbb{C}^3$:

- type A_n $(n \geq 1)$: $z^2 + x^2 + y^{n+1} = 0$;
- type D_n $(n \geq 4)$: $z^2 + y(x^2 + y^{n-2}) = 0$;
- type E_6: $z^2 + x^3 + y^4 = 0$;
- type E_7: $z^2 + x(x^2 + y^3) = 0$;
- type E_8: $z^2 + x^3 + y^5 = 0$.

Any irreducible component of the exceptional set of the minimal resolution of a rational double point is a non-singular rational curve with the self-intersection number -2. The dual graph is defined by associating each irreducible component to a vertex and by joining two vertices by an edge if the corresponding irreducible curves meet. The obtained dual graph coincides with a Dynkin diagram of type A_n, D_n, E_6, E_7, E_8 (Figures 1.1, 1.2), respectively. The equation obtained by removing the term z^2 in each of the above equations defines a singular curve on $\mathbb{C}^2$. Thus each of the above equations means that a rational double point appears on the double covering branched along this singular curve.

A rational double point also appears on the quotient surface by a finite group. Consider the natural action of $\mathrm{GL}(2, \mathbb{C})$ on $\mathbb{C}^2$. Let $G \subset \mathrm{GL}(2, \mathbb{C})$ be a finite group and assume that G fixes only the origin. Then the quotient $\mathbb{C}^2/G$ has a singularity, and it is a rational double point if $G \subset \mathrm{SL}(2, \mathbb{C})$. For example, if G is a cyclic group of order n in $\mathrm{SL}(2, \mathbb{C})$, then $\mathbb{C}^2/G$ has a rational double point of type A_{n-1}.

As we showed in the proof of Theorem 4.21, a holomorphic 2-form on the open set deleting the singular point can be extended to a holomorphic 2-form on the minimal resolution. This also holds in other rational double points (e.g., Barth, Hulek, Peters, Van de Ven [BHPV, Chap. III, Prop. 3.5, Thm. 7.2]). We will use this fact later.

Exercise 4.23. Show that the singularity of the quotient surface of $\mathbb{C}^2$ by $-1_{\mathbb{C}^2}$ appearing in the Kummer surface is analytically isomorphic to the equation of type A_1 in Remark 4.22.

Example 4.24. Consider the case that a 2-dimensional complex torus is a product $E \times F$ of two elliptic curves E, F. The automorphism -1_A of order 2 is the product $-1_A = (-1_E, -1_F)$ of automorphisms $-1_E, -1_F$ of E, F of order 2. If we respectively denote by $\{p_1, \ldots, p_4\}$, $\{q_1, \ldots, q_4\}$ the sets of the points of E, F of order 2, then the fixed points of -1_A are 16 points (p_i, q_j). The quotient $(E \times F)/\pm 1_A$ has 16 rational double points of type A_1, and a $K3$ surface, as the minimal resolution, is obtained (Theorem 4.21). We denote this $K3$ surface by $\mathrm{Km}(E \times F)$. We have 8 elliptic curves $E \times \{q_j\}$, $\{p_i\} \times F$ invariant by -1_A, and -1_A has 4 fixed points on each elliptic curve. Therefore the images of these elliptic curves on $\mathrm{Km}(E \times F)$ are non-singular rational curves. Thus there exist 24 non-singular rational curves on $\mathrm{Km}(E \times F)$, that is, exceptional curves N_{ij} corresponding to 16 points (p_i, q_j) of order 2, the images E_j of $E \times \{q_j\}$ and F_i of $\{p_i\} \times F$ (see Figure 4.1).

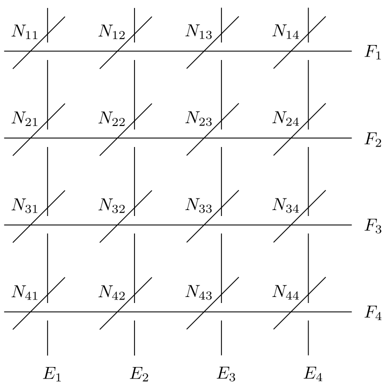


Figure 4.1.

Exercise 4.25. Show that in Example 4.24, the map

$$\pi\colon \mathrm{Km}(E \times F) \to F/\{\pm 1_F\} = \mathbb{P}^1$$

induced from the projection $E \times F \to F$ gives the structure of an elliptic fibration, and determine the singular fibers of π.

Exercise 4.26. Determine the rank of the sublattice of the Néron–Severi lattice generated by 24 non-singular rational curves in Example 4.24.

Exercise 4.27. Let τ be the automorphism of $\mathrm{Km}(E \times F)$ induced by the automorphism $(-1_E, 1_F)$ of $E \times F$ in Example 4.24. Then determine the set of fixed points of τ. Moreover, show that the quotient surface $\mathrm{Km}(E \times F)/\langle\tau\rangle$ by τ contains 16 exceptional curves, and $\mathbb{P}^1 \times \mathbb{P}^1$ is obtained from $\mathrm{Km}(E \times F)/\langle\tau\rangle$ by blowing down these exceptional curves.

4.4 The Kummer surface associated with a curve of genus 2

Let C be a compact Riemann surface of genus 2. Recall that C is a hyperelliptic curve and is the double covering of the projective line given by

$$y^2 = \prod_{i=0}^{5}(x - \xi_i). \tag{4.4}$$

Here $x \in \mathbb{P}^1$ is an inhomogeneous coordinate and $p_i = (\xi_i, 0) \in C$ are 6 ramification points of the double covering. Let $J(C)$ be the Jacobian of C which is a 2-dimensional abelian variety and $J(C) = \mathrm{Pic}^0(C)$ by Abel's theorem where $\mathrm{Pic}^0(C)$ is the subgroup of the Picard group consisting of divisors of degree 0. The covering transformation of the double covering $C \to \mathbb{P}^1$ induces an automorphism ι of $J(C)$ of order 2 whose fixed points are the points of order 2,

$$\mu_i = p_i - p_0 \quad (0 \le i \le 5), \qquad \mu_{ij} = p_i + p_j - 2p_0 \quad (1 \le i < j \le 5).$$

The image of C under the Abel–Jacobi map and its translations by μ_i, μ_{ij},

$$\Theta = \{p - p_0 \in J(C) : p \in C\}, \quad \Theta_i = \Theta + \mu_i, \quad \Theta_{ij} = \Theta + \mu_{ij},$$

are called *theta divisors*. The divisor Θ contains 6 points $\{\mu_i : 0 \le i \le 5\}$ of order 2, Θ_i contains $\{\mu_0, \mu_i, \mu_{ij} : j \ne 0, i\}$ and Θ_{ij} contains $\{\mu_i, \mu_j, \mu_{ij}, \mu_{kl} : k, l \ne 0, i, j\}$. Conversely, the points μ_i, μ_{ij} of order 2 are contained in 6 theta divisors $\{\Theta, \Theta_i, \Theta_{ij} : j \ne 0, i\}$, $\{\Theta_i, \Theta_j, \Theta_{ij}, \Theta_{kl} : k, l \ne 0, i, j\}$, respectively. For simplicity, we may denote the subscripts i, ij by $\alpha, \beta, \gamma, \delta, \ldots$, etc. if there is no confusion.

The automorphism ι of $J(C)$ of order 2 preserves each theta divisor and fixes 6 points of order 2 on it, and hence the quotient of a theta divisor by ι is a non-singular rational curve. Therefore, if we denote by $\bar{X}$ the quotient $J(C)/\langle\iota\rangle$ of $J(C)$, $\bar{X}$ has 16 rational double points n_α of type A_1 corresponding to 16 points μ_α of order 2 and contains 16 non-singular rational curves $\bar{T}_\alpha$ which are the images of Θ_α. Let X be the non-singular minimal model of $\bar{X}$ on which there exist 32 non-singular rational curves N_α, T_α. Here N_α is the exceptional curve over n_α and T_α is the proper transform of $\bar{T}_\alpha$. Both $\{N_\alpha\}$ and $\{T_\alpha\}$ are sets of 16 mutually disjoint non-singular rational curves, and each member of one set meets exactly 6 members of the other set. This follows from the incidence relation between $\{\Theta_\alpha\}$ and $\{\mu_\beta\}$ mentioned above. It is said that the 32 curves $(\{N_\alpha\}, \{T_\alpha\})$ form a (16_6)-*configuration*.

The map

$$\varphi_{|2\Theta|} \colon J(C) \to \mathbb{P}^3 \tag{4.5}$$

associated with the complete linear system $|2\Theta|$ is the double covering ramified at 16 points of order 2 onto its image $\bar{X}$ which is a quartic surface with 16 rational double points of type A_1 in $\mathbb{P}^3$. Here n_α are 16 rational double points and $\bar{T}_\alpha$ are conics.

We fix n_α among 16 nodes and consider the projection

$$\mathbb{P}^3 \setminus \{n_\alpha\} \to \mathbb{P}^2$$

from n_α. Since n_α is a double point of $\bar{X}$ and $\bar{X}$ is a quartic surface, any line passing through n_α meets with $\bar{X}$ at two points other than n_α. This implies that the projection induces a double covering

$$\pi \colon \bar{X} \to \mathbb{P}^2.$$

For simplicity we assume $n_\alpha = n_0$. Then the images of $\bar{T}_i$ $(0 \le i \le 5)$ under π are lines L_i, and L_i and L_j meet at the image of n_{ij}. The image of n_0 is a conic tangent to each L_i. We call X, $\bar{X}$ respectively the *Kummer surface*, the *Kummer quartic surface* associated with a curve C of genus 2, and denote X by $\mathrm{Km}(C)$.

Much research was done during the 19th and at the beginning of the 20th centuries on $\mathrm{Km}(C)$, and it is known that the Kummer surface has a rich structure. In the following we introduce part of it.

Let $(X_0, X_1, X_2, X_3, X_4, X_5)$ be homogeneous coordinates of $\mathbb{P}^5$ and let

$$Q_1 \colon \sum_{i=0}^{5} X_i^2 = 0, \quad Q_2 \colon \sum_{i=0}^{5} \xi_i X_i^2 = 0, \quad Q_3 \colon \sum_{i=0}^{5} \xi_i^2 X_i^2 = 0. \tag{4.6}$$

Here $\xi_0, \ldots, \xi_5$ are distinct complex numbers. Then $Y = Q_1 \cap Q_2 \cap Q_3$ is non-singular and hence is a $K3$ surface (Example 4.1).

Exercise 4.28. Show that $Y = Q_1 \cap Q_2 \cap Q_3$ is non-singular.

On the other hand, we have a Kummer surface $\mathrm{Km}(C)$ associated with the curve C of genus 2 defined by equation (4.4). In fact, it is known that Y and $\mathrm{Km}(C)$ are isomorphic (Griffiths, Harris [GH, final chapter]). Also 3 quadric hypersurfaces define a family of quadric hypersurfaces

$$\mathcal{Q} = \{Q_{(x,y,z)} : Q_{(x,y,z)} = xQ_1 + yQ_2 + zQ_3\}_{(x,y,z)\in\mathbb{P}^2}.$$

Let

$$D = \{(x, y, z) \in \mathbb{P}^2 : \det(Q_{(x,y,z)}) = 0\}$$

be the set of singular members of $\mathcal{Q}$, that is, the set of quadrics defined by singular symmetric matrices of degree 6. Then D is a sextic curve in $\mathbb{P}^2$ given by

$$\prod_{i=0}^{5}(x + \xi_i y + \xi_i^2 z) = 0.$$

Obviously, D is the union of 6 lines of

$$\ell_i : x + \xi_i y + \xi_i^2 z = 0.$$

Note that the line ℓ_i is tangent to the conic $4xz - y^2 = 0$ at the point $(\xi_i^2, -2\xi_i, 1)$. The double covering of $\mathbb{P}^2$ branched along D has 15 singularities over 15 double points of D, and its minimal non-singular model, denoted by Z, is a $K3$ surface (Example 4.1, Remark 4.22). In general, a $K3$ surface that is the minimal non-singular model of the double covering of $\mathbb{P}^2$ branched along 6 lines is a Kummer surface if there exists a conic tangent to 6 lines. In our case, it is known that Z is isomorphic to $\mathrm{Km}(C)$.

A Kummer surfaces has a rich geometric structure as mentioned above, but this is not all. The Kummer surface was originally discovered from a geometry of lines in the 19th century. We recall this briefly. Let $G = G(1, 3)$ be the set of all lines in $\mathbb{P}^3$. Then G is a Grassmann variety. Any line ℓ in $\mathbb{P}^3$ is determined by two points (u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) on it. By considering the minors p_{ij} of degree 2 consisting of the ith and jth rows of the matrix

$$\begin{pmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \end{pmatrix},$$

we obtain a point

$$(X_0, X_1, X_2, X_3, X_4, X_5) = (p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23})$$

in $\mathbb{P}^5$ which is independent of the choice of two points on ℓ, and satisfies the relation

$$X_0X_3 + X_1X_4 + X_2X_5 = 0. \tag{4.7}$$

We call $(p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23})$ the *Plücker coordinates* of ℓ.

Exercise 4.29. Show that the Plücker coordinates are independent of the choice of two points on ℓ and satisfy relation (4.7).

The Grassmann variety G is realized as a non-singular quadric hypersurface in $\mathbb{P}^5$ defined by equation (4.7). Now we consider another non-singular quadric hypersurface Q in $\mathbb{P}^5$ such that $G \cap Q$ is non-singular. In this case, by changing the coordinates we may assume that $G = Q_1$, $Q = Q_2$ (e.g., Mabuchi, Mukai [MM, §6]). Here Q_1, Q_2 are quadric hypersurfaces given by equations (4.6). The intersection of two quadrics $G \cap Q$ was classically called a quadratic line complex. We denote by $\sigma(p)$ the set of lines through a point $p \in \mathbb{P}^3$. Then $\sigma(p) \subset G \subset \mathbb{P}^5$ is a 2-dimensional linear space and

$$\sigma(p) \cap G \cap Q = \sigma(p) \cap Q$$

is a conic in $\sigma(p)$ ($\cong \mathbb{P}^2$). In fact, the quartic surface $\bar{X}$ defined by the image of the map (4.5) coincides with the set of points p such that the conic $\sigma(p) \cap Q$ decomposes into two lines:

$$\bar{X} = \{p \in \mathbb{P}^3 : \det(\sigma(p) \cap Q) = 0\}.$$

Instead of $\sigma(p)$, we take a plane h in $\mathbb{P}^3$ and denote by $\sigma(h)$ the set of lines lying on h. Then we get the dual of $\bar{X}$ which is projectively isomorphic to $\bar{X}$ itself:

$$\bar{X}^* = \{h \in (\mathbb{P}^3)^* : \det(\sigma(h) \cap Q) = 0\}.$$

Both $\bar{X}$, $\bar{X}^*$ have 16 rational double points of type A_1. These 16 singularities correspond to the case that $\sigma(p) \cap Q$, $\sigma(h) \cap Q$ are double lines. Recall that there exist 32 non-singular rational curves on the Kummer surface $\mathrm{Km}(C)$ which are divided into two sets consisting of 16 disjoint curves. The surface $\bar{X}$ is obtained by blowing down a set of 16 non-singular rational curves to 16 rational double points of type A_1, and its dual $\bar{X}^*$ is obtained by blowing down the other set of 16 non-singular rational curves to 16 rational double points of type A_1. For more details, we refer the reader to the last chapter of Griffiths, Harris [GH].[2]

4.5 Torelli theorem for 2-dimensional complex tori

In Chapter 6 we will give the Torelli-type theorem for Kummer surfaces by reducing its proof to the case of 2-dimensional complex tori. Therefore we prepare the Torelli theorem for 2-dimensional complex tori here.

Let $V = \mathbb{C}^2$ be a 2-dimensional complex vector space. We denote by Γ a free abelian group V of rank 4 generated by linearly independent elements v_1, v_2, v_3, v_4

[2]Added in English translation: See also Chapter 12.

over $\mathbb{R}$. The group Γ is a discrete subgroup of V acting on V by translations, and the quotient $A = V/\Gamma$ has the structure of an abelian group and also the structure of a complex manifold. The surface A is a complex torus. The dual $\mathrm{Hom}(\Gamma, \mathbb{Z})$ of Γ is denoted by Γ^*. Note that $\pi_1(A) = \Gamma$ and its abelianization is the first homology group, and hence $H_1(A, \mathbb{Z}) = \Gamma$. It follows from the universal coefficient theorem that $H^1(A, \mathbb{Z}) \cong \Gamma^*$. Moreover, by applying the Künneth formula to a decomposition $A \cong (S^1)^4$, we have

$$H^2(A, \mathbb{Z}) \cong \wedge^2(\Gamma^*), \quad H^4(A, \mathbb{Z}) \cong \wedge^4(\Gamma^*)$$

(Mumford [Mum]). Denote by $\{u^1, u^2, u^3, u^4\}$ the dual basis of the basis $\{v_1, v_2, v_3, v_4\}$ of Γ, that is, $u^i(v_j) = \delta_{ij}$. If $u^{ij} = u^i \wedge u^j$, then $\{u^{ij} : 1 \le i < j \le 4\}$ is a basis of $H^2(A, \mathbb{Z})$. The cup product

$$\langle\ ,\ \rangle \colon H^2(A, \mathbb{Z}) \times H^2(A, \mathbb{Z}) \to H^4(A, \mathbb{Z}) \cong \mathbb{Z}$$

induces the structure of a lattice. Here $H^4(A, \mathbb{Z}) \cong \mathbb{Z}$ is induced from the natural orientation of the complex manifold. Up to sign this lattice structure corresponds to

$$\langle\ ,\ \rangle \colon \wedge^2(\Gamma^*) \times \wedge^2(\Gamma^*) \to \mathbb{Z},$$

defined by fixing an isomorphism

$$\alpha \colon \wedge^4(\Gamma^*) \to \mathbb{Z} \tag{4.8}$$

and by putting $\langle u, u' \rangle = \alpha(u \wedge u')$ for $u, u' \in \wedge^2(\Gamma^*)$. In the following, we assume that the basis $\{u^1, u^2, u^3, u^4\}$ satisfies

$$\langle u^{12}, u^{34} \rangle = 1. \tag{4.9}$$

Then the matrix of the bilinear form $\langle\ ,\ \rangle$ with respect to the basis

$$\{u^{12}, u^{13}, u^{14}, u^{34}, u^{42}, u^{23}\} \tag{4.10}$$

is given by

$$\begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}. \tag{4.11}$$

Here I_3 denotes the identity matrix of degree 3.

Definition 4.30. Let A, A' be complex tori. The *determinant* $\det(\phi)$ of an isomorphism of lattices

$$\phi \colon H^2(A, \mathbb{Z}) \to H^2(A', \mathbb{Z})$$

is defined as follows. Take a basis of $H^2(A,\mathbb{Z})$ and of $H^2(A',\mathbb{Z})$ satisfying (4.9) and define $\det(\phi)$ as the determinant of the matrix of ϕ with respect to these bases. This matrix is a regular matrix with integer coefficients and its inverse matrix has integral coefficients, and hence $\det(\phi) = \pm 1$. Thus $\det(\phi)$ is independent of the choice of a basis satisfying (4.9).

Next we give a characterization of ϕ in terms of the value $\det(\phi)$. Let k be the rational numbers field $\mathbb{Q}$ or the finite field $\mathbb{F}_2$, and let us denote $\Gamma^* \otimes k$ simply by Γ^*_k. A subspace W of $\wedge^2(\Gamma^*_k)$ is called *isotropic* if the restriction of the quadratic form $Q(x) = \langle x, x\rangle$ to W is identically equal to 0. Since the lattice $\wedge^2(\Gamma^*_k)$ has signature $(3,3)$, the dimension of an isotropic subspace is at most 3. For example, $\{u^{12}, u^{13}, u^{14}\}$, $\{u^{34}, u^{42}, u^{23}\}$ each generate a 3-dimensional isotropic subspace.

Denote by $G(1,3)$ the Grassmann variety of lines in 3-dimensional projective space $\mathbb{P}(\Gamma^*_k)$. Under the Plücker embedding

$$G(1,3) \to \mathbb{P}(\wedge^2(\Gamma^*_k)),$$

$G(1,3)$ is isomorphic to the quadric hypersurface $\{x : Q(x) = 0\}$ in $\mathbb{P}(\wedge^2(\Gamma^*_k))$ (see equation (4.7)). That is, an isotropic line in $\wedge^2(\Gamma^*_k)$ corresponds to a point on $G(1,3)$. Now $G(1,3)$ contains planes corresponding to 3-dimensional isotropic subspaces of $\wedge^2(\Gamma^*_k)$, and there are two families of planes in $G(1,3)$ (see Griffiths, Harris [GH, Chap. 6] for a general fact): a plane consisting of lines through a point p in $\mathbb{P}(\Gamma^*_k)$,

$$\pi_p = \{\ell \subset \mathbb{P}(\Gamma^*_k) : \ell \text{ is a line, } p \in \ell\},$$

and a plane consisting of lines lying on a plane H in $\mathbb{P}(\Gamma^*_k)$,

$$\pi_H = \{\ell \subset \mathbb{P}(\Gamma^*_k) : \ell \text{ is a line, } \ell \subset H\}.$$

For example, the 3-dimensional isotropic subspace generated by $\{u^{12}, u^{13}, u^{14}\}$ corresponds to lines through $ku^1 \in \mathbb{P}(\Gamma^*_k)$. A family of this type can be written as

$$u \wedge \Gamma^*_k \quad (u \in \Gamma^*_k). \tag{4.12}$$

The 3-dimensional isotropic subspace generated by $\{u^{34}, u^{42}, u^{23}\}$ corresponds to lines lying on the plane in $\mathbb{P}(\Gamma^*_k)$ generated by u^2, u^3, u^4. An isomorphism of $\wedge^2\Gamma^*$ changing u^{12} and u^{34} in the basis (4.10) and fixing others switches two families of planes. This corresponds to the determinant of the matrix of the isomorphism being -1. We now conclude the following.

Lemma 4.31. *Let Γ, Γ' be free abelian groups of rank 4 and let $\phi\colon \wedge^2\Gamma^* \to \wedge^2(\Gamma')^*$ be an isomorphism of lattices. Then ϕ preserves two families of planes if the determinant $\det(\phi)$ is equal to $+1$, and changes the families if $\det(\phi) = -1$.*

The next lemma will be key in the proof of the Torelli-type theorem for Kummer surfaces.

Lemma 4.32. *Let* Γ, Γ' *be free abelian groups of rank* 4 *and let*

$$\phi\colon \wedge^2 \Gamma^* \to \wedge^2(\Gamma')^*$$

be an isomorphism of lattices. Then the following are equivalent:

(1) $\det(\phi) = 1$.

(2) *There exists an isomorphism* $\psi\colon \Gamma^* \to (\Gamma')^*$ *satisfying* $\phi = \pm\psi \wedge \psi$.

(3) *There exists an isomorphism* $\psi_2\colon \Gamma^* \otimes \mathbb{F}_2 \to (\Gamma')^* \otimes \mathbb{F}_2$ *satisfying* $\phi \bmod 2 = \psi_2 \wedge \psi_2$.

Proof. We show that (1) implies (2). By the assumption, $\phi\otimes\mathbb{Q}$ preserves two families of planes (Lemma 4.31). Since it preserves a plane of type (4.12), if the set of lines through a point $\mathbb{Q}u \in \mathbb{P}(\Gamma^*_{\mathbb{Q}})$ is sent to the set of lines through a point $\mathbb{Q}\bar{\psi}(u) \in \mathbb{P}((\Gamma')^*_{\mathbb{Q}})$ by ϕ, then we have a bijection

$$\bar{\psi}\colon \mathbb{P}(\Gamma^*_{\mathbb{Q}}) \to \mathbb{P}((\Gamma')^*_{\mathbb{Q}}).$$

The set of lines through another point $\mathbb{Q}u'$ is sent to the set of lines through the point $\mathbb{Q}\bar{\psi}(u')$ by ϕ, and hence the line $\ell_{uu'}$ in $\mathbb{P}(\Gamma^*_{\mathbb{Q}})$ through $\mathbb{Q}u$ and $\mathbb{Q}u'$ is sent to the line $\ell_{\bar{\psi}(u)\bar{\psi}(u')}$ in $\mathbb{P}((\Gamma')^*_{\mathbb{Q}})$ through $\bar{\psi}(u)$ and $\bar{\psi}(u')$ by ϕ. Similarly we can prove that $\bar{\psi}$ sends $\ell_{uu'}$ to $\ell_{\bar{\psi}(u)\bar{\psi}(u')}$ by considering the image of a line through a point on $\ell_{uu'}$ under ϕ. Thus $\bar{\psi}$ sends lines in $\mathbb{P}(\Gamma^*_{\mathbb{Q}})$ to lines in $\mathbb{P}((\Gamma')^*_{\mathbb{Q}})$. It is known that a bijection between projective spaces preserving lines is a projective transformation as a characterization of projective transformations (see, e.g., Kawada [Ka, §5.2]). Now we conclude that $\bar{\psi}$ is induced from an isomorphism

$$\psi\colon \Gamma^*_{\mathbb{Q}} \to (\Gamma')^*_{\mathbb{Q}}.$$

By definition of ψ, we obtain

$$\phi(u \wedge \Gamma^*_{\mathbb{Q}}) = \psi(u) \wedge \Gamma^*_{\mathbb{Q}} \quad (u \in \Gamma^*_{\mathbb{Q}}),$$

and hence

$$\phi \otimes \mathbb{Q} = \lambda\psi \wedge \psi \quad (\lambda \in \mathbb{Q}).$$

The elementary divisor theorem implies that there exist a basis e_1, e_2, e_3, e_4 of Γ^* and a basis e'_1, e'_2, e'_3, e'_4 of $(\Gamma')^*$ satisfying $\psi(e_i) = d_i e'_i$, $d_i \in \mathbb{Q}$ $(1 \le i \le 4)$. If necessary by multiplying ψ by an integer, d_i is an integer and their maximum common divisor is 1. Then $\phi(e_i \wedge e_j) = \lambda d_i d_j e'_i \wedge e'_j$ and $\lambda d_i d_j = \pm 1$ because ϕ is an isomorphism. Since the maximum common divisor of $d_1, \ldots, d_4$ is 1, $d_i = \pm 1$, $\lambda = \pm 1$, and hence ψ is the desired one.

Obviously, (3) follows from (2). Let's prove (3) implies (1). To do this, it suffices to see that it preserves planes of type (4.12). It is enough to show this in the situation of modulo 2, but this holds by (3) and hence assertion (1) follows. □

Remark 4.33. In Lemma 4.32(2), ψ is uniquely determined up to sign. In fact, if $\phi = \pm\psi \wedge \psi = \pm\psi' \wedge \psi'$, then $\psi(\ell) = \psi'(\ell)$ for any line $\ell \subset \mathbb{P}(\Gamma_{\mathbb{Q}})$. By considering the intersection of lines, we have $\psi = c\psi'$, $c \in \mathbb{Q}$. Since $c^2 = \pm 1$, we obtain $c = \pm 1$.

To state the Torelli theorem for complex tori, we recall the Hodge structure of a complex torus (see, e.g., Mumford [Mum]). Let $A = V/\Gamma$ be a complex torus.

Proposition 4.34. *The following hold*:

(1) $H^1(A, \mathbb{C}) \cong H^0(A, \Omega^1_A) \oplus H^1(A, \mathcal{O}_A)$.

(2) $H^0(A, \Omega^1_A) \cong V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$.

(3) $H^2(A, \mathbb{C}) \cong H^0(A, \Omega^2_A) \oplus H^1(A, \Omega^1_A) \oplus H^2(A, \mathcal{O}_A)$.

(4) $H^0(A, \Omega^2_A) \cong \mathbb{C}$.

Now, to $\gamma \in H_1(A, \mathbb{Z})$, by associating the map

$$\gamma\colon H^0(A, \Omega^1_A) \to \mathbb{C}, \quad \omega \to \int_\gamma \omega,$$

we can consider $\Gamma = H_1(A, \mathbb{Z})$ as a subgroup of $H^0(A, \Omega^1_A)^*$ and obtain an isomorphism

$$V/\Gamma \cong H^0(A, \Omega^1_A)^*/\Gamma.$$

Theorem 4.35 (Torelli theorem for complex tori). *Let $A = V/\Gamma$, $A' = V'/\Gamma'$ be two complex tori and let*

$$\phi\colon H^2(A, \mathbb{Z}) \to H^2(A', \mathbb{Z})$$

be an isomorphism of lattices satisfying the following conditions:

(i) $(\phi \otimes \mathbb{C})(H^0(A, \Omega^2_A)) = H^0(A', \Omega^2_{A'})$.

(ii) *There exists an isomorphism* $\psi_2\colon \Gamma^* \otimes \mathbb{F}_2 \to (\Gamma')^* \otimes \mathbb{F}_2$ *with* $\phi \bmod 2 = \psi_2 \wedge \psi_2$.

Then there exists an isomorphism $\varphi\colon A' \to A$ of complex manifolds with $\varphi^* = \phi$. Here φ is unique up to $\pm 1_A$.

Proof. It follows from Lemma 4.32 that there exists an isomorphism $\psi\colon H^1(A,\mathbb{Z}) \to H^1(A',\mathbb{Z})$, unique up to sign, with $\phi = \pm\psi \wedge \psi$ (Lemma 4.32, Remark 4.33). We show that the induced isomorphism $\psi_{\mathbb{C}}\colon H^1(A,\mathbb{C}) \to H^1(A',\mathbb{C})$ preserves Hodge decompositions, that is,

$$\psi_{\mathbb{C}}(H^0(A,\Omega^1_A)) = H^0(A,\Omega^1_{A'}).$$

Let ω_1, ω_2 be a basis of $H^0(A,\Omega^1_A)$, and let

$$\psi_{\mathbb{C}}(\omega_1) = \eta_1 + \bar{\tau}_1, \quad \psi_{\mathbb{C}}(\omega_2) = \eta_2 + \bar{\tau}_2.$$

Here $\eta_1, \eta_2, \tau_1, \tau_2 \in H^0(A,\Omega^1_{A'})$. Then

$$\pm\phi_{\mathbb{C}}(\omega_1 \wedge \omega_2) = \psi_{\mathbb{C}}(\omega_1) \wedge \psi_{\mathbb{C}}(\omega_2) = \eta_1 \wedge \eta_2 + \eta_1 \wedge \bar{\tau}_2 - \eta_2 \wedge \bar{\tau}_1 + \bar{\tau}_1 \wedge \bar{\tau}_2.$$

Since $\phi_{\mathbb{C}}$ preserves holomorphic 2-forms,

$$\eta_1 \wedge \bar{\tau}_2 - \eta_2 \wedge \bar{\tau}_1 = 0, \quad \tau_1 \wedge \tau_2 = 0.$$

If $\tau_1 \neq 0$, then we have $\tau_2 = a\tau_1$, $a \in \mathbb{C}$. Therefore $\bar{a}\eta_1 = \eta_2$. This means that $\phi_{\mathbb{C}}(\omega_1 \wedge \omega_2) = 0$, which contradicts the fact that ϕ is an isomorphism. Thus we have $\tau_1 = 0$. Similarly we have $\tau_2 = 0$ and obtain an isomorphism

$$\psi_{\mathbb{C}}\colon H^0(A,\Omega^1_A) \to H^0(A',\Omega^1_{A'}).$$

This induces an isomorphism

$$\varphi\colon A' = H^0(A',\Omega^1_{A'})^*/H_1(A',\mathbb{Z}) \to A = H^0(A,\Omega^1_A)^*/H_1(A,\mathbb{Z}),$$

and by construction we have $\varphi^* = \phi$. □

Remark 4.36. Let E, F be elliptic curves and let $A = E \times F$. We assume that the Néron–Severi lattice of A is generated by $e = E \times \{0\}$ and $f = \{0\} \times F$. Then E and F are not isomorphic. If they are isomorphic, then the graph of an isomorphism gives a class d in the Néron–Severi lattice linearly independent of e, f, which is a contradiction. The classes e, f generate a primitive sublattice U of the unimodular lattice $H^2(A,\mathbb{Z}) \cong U^{\oplus 3}$. Let N be its orthogonal complement. Since U is unimodular, $H^2(A,\mathbb{Z}) \cong U \oplus N$. Now consider an isomorphism $\phi\colon H^2(A,\mathbb{Z}) \to H^2(A,\mathbb{Z})$ of lattices defined by

$$\phi(e) = f, \quad \phi(f) = e, \quad \phi|N = 1_N.$$

Then ϕ is not induced by any isomorphism of A. If not, it gives an isomorphism between E and F which contradicts the fact that E and F are not isomorphic. In this case, $\det(\phi) = -1$ and hence it does not satisfy condition (ii) in Theorem 4.35 (Lemma 4.32).

Exercise 4.37. Show that if E and F are isomorphic, then d and e, f are independent in the Néron–Severi lattice.

Remark 4.38. In Piatetskii-Shapiro, Shafarevich [PS], condition (ii) in the Torelli theorem (Theorem 4.35) for complex tori is not clearly stated. This was pointed out by M. Rapoport and T. Shioda. The Torelli theorem for complex tori in this section follows from Shioda [Shi]. We have also referred to Beauville [Be3].

5

Bounded symmetric domains of type IV and deformations of complex structures

In this chapter, we first recall the upper half-plane that appeared as the period domain of elliptic curves, and then introduce a bounded symmetric domain of type IV as a generalization of the upper half-plane. A bounded symmetric domain of type IV appears as the period domain of polarized $K3$ surfaces. In the latter half of the chapter, we introduce the deformation theory of compact complex manifolds. This theory will be necessary to define the period map for $K3$ surfaces.

5.1 Bounded symmetric domains of type IV

5.1.1 The upper half-plane. We denote by $\mathrm{Im}(z)$ the imaginary part of a complex number z. Now put

$$H^+ = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\} \subset H = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) \neq 0\}$$

and call H^+ the *upper half-plane*. Obviously, H^+ is a complex manifold. We define

$$\mathrm{GL}(2,\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d \in \mathbb{Z},\ ad - bc = \pm 1 \right\},$$
$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{Z}) : ad - bc = 1 \right\}.$$

Then the groups $\mathrm{SL}(2,\mathbb{Z})$, $\mathrm{GL}(2,\mathbb{Z})$ act properly discontinuously on H^+, H, respectively, as linear fractional transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}.$$

The quotient space $H^+/\mathrm{SL}(2,\mathbb{Z})$ has the structure of a complex manifold and is isomorphic to $\mathbb{C}$. The quotient space is not compact, but it has a compactification $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ by adding a point. We understand this compactification as follows. First, the upper half-plane is holomorphically isomorphic to an open disc

$$\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$$

under the Cayley transformation

$$\tau \to z = \frac{\tau - \sqrt{-1}}{\tau + \sqrt{-1}}.$$

The point $\sqrt{-1}$ is sent to the origin, and $z \to 1$ if $\mathrm{Im}(\tau) \to \infty$. The real axis which is the boundary of the upper half-plane is sent to the boundary $|z| = 1$ of D except for $\{1\}$. Rational points on the real axis and $\mathrm{Im}(\tau) = \infty$ form an orbit under the action of $\mathrm{SL}(2, \mathbb{Z})$. We define the following topology on the set $H^+ \cup \mathbb{Q} \cup \{\infty\}$. As a closed neighborhood of ∞ we take $\{\tau : \mathrm{Im}(\tau) \geq k\}$ $(k > 0)$. As a closed neighborhood of a rational point x, we employ a disc $|\tau - (x + \sqrt{-1}\,k)| \leq k$ tangent to the real axis. Then as the quotient of $H^+ \cup \mathbb{Q} \cup \{\infty\}$ by $\mathrm{SL}(2, \mathbb{Z})$, we have $\mathbb{P}^1$.

Exercise 5.1. What is the image of a closed neighborhood $\{\tau : \mathrm{Im}(\tau) \geq k\}$ of ∞ under the Cayley transformation?

In the theory of polarized $K3$ surfaces, a generalization of the upper half-plane, called a bounded symmetric domain of type IV associated with a lattice of signature $(2, n)$, appears. We first reconstruct H^+ from a lattice of signature $(2, 1)$.

Consider a lattice $L = \langle 2 \rangle \oplus U$ of signature $(2, 1)$. Here $\langle 2 \rangle$ is a lattice of rank 1 generated by an element of norm 2. Let e_1 be a basis of $\langle 2 \rangle$ and let e_2, e_3 be a basis of U. Obviously, $e_1^2 = 2$. We assume that $e_2^2 = e_3^2 = 0$, $\langle e_2, e_3 \rangle = 1$. By denoting elements of $L \otimes \mathbb{C}$ by $z = z_1 e_1 + z_2 e_2 + z_3 e_3$, we define the subset $\Omega(L)$ of the projective plane $\mathbb{P}(L \otimes \mathbb{C}) = \mathbb{P}^2$ by

$$\Omega(L) = \{z \in \mathbb{P}(L \otimes \mathbb{C}) : \langle z, z \rangle = 0,\ \langle z, \bar{z} \rangle > 0\}. \tag{5.1}$$

Here, for simplicity, we use the same symbol z for a point of $L \otimes \mathbb{C}$ and the corresponding point in $\mathbb{P}(L \otimes \mathbb{C})$. The domain $\Omega(L)$ is an open set of a non-singular conic in $\mathbb{P}^2$. By using the coordinates, the defining equation (5.1) can be represented by

$$\langle z, z \rangle = 2z_1^2 + 2z_2 z_3 = 0, \quad \langle z, \bar{z} \rangle = 2|z_1|^2 + z_2 \bar{z}_3 + \bar{z}_2 z_3 > 0.$$

Since $z_3 \neq 0$ if $z \in \Omega(L)$, we may assume that $z_3 = 1$. By combining the two obtained formulae

$$z_1^2 + z_2 = 0, \quad |z_1|^2 + \mathrm{Re}(z_2) > 0,$$

we have $\mathrm{Im}(z_1)^2 > 0$, that is, $\mathrm{Im}(z_1) \neq 0$. Thus $\Omega(L)$ is isomorphic to the domain H obtained from $\mathbb{C}$ by removing the real axis.

The point corresponding to a rational point $z_1 = q/p$ $((p, q) = 1)$ on the real axis is

$$\left(\frac{q}{p} : -\frac{q^2}{p^2} : 1 \right) \in \mathbb{P}(L \otimes \mathbb{C}),$$

and ∞ corresponds to $(0:1:0)$. We remark that these correspond to primitive, isotropic elements $(pq, -q^2, p^2)$ $(p \neq 0)$ and $(0,1,0)$ in L.

A point τ in H corresponds to $(\tau : -\tau^2 : 1) \in \Omega(L)$, and the action of $\mathrm{GL}(2,\mathbb{Z})$ on H induces that of $\Omega(L)$,

$$(\tau : -\tau^2 : 1) \to \left(\frac{a\tau + b}{c\tau + d} : -\left(\frac{a\tau + b}{c\tau + d}\right)^2 : 1 \right). \tag{5.2}$$

Here $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{Z})$. The right-hand side of formula (5.2) coincides with

$$\left((ad+bc)\tau + ac\tau^2 + bd : -2ab\tau - a^2\tau^2 - b^2 : 2cd\tau + c^2\tau^2 + d^2\right).$$

Therefore the action of $\mathrm{GL}(2,\mathbb{Z})$ on $\Omega(L)$ coincides with the natural action of an element

$$\begin{pmatrix} ad+bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ 2cd & -c^2 & d^2 \end{pmatrix} \tag{5.3}$$

of $\mathrm{O}(L)$ on $\Omega(L)$. Moreover, we can easily check that $\mathrm{SL}(2,\mathbb{Z})$ preserves each of the connected components of $\Omega(L)$.

Exercise 5.2. Show that the matrix in formula (5.3) is an element of $\mathrm{O}(L)$.

5.1.2 Bounded symmetric domains of type IV. Now we give a generalization of the upper half-plane. Let L be a lattice of signature $(2,n)$ and fix it. In this case, we also define $\Omega(L)$ by (5.1). Then $\Omega(L)$ is an open set of a non-singular quadratic hypersurface of $\mathbb{P}^{n+1}$ and is an n-dimensional complex manifold because the lattice is non-degenerate. We take a basis

$$e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}$$

of $L \otimes \mathbb{Q}$ such that

$$\langle z, z' \rangle = z_1 z_1' - \sum_{i=2}^{n} z_i z_i' + z_{n+1} z_{n+2}' + z_{n+2} z_{n+1}'$$

for $z = \sum_{i=1}^{n+2} z_i e_i$, $z' = \sum_{i=1}^{n+2} z_i' e_i$. As in the case of the upper half-plane, we may assume that $z_{n+2} = 1$ for $z \in \Omega(L)$. By combining this with definition (5.1), $\Omega(L)$ is an open set in $\mathbb{C}^n$ defined by the inequality

$$\mathrm{Im}(z_1)^2 - \sum_{i=2}^{n} \mathrm{Im}(z_i)^2 > 0.$$

The domain $\Omega(L)$ consists of two connected components according to $\mathrm{Im}(z_1)$ being positive or negative. A connected component, denoted by $\mathcal{D}(L)$, is called a *bounded symmetric domain of type* IV, or more precisely of type IV_n. The case $n = 1$ is the upper half-plane.

Remark 5.3. For the definition of the above bounded domain, it suffices to consider a real quadratic form of signature $(2, n)$, not a lattice L.

Remark 5.4. As classical irreducible bounded symmetric domains other than of type IV_n, there exist those of type $\mathrm{I}_{m,n}$ $(n \geq m \geq 1)$, of type II_m $(m \geq 2)$, and of type III_m $(m \geq 1)$ where

$$\begin{aligned} \mathrm{I}_{m,n} &= \{Z : Z \in M_{n,m}(\mathbb{C}),\ E_m - Z^* Z > 0\}, \\ \mathrm{II}_m &= \{Z : Z \in M_m(\mathbb{C}),\ {}^t Z = -Z,\ E_m - Z^* Z > 0\}, \\ \mathrm{III}_m &= \{Z : Z \in M_m(\mathbb{C}),\ {}^t Z = Z,\ E_m - Z^* Z > 0\}. \end{aligned}$$

Here $M_{n,m}(\mathbb{C})$, $M_m(\mathbb{C})$ are the sets of all (n, m)-, (m, m)-complex matrices, respectively, E_m is the mth identity matrix, Z^* is the complex conjugate of the transposition of Z, and $A > 0$ means that a hermitian matrix A is positive definite. By definition, there are isomorphisms between the cases of the smallest dimension:

$$\mathrm{I}_{1,1} \cong \mathrm{III}_1 \cong \mathrm{IV}_1 \cong H^+.$$

Moreover, it is known that

$$\mathrm{III}_2 \cong \mathrm{IV}_3, \quad \mathrm{I}_{2,2} \cong \mathrm{IV}_4.$$

The first isomorphism is induced from the correspondence between the Hodge structure on $H^1(A, \mathbb{Z})$ of an abelian surface A and that on $H^2(A, \mathbb{Z}) = \wedge^2 H^1(A, \mathbb{Z})$ (see Section 4.5). On the other hand, $\mathrm{I}_{1,n}$ is nothing but a *complex ball*

$$\sum_{i=1}^{n} |z_i|^2 < 1.$$

Now consider the action of the orthogonal group $\mathrm{O}(L)$ of the lattice L on $\Omega(L)$. Denote by $\mathrm{O}(L)^+$ the subgroup of index 2 preserving $\mathcal{D}(L)$. In the case $n = 1$, it is nothing but $\mathrm{SL}(2, \mathbb{Z})$. Let Γ be a subgroup of $\mathrm{O}(L)^+$ of finite index. It is known that the action of Γ on $\mathcal{D}(L)$ is properly discontinuous (Definition 2.4). This implies that the quotient space $\mathcal{D}(L)/\Gamma$ is Hausdorff. Here let's check directly that the stabilizer subgroup

$$\Gamma_a = \{\gamma \in \Gamma : \gamma(a) = a\}$$

of a point $a \in \mathcal{D}(L)$ is finite. First, we can prove that the subspace $E(a)$ of $L \otimes \mathbb{R}$ generated by $\mathrm{Re}(a)$, $\mathrm{Im}(a)$ is positive definite by the same argument as in (4.2). Since the signature of L is $(2, n)$, the orthogonal complement $E(a)^\perp$ of $E(a)$ in $L \otimes \mathbb{R}$ is negative definite. Since Γ_a preserves $E(a)$ and $E(a)^\perp$, it is a subgroup of the compact group. On the other hand, Γ_a is discrete, and hence finite.

Moreover, the action of Γ_a around the neighborhood of a can be linearized by changing coordinates. In fact, if we denote by γ' the linear transformation of $\gamma \in \Gamma_a$ on the tangent space at a, by defining

$$\sigma(z) = \frac{1}{|\Gamma_a|} \sum_{\gamma \in \Gamma_a} \gamma'^{-1} \gamma(z) \tag{5.4}$$

we can prove that σ acts on the tangent space at a trivially and $\sigma\gamma = \gamma'\sigma$ $(\gamma \in \Gamma_a)$. Thus a neighborhood of the image of a in $\mathcal{D}(L)/\Gamma$ is homeomorphic to the quotient $\mathbb{C}^n/G$ of $\mathbb{C}^n$ by a finite subgroup $G \subset \mathrm{GL}(\mathbb{C}^n)$. Moreover, the following holds. For more details we refer the reader, for example, to Cartan [Ca].

Proposition 5.5. *The quotient space $\mathcal{D}(L)/\Gamma$ has the structure of a normal complex analytic space.*

The quotient space $\mathcal{D}(L)/\Gamma$ is not compact in general as in the case of H^+/Γ. A canonical compactification, called the *Baily–Borel compactification* or the *Satake–Baily–Borel compactification*, is known. Its (rational) boundary components added are projective spaces $\mathbb{P}(T \otimes \mathbb{C})$ associated with primitive isotropic sublattices T as in the case of the upper half-plane. Since the signature of L is $(2, n)$, for $n \geq 2$, T has rank 1 or 2 if it exists, and the corresponding boundary component is one point or the upper half-plane. The quotient space of the union of $\mathcal{D}(L)$ and all (rational) boundary components with a certain topology by Γ has the structure of a normal projective variety which is the Baily–Borel compactification of $\mathcal{D}(L)/\Gamma$.

The boundary of the Baily–Borel compactification has a high codimension and the singularity at the boundary is complicated. On the other hand, this compactification can be embedded into a projective space by automorphic forms on $\mathcal{D}(L)$ with respect to Γ. In other words, it is defined as the homogeneous spectrum Proj of the graded ring of automorphic forms on $\mathcal{D}(L)$ with respect to Γ. Here the homogeneous degree is given by the weight of automorphic forms. Thus it depends only on Γ, and in this sense, it is canonically defined.

Remark 5.6. The Baily–Borel compactification is minimal in the following sense. Let

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}, \quad \Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

Then if a holomorphic map

$$f : \Delta \times \cdots \times \Delta \times \Delta^* \times \cdots \times \Delta^* \to \mathcal{D}(L)/\Gamma$$

is locally liftable, it can be extended to a holomorphic map

$$\bar{f}\colon \Delta \times \cdots \times \Delta \times \Delta \times \cdots \times \Delta \to \overline{\mathcal{D}(L)/\Gamma}.$$

Here $\overline{\mathcal{D}(L)/\Gamma}$ is the Baily–Borel compactification. In particular, consider the case that $\mathcal{D}(L)/\Gamma$ is non-singular and X is any non-singular compactification with a normal crossing divisor as its boundary. Then the identity map $\mathcal{D}(L)/\Gamma \to \mathcal{D}(L)/\Gamma$ can be extended to a surjective holomorphic map

$$X \to \overline{\mathcal{D}(L)/\Gamma}.$$

Remark 5.7. As a generalization of the upper half-plane H^+, we mentioned the bounded symmetric domain of III_m in Remark 5.4, which is also called the Siegel upper half-space of degree m and is denoted by $\mathfrak{H}_m$. The domain $\mathfrak{H}_m$ appears as the period domain of m-dimensional abelian varieties, the higher-dimensional analogue of elliptic curves. The symplectic group $\mathrm{Sp}(2m,\mathbb{Z})$, as a generalization of $\mathrm{SL}(2,\mathbb{Z})$, acts on it and the quotient $\mathfrak{H}_m/\mathrm{Sp}(2m,\mathbb{Z})$ is the moduli space of principally polarized abelian varieties. In this case, Satake [Sa] first discovered the canonical compactification, called Satake's compactification. Later this was generalized to the case of the quotient of a bounded symmetric domain by an arithmetic subgroup by Baily, Borel [BB].

5.2 Deformations of complex structures and the Kodaira–Spencer map

First of all, we recall an outline of deformation theory of complex manifolds. Here we consider only smooth deformations which will be used later.

Definition 5.8. Let $\mathcal{Y}$, B be connected complex manifolds. A holomorphic map $\pi\colon \mathcal{Y} \to B$ is called a *complex analytic family* if the following two conditions hold:

(i) π is proper, that is, $\pi^{-1}(K)$ is compact if $K \subset B$ is so.

(ii) The rank of the Jacobian matrix $J(\pi)$ of π is equal to $\dim B$.

Note that $Y_t = \pi^{-1}(t)$ is a submanifold $\mathcal{Y}$ for any $t \in B$. We call $\mathcal{Y}$ a *deformation family* of Y_t, and $Y_{t'}$ ($t' \in B$) a *deformation* of Y_t.

Example 5.9. We give a complex analytic family of elliptic curves. For $\tau \in H^+$, we put

$$\Gamma_\tau = \mathbb{Z} + \mathbb{Z}\tau, \quad E_\tau = \mathbb{C}/\Gamma_\tau.$$

The action of $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ on $\mathbb{C} \times H^+$ defined by

$$(m, n) \colon (z, \tau) \to (z + m + n\tau, \tau)$$

is properly discontinuous and has no fixed points. Therefore the quotient space $\mathcal{E} = (\mathbb{C} \times H^+)/\Gamma$ is a complex manifold and the projection

$$\pi \colon \mathcal{E} \to H^+$$

induces a complex analytic family.

Example 5.10. Consider Example 3.17. Let N be the connected open set of $\mathbb{C}^{21}$ consisting of τ satisfying two conditions (3.7), (3.8). By considering $g(u)$, $h(u)$ as functions of τ, we define the submanifold $\mathcal{Y}$ of $W \times N$ by equations (3.5). Then $\mathcal{Y} \to N$ is a complex analytic family of elliptic $K3$ surfaces Y_τ.

Any deformation of a compact complex manifold does not change the underlying differentiable structure. That is, the following holds (see, e.g., Kodaira [Kod3, Thm. 2.3]).

Theorem 5.11. *Let $\pi \colon \mathcal{Y} \to B$ be a complex analytic family. Then $Y_t = \pi^{-1}(t)$ and $Y_{t'} = \pi^{-1}(t')$ $(t, t' \in B)$ are diffeomorphic.*

Let $\pi \colon \mathcal{Y} \to B$ be a complex analytic family and let Θ_{Y_t} be the sheaf of germs of holomorphic vector fields on a fiber $Y_t = \pi^{-1}(t)$. Let U be a neighborhood of $t \in B$ with local coordinates $t = (t_1, \ldots, t_m)$ where $\dim B = m$. Then by condition (ii), there exists an open covering $\{\mathcal{U}_i\}_{i \in I}$ of $\pi^{-1}(U)$ with local coordinates $\mathcal{U}_i = U_i \times U$, $(z_i, t) = (z_i^1, \ldots, z_i^n, t_1, \ldots, t_m)$ such that

$$z_i^\ell = f_{ij}^\ell(z_j, t) \quad (\ell = 1, \ldots, n), \tag{5.5}$$

where f_{ij}^ℓ is a holomorphic function on $\mathcal{U}_i \cap \mathcal{U}_j$. Since

$$Y_t = \bigcup_{i \in I} Y_t \cap \mathcal{U}_i \cong \bigcup_{i \in I} U_i,$$

U_i does not depend on $t \in U$, but the patching (5.5) of U_i and U_j depends on t. Now we define

$$\theta_{ij}^\alpha = \sum_{\ell=1}^{n} \frac{\partial f_{ij}^\ell(z_j, t)}{\partial t_\alpha} \frac{\partial}{\partial z_i^\ell} \quad (\alpha = 1, \ldots, m).$$

Then θ_{ij}^α does not depend on the choice of local coordinates (see, e.g., Morrow, Kodaira [MK, Prop. 3.1]) and

$$\theta_{ij}^\alpha \in H^0(U_i \cap U_j, \Theta_{Y_t}).$$

On $U_i \cap U_j \cap U_k$, we have $z_i^\ell = f_{ij}^\ell(z_j, t) = f_{ik}^\ell(z_k, t) = f_{ik}^\ell(f_{kj}(z_j, t), t)$, and by differentiating this, we obtain

$$\frac{\partial f_{ij}^\ell}{\partial t_\alpha} = \frac{\partial f_{ik}^\ell}{\partial t_\alpha} + \sum_{p=1}^{n} \frac{\partial f_{ik}^\ell}{\partial z_k^p} \frac{\partial f_{kj}^p}{\partial t_\alpha}.$$

Therefore

$$\theta_{ij}^\alpha = \sum_\ell \frac{\partial f_{ij}^\ell}{\partial t_\alpha} \frac{\partial}{\partial z_i^\ell} = \sum_l \frac{\partial f_{ik}^l}{\partial t_\alpha} \frac{\partial}{\partial z_i^\ell} + \sum_p \frac{\partial f_{kj}^p}{\partial t_\alpha} \frac{\partial}{\partial z_k^p} = \theta_{ik}^\alpha + \theta_{kj}^\alpha.$$

In this formula, by putting $i = k$ we have $\theta_{ii}^\alpha = 0$. Thus it follows that $\theta_{ij}^\alpha = -\theta_{ji}^\alpha$ on $U_i \cap U_j$, and hence $\{\theta_{ij}^\alpha\}$ is a 1-cocycle.

Definition 5.12. We denote by

$$\frac{\partial Y_t}{\partial t_\alpha} \in H^1(Y_t, \Theta_{Y_t})$$

the cohomology class of $\{\theta_{ij}^\alpha\}$, and for

$$\frac{\partial}{\partial t} = \sum_{\alpha=1}^{m} a_\alpha \frac{\partial}{\partial t_\alpha} \in T_t(B) \quad (a_\alpha \in \mathbb{C})$$

we define

$$\frac{\partial Y_t}{\partial t} = \sum_{\alpha=1}^{m} a_\alpha \frac{\partial Y_t}{\partial t_\alpha},$$

which is called an *infinitesimal deformation* of Y_t. The linear map

$$\rho_t \colon T_t(B) \to H^1(Y_t, \Theta_{Y_t}) \tag{5.6}$$

defined by $\rho_t(\frac{\partial}{\partial t}) = \frac{\partial Y_t}{\partial t}$ is called the *Kodaira–Spencer map*.

Definition 5.13. A complex analytic family $\pi \colon \mathcal{Y} \to B$ is called *complete* at $t_0 \in B$ if for any deformation family

$$\pi' \colon \mathcal{Y}' \to B', \quad Y = \pi^{-1}(s_0), \quad s_0 \in B'$$

of $Y = \pi^{-1}(t_0)$, there exist a neighborhood $B'' \subset B'$ of s_0 and a holomorphic map $f \colon B'' \to B$ with $f(s_0) = t_0$ such that the complex analytic family π' is the pullback of π by f, that is, π' is isomorphic to the fiber product $\mathcal{Y} \times_B B''$.

The next two theorems are due to Kodaira, Spencer [KS1], Kodaira, Nirenberg, Spencer [KNS].

Theorem 5.14. *Let $\pi\colon \mathcal{Y} \to B$ be a complex analytic family. Suppose that the Kodaira–Spencer map is surjective at a point $t \in B$. Then π is complete at t.*

Theorem 5.15. *Let Y be a compact complex manifold with $H^2(Y, \Theta_Y) = 0$. Then there exists a deformation family $\pi\colon \mathcal{Y} \to B$, $Y = \pi^{-1}(t_0)$ of Y such that the Kodaira–Spencer map $\rho_{t_0}\colon T_{t_0}(B) \to H^1(Y, \Theta_Y)$ is isomorphic.*

Now we consider the case of $K3$ surfaces.

Proposition 5.16. *Any deformation of a $K3$ surface X is a $K3$ surface.*

Proof. Consider a deformation family $\pi\colon \mathcal{X} \to \mathcal{B}$, $X = X_{t_0} = \pi^{-1}(t_0)$ ($t_0 \in B$) of the $K3$ surface X. It follows from Theorem 5.11 that topological invariants of every fiber X_t do not change, and hence $b_1(X_t) = b_1(X_{t_0}) = 0$. Therefore, by Theorem 3.5, we have $p_g(X_t) = 1$, $q(X_t) = 0$. On the other hand, the Chern class $c_1(X_{t_0}) \in H^2(X_{t_0}, \mathbb{Z})$ is invariant under deformation, and hence $c_1(X_t) = c_1(X_{t_0}) = 0$. Thus we obtain that $K_{X_t} = 0$. □

Let X be a $K3$ surface and $(U, z = (z_1, z_2))$ local coordinates of X. By definition, there exists a nowhere-vanishing holomorphic 2-form ω_X on X unique up to a constant. If we write

$$\omega_X = \frac{1}{2}\sum_{i,j=1}^{2} f_{ij}(z)\, dz_i \wedge dz_j \in \Gamma(U, \Omega_X^2), \quad f_{ij} = -f_{ji},$$

then $f_{ij}(z) \neq 0$ for any $z \in U$. To

$$\theta = \sum_{i=1}^{2} \theta_i(z) \frac{\partial}{\partial z_i} \in \Gamma(U, \Theta_X)$$

we associate

$$\eta = \sum_{i,j=1}^{2} \theta_i f_{ij}\, dz_j,$$

which gives an isomorphism $\Theta_X \cong \Omega_X^1$ of sheaves. In particular,

$$H^k(X, \Theta_X) \cong H^k(X, \Omega_X^1). \tag{5.7}$$

Lemma 5.17. $\dim H^0(X, \Theta_X) = \dim H^2(X, \Theta_X) = 0$, $\dim H^1(X, \Theta_X) = 20$.

Proof. Let $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$. Then by definition we have $h^{1,0}(X) = h^{0,1}(X) = 0$, $h^{2,0}(X) = 1$. By Serre duality we have $h^{1,2}(X) = h^{2,1}(X) = 0$, $h^{0,2}(X) = 1$. By $h^{0,2}(X) + h^{1,1}(X) + h^{2,0}(X) = b_2(X) = 22$, we obtain $h^{1,1}(X) = 20$. The assertion now follows from (5.7). □

By combining this with Theorems 5.14 and 5.15, we have the following.

Corollary 5.18. *Let X be a $K3$ surface. Then X has a complete complex analytic family, as its deformation family, whose Kodaira–Spencer map is isomorphic.*

Remark 5.19. As mentioned above, any $K3$ surface has a complete deformation family. Moreover, it is known that the map f in Definition 5.13 of completeness is unique by the property $H^0(X, \Theta_X) = 0$ (in this case, a complex analytic family is called universal). In this book, keeping to the necessary minimum, we consider only the case that the base space B of a deformation family is non-singular and the map π is smooth. For further development, for example, see Barth, Hulek, Peters, Van de Ven [BHPV].

Example 5.20. Consider a hypersurface S in a projective space $\mathbb{P}^n$ of degree m. For simplicity, we assume $m \geq 3$. Then S is the set of zeros of a homogeneous polynomial of degree m in $n+1$ variables. The dimension of the vector space over $\mathbb{C}$ consisting of homogeneous polynomials of degree m, by counting the number of monomials $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$ $(\sum_{k=0}^n i_k = m)$ of degree m, is given by $\binom{n+m}{m}$. Up to constant multiplication and the action of the projective transformation group $\mathrm{PGL}(n, \mathbb{C})$, hypersurfaces of degree m form an $\left(\binom{n+m}{m} - (n+1)^2\right)$-dimensional family. On the other hand, except for the case $n = 3$, $m = 4$, it is known that

$$\dim H^1(S, \Theta_S) = \binom{n+m}{m} - (n+1)^2$$

(Kodaira, Spencer [KS2, II, Thm. 14.2]). In the case $n = 3$, $m = 4$, S is a $K3$ surface. Contrary to the above fact that quartic surfaces form a 19-dimensional family, as we proved in Lemma 5.17, we have $H^1(S, \Theta_S) = 20$.

Finally, we give a representation of an infinitesimal deformation $\rho_t(\frac{\partial}{\partial t})$ in terms of a Dolbeault cohomology class. Let Y be a compact complex manifold and let $\pi\colon \mathcal{Y} \to B$ be a complex analytic family with $\pi^{-1}(t_0) = Y_{t_0} = Y$. Let $\{\mathcal{U}_i\}_{i \in I}$ be an open covering of $\mathcal{Y}$ and let $\mathcal{U}_i = U_i \times U$, $(z_i, t) = (z_i^1, \ldots, z_i^n, t_1, \ldots, t_m)$ be local coordinates. Recall that $z_i^\ell = f_{ij}^\ell(z_j, t)$ $(\ell = 1, \ldots, n)$ on $\mathcal{U}_i \cap \mathcal{U}_j$ (formula (5.5)). Let x be local coordinates of Y. As a differentiable manifold, $\mathcal{Y} \cong Y \times B$ (Theorem 5.11), and hence $z_i^l = z_i^l(x, t)$ is a C^∞-function in variables (x, t), and $z_i^l(x, t_0)$ is holomorphic in x.

Let T_Y be the holomorphic tangent bundle of Y and let $\mathcal{A}^{0,q}(T_Y)$ be the sheaf of germs of $(0,q)$-forms valued in T_Y. In terms of local coordinates $x = (x^1, \dots, x^n)$, a section φ of $\mathcal{A}^{0,q}(T_Y)$ can be expressed as

$$\varphi = \sum_{\alpha=1}^{n} \varphi_\alpha \frac{\partial}{\partial x^\alpha}, \quad \varphi_\alpha = \frac{1}{q!} \sum \varphi_\alpha^{j_1,\dots,j_q}(x)\, d\bar{x}^{j_1} \wedge \cdots \wedge d\bar{x}^{j_q}.$$

There exists a fine resolution of the tangent sheaf Θ_Y,

$$0 \longrightarrow \Theta_Y \longrightarrow \mathcal{A}^{0,0}(T_Y) \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1}(T_Y) \longrightarrow \cdots,$$

where $\bar{\partial}\varphi = \sum_\alpha \bar{\partial}\varphi_\alpha \frac{\partial}{\partial x^\alpha}$. Therefore by defining

$$Z_{\bar{\partial}}^{0,q}(T_Y) = \left\{\varphi \in \Gamma(Y, \mathcal{A}^{0,q}(T_Y)) : \bar{\partial}(\varphi) = 0\right\},$$

we have an isomorphism

$$H^1(Y, \Theta) \cong Z_{\bar{\partial}}^{0,1}(T_Y)/\bar{\partial}\Gamma(Y, \mathcal{A}^{0,0}(T_Y)).$$

Let θ_{ij} be a cocycle defining $\left(\frac{\partial Y_t}{\partial t}\right)_{t=t_0}$, and let η be the vector-valued $(0,1)$-form corresponding to θ_{ij}. Then η is given as follows. Take a $\xi_i \in \Gamma(U_i, \mathcal{A}^{0,0}(T_Y))$ satisfying $\theta_{ij} = \xi_j - \xi_i$ on $U_i \cap U_j$. Then $\eta = \bar{\partial}\xi_i = \bar{\partial}\xi_j$. The next lemma will be used in the proof of the local isomorphism of the period map of $K3$ surfaces.

Lemma 5.21. *An infinitesimal deformation* $\left(\frac{\partial Y_t}{\partial t}\right)_{t=t_0} \in H^1(Y, \Theta)$ *corresponds to a vector-valued* $(0,1)$*-form given by*

$$-\sum_\ell \bar{\partial}\left(\frac{\partial z_i^\ell(x,t)}{\partial t}\right)_{t=t_0} \left(\frac{\partial}{\partial z_i^\ell}\right)$$

modulo $\bar{\partial}\Gamma(Y, \mathcal{A}^{0,0}(T_Y))$.

Proof. By differentiating the equation

$$z_i^\ell(x,t) = f_{ij}^\ell(z_j(x,t), t) \quad (\ell = 1, \dots, n),$$

we have

$$\left(\frac{\partial z_i^\ell}{\partial t}\right)_{t=t_0} = \sum_m \left(\frac{\partial f_{ij}^\ell}{\partial z_j^m}\right)\left(\frac{\partial z_j^m}{\partial t}\right)_{t=t_0} + \left(\frac{\partial f_{ij}^\ell}{\partial t}\right)_{t=t_0}.$$

Therefore

$$
\begin{aligned}
\theta_{ij} &= \sum_{\ell} \left(\frac{\partial f_{ij}^{\ell}}{\partial t}\right)_{t=t_0} \left(\frac{\partial}{\partial z_i^{\ell}}\right) \\
&= \sum_{\ell} \left(\frac{\partial z_i^{\ell}}{\partial t}\right)_{t=t_0} \left(\frac{\partial}{\partial z_i^{\ell}}\right) - \sum_{\ell,m} \left(\frac{\partial z_j^{m}}{\partial t}\right)_{t=t_0} \left(\frac{\partial z_i^{\ell}}{\partial z_j^{m}}\right)\left(\frac{\partial}{\partial z_i^{\ell}}\right) \\
&= \sum_{\ell} \left(\frac{\partial z_i^{\ell}}{\partial t}\right)_{t=t_0} \left(\frac{\partial}{\partial z_i^{\ell}}\right) - \sum_{m} \left(\frac{\partial z_j^{m}}{\partial t}\right)_{t=t_0} \left(\frac{\partial}{\partial z_j^{m}}\right).
\end{aligned}
$$

Now by defining

$$
\xi_i = -\sum_{\ell} \left(\frac{\partial z_i^{\ell}}{\partial t}\right)_{t=t_0} \left(\frac{\partial}{\partial z_i^{\ell}}\right),
$$

we obtain $\theta_{ij} = \xi_j - \xi_i$ on $U_i \cap U_j$, and hence the assertion. □

Remark 5.22. The reference for this section is Morrow, Kodaira [MK].

6

The Torelli-type theorem for $K3$ surfaces

We will state the main theme in this book: the Torelli-type theorem for Kähler $K3$ surfaces and its proof. First of all, we formulate the Torelli-type theorem for Kähler $K3$ surfaces and for projective $K3$ surfaces, respectively. Next we show the local Torelli theorem which says that the period map is locally isomorphic. Then we give a proof of the Torelli-type theorem for Kummer surfaces reducing it to the case of the Torelli theorem for complex tori. Also we show that the set of periods of Kummer surfaces is dense in the period domain. With this preparation, the Torelli-type theorem for $K3$ surfaces is proved as follows. Suppose that the periods of two $K3$ surfaces coincide. Then it follows from the local Torelli theorem, the density of the periods of Kummer surfaces, and the Torelli theorem for Kummer surfaces that a complete complex analytic family of each $K3$ surface contains the same Kummer surfaces whose periods converge to the period of the given $K3$ surface. Then the proof will be completed to show that the limit of the graphs of isomorphisms between Kummer surfaces induces an isomorphism between two given $K3$ surfaces.

6.1 Periods of $K3$ surfaces and the Torelli-type theorem

In this chapter we assume that every $K3$ surface is Kähler. First, we consider when two $K3$ surfaces are isomorphic. Let X, X' be $K3$ surfaces. If an isomorphism $\varphi\colon X' \to X$ exists, then it induces an isomorphism

$$\varphi^*\colon H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$$

of lattices and the pullback $\varphi^*(\omega_X)$ of a non-zero holomorphic 2-form ω_X on X is a non-zero holomorphic 2-form on X'. Since such 2-forms coincide up to a constant, we have

$$\varphi^*(\omega_X) = c \cdot \omega_{X'},$$

where c is a non-zero constant. Thus φ^* induces an isomorphism $H^{1,1}(X,\mathbb{R}) \to H^{1,1}(X',\mathbb{R})$. Moreover, φ sends effective divisors to effective divisors, and hence

$$\varphi^*(D(X)) = D(X').$$

Here $D(X)$, $D(X')$ are the Kähler cones of X, X' respectively. The Torelli-type theorem for $K3$ surfaces claims its converse.

Theorem 6.1 (Torelli-type theorem for $K3$ surfaces). *Let X, X' be $K3$ surfaces and let ω_X, $\omega_{X'}$ be non-zero holomorphic 2-forms on X, X' respectively. Suppose that an isomorphism of lattices $\phi\colon H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ satisfies the two conditions*

(a) $\phi(\omega_X) \in \mathbb{C}\omega_{X'}$,

(b) $\phi(D(X)) = D(X')$.

Then there exists a unique isomorphism $\varphi\colon X' \to X$ of complex manifolds with $\varphi^* = \phi$.

Corollary 6.2 (Weak Torelli theorem for $K3$ surfaces). *Let X, X' be $K3$ surfaces. Suppose that an isomorphism of lattices $\phi\colon H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ satisfies the condition $\phi(\omega_X) \in \mathbb{C}\omega_{X'}$. Then X and X' are isomorphic.*

Proof. It is enough to show that ϕ induces an isomorphism of lattices satisfying conditions (a), (b) in Theorem 6.1. First, if necessary considering $-\phi$, we may assume that $\phi(P(X)^+) \subset P(X')^+$. Then both $D(X')$ and $\phi(D(X))$ are fundamental domains of $W(X')$ and hence it follows from Theorem 2.9 that there exists an element w in $W(X')$ with $w \circ \phi(D(X)) = D(X')$. By Remark 4.18, $w(\omega_{X'}) = \omega_{X'}$ and thus $w \circ \phi$ is the desired one. □

Next we state the Torelli-type theorem for projective $K3$ surfaces.

Lemma 6.3. *Let X, X' be projective $K3$ surfaces. Suppose that an isomorphism of lattices*

$$\phi\colon H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$$

satisfies $\phi(\omega_X) \in \mathbb{C}\omega_{X'}$. Then the following three conditions are equivalent:

(i) *ϕ sends any effective divisor to an effective divisor.*

(ii) *ϕ sends any ample divisor to an ample divisor.*

(iii) $\phi(D(X)) \subset D(X')$.

Proof. The equivalence of (ii) and (iii) follows from the fact that $D(X) \cap H^2(X,\mathbb{Z})$ is the set of ample classes (Remark 4.19). By the definition of $D(X)$, (i) implies (iii). Thus it suffices to see that condition (ii) implies that the image of an irreducible curve C by ϕ is effective. By Lemma 4.16, $\phi(C)$ or $-\phi(C)$ is effective, and by considering the intersection number of $\phi(C)$ with an ample class, $\phi(C)$ is effective. □

Definition 6.4. Let X be a projective $K3$ surface and H an ample divisor on X. We assume that H is primitive, that is, if we denote by the same symbol H its cohomology class, it is primitive in the Néron–Severi lattice S_X. The pair (X, H) is called a *polarized* $K3$ *surface* and $H^2 = 2d$ the *degree of polarization*.

It follows from Lemma 6.3 that in the case of polarized $K3$ surfaces, Theorem 6.1 is stated as in the following.

Theorem 6.5 (Torelli-type theorem for polarized $K3$ surfaces). *Let* (X, H)*,* (X', H') *be polarized* $K3$ *surfaces. Suppose that an isomorphism of lattices* $\phi\colon H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ *satisfies the two conditions*

(a) $\phi(\omega_X) \in \mathbb{C}\omega_{X'}$,

(b′) $\phi(H) = H'$.

Then there exists a unique isomorphism $\varphi\colon X' \to X$ *of complex manifolds with* $\varphi^* = \phi$.

As mentioned in the introduction, in the case of an elliptic curve its period is defined as a point in the upper half-plane by taking a basis of the homology group. We can define the period domain of $K3$ surfaces similarly. First, we take an even unimodular lattice L of signature $(3, 19)$ and fix it. The isomorphism class of L is unique by Theorem 1.27.

Definition 6.6. Following the defining formula (5.1) of a bounded symmetric domain of type IV, we define

$$\Omega = \{\omega \in \mathbb{P}(L \otimes \mathbb{C}) : \langle \omega, \omega \rangle = 0,\ \langle \omega, \bar{\omega} \rangle > 0\} \tag{6.1}$$

and call it the *period domain* of $K3$ surfaces. As mentioned in (5.1) we use the same symbol ω for a point in $L \otimes \mathbb{C}$ and its corresponding point in $\mathbb{P}(L \otimes \mathbb{C})$. It follows from Theorem 4.5 that there exists an isomorphism of lattices

$$\alpha_X\colon H^2(X, \mathbb{Z}) \to L$$

for each $K3$ surface X. We call a pair (X, α_X) a *marked* $K3$ *surface*. By the Riemann condition (4.1), we have $\alpha_X(\omega_X) \in \Omega$. The point $\alpha_X(\omega_X)$ is called the *period* of a marked $K3$ surface (X, α_X). Since L has rank 22, Ω is an open set of a non-singular hypersurface of degree 2 in a 21-dimensional projective space, and thus is a 20-dimensional complex manifold.

Remark 6.7. The domain Ω is not a bounded symmetric domain and the quotient $\Omega/\mathrm{O}(L)$ does not have the structure of a complex analytic space. It is known that Ω is connected (Beauville [Be3, VII, Lem. 2]).

Definition 6.8. Let $\mathcal{M}$ be the set of isomorphism classes of marked $K3$ surfaces (X, α_X). Associating $\alpha_X(\omega_X)$ to (X, α_X), we have a (set-theoretical) map

$$\lambda\colon \mathcal{M} \to \Omega. \tag{6.2}$$

We call λ the *period map* of marked $K3$ surfaces.

If $\omega \in \Omega$ is the period of a $K3$ surface X, then the Torelli-type theorem (Theorem 6.1) and its Corollary 6.2 claim that the fiber $\lambda^{-1}(\omega)$ is the set of markings of X. The surjectivity of the period map is stated as in the following theorem.

Theorem 6.9. *For any point $\omega \in \Omega$, there exists a marked $K3$ surface (X, α_X) satisfying $\alpha_X(\omega_X) = \omega$.*

Remark 6.10. One can introduce the structure of a non-singular analytic space on $\mathcal{M}$ by patching complete deformation families of $K3$ surfaces. However, this space is not Hausdorff. As a concrete example to show the non-Hausdorffness, a 3-dimensional family of quartic surfaces, due to Atiyah [At], is famous. The detail is given, for example, in Barth, Hulek, Peters, Van de Ven [BHPV, Chap. VIII, §12].

We also state the period domain for projective $K3$ surfaces. Take a primitive element $h \in L$ with $h^2 = 2d$ and fix it. Denote by L_{2d} the orthogonal complement of h in L. The lattice L_{2d} has the signature $(2, 19)$. The isomorphism class of L_{2d} is independent of the choice of h by Lemma 1.45. Let (X, H) be a polarized $K3$ surface of degree $2d$ and let ω_X be a non-zero holomorphic 2-form on X. Then it follows from Lemma 1.45 that there exists an isomorphism of lattices

$$\alpha_X\colon H^2(X, \mathbb{Z}) \to L, \quad \alpha_X(H) = h. \tag{6.3}$$

We call the pair (X, H, α_X) a *marked polarized $K3$ surface*. Since ω_X is perpendicular to H with respect to the cup product, $\alpha_X(\omega_X)$ is contained in $L_{2d} \otimes \mathbb{C}$. We now define

$$\Omega_{2d} = \{\omega \in \mathbb{P}(L_{2d} \otimes \mathbb{C}) : \langle \omega, \omega \rangle = 0,\ \langle \omega, \bar{\omega} \rangle > 0\}. \tag{6.4}$$

Then $\alpha_X(\omega_X)$ is contained in Ω_{2d}. The rank of L_{2d} is 21 and hence Ω_{2d} is a 19-dimensional complex manifold. The domain Ω_{2d} is the *period domain* for the pairs $((X, H), \alpha_X)$. As mentioned in Section 5.1.2, Ω_{2d} is a disjoint union of two bounded symmetric domains of type IV. Let

$$\Gamma_{2d} = \{\gamma \in \mathrm{O}(L) : \gamma(h) = h\}.$$

Then we have $\Gamma_{2d} = \widetilde{\mathrm{O}}(L_{2d})$ which is a subgroup of $\mathrm{O}(L_{2d})$ of finite index (see Definition 1.16). Two connected components of Ω_{2d} are exchanged by the action of an element in Γ_{2d}. The group Γ_{2d} acts on Ω_{2d} properly discontinuously and in particular the quotient space Ω_{2d}/Γ_{2d} has the structure of a complex analytic space (Proposition 5.5). Let $\mathcal{M}_{2d}$ be the set of isomorphism classes of polarized $K3$ surfaces of degree $2d$. To each marked polarized $K3$ surface (X, H, α_X) of degree $2d$, we associate $\alpha_X(\omega_X)$ mod Γ_{2d} in Ω_{2d}/Γ_{2d} which is independent of the choice of a marking α_X. Therefore we have obtained the map

$$\lambda_{2d}\colon \mathcal{M}_{2d} \to \Omega_{2d}/\Gamma_{2d}.$$

The injectivity of this map is claimed by the Torelli-type theorem for projective $K3$ surfaces.

Remark 6.11. It is known that $\mathcal{M}_{2d}$ is constructed as an algebraic variety and the map λ_{2d} is a morphism of algebraic varieties (see Section 7.3).

We need to be careful of the surjectivity in the projective case. For $\omega \in \Omega_{2d}$, there exists a marked $K3$ surface (X, α_X) satisfying $\alpha_X(\omega_X) = \omega$. Then $\alpha_X^{-1}(h)$ is represented by a divisor H on X with $H^2 = 2d$. If $\langle H, \delta\rangle \neq 0$ for any $\delta \in \Delta(X)$, then by applying suitable reflections we may assume that H is ample and hence obtain a polarized $K3$ surface (X, H). However, it may happen that $\langle H, \delta\rangle = 0$. The condition $H^2 > 0$ implies that the number of such δ is finite. We need to include this type of polarization H. Geometrically the linear system $|mH|$ gives a birational embedding of X into a projective space whose image is an algebraic surface obtained from X by contracting a finite number of non-singular rational curves to rational double points. We relax the definition of polarized $K3$ surfaces (X, H) allowing this type of polarization H, and then can state the surjectivity of the period map of polarized $K3$ surfaces as follows.

Theorem 6.12. *For any $\omega \in \Omega_{2d}$, there exists a marked polarized $K3$ surface (X, α_X) satisfying $\alpha_X(\omega_X) = \omega$ in the above sense.*

Exercise 6.13. Consider a divisor H on a $K3$ surface X with $H^2 > 0$. Show that the sublattice generated by classes $\delta \in \Delta(X)$ with $\langle H, \delta\rangle = 0$ in the Néron–Severi lattice is a root lattice.

6.2 Local isomorphism of the period map (local Torelli theorem)

Definition 6.14. For any $K3$ surface X, we consider a complex analytic family $\pi\colon \mathcal{X} \to B$ of X with $X = X_{t_0}$ ($t_0 \in B$) given in Corollary 5.18 and assume that the base space B is contractible. Then the locally constant sheaf $R^2\pi_*(\mathbb{Z})$ is trivial. Let

$$\alpha\colon R^2\pi_*(\mathbb{Z}) \cong L$$

be an isomorphism of sheaves where L is a constant sheaf over B. Thus we can consider that each fiber of the complex analytic family π is a marked $K3$ surface (X_t, α_{X_t}). Associating $\lambda(t) = \alpha_{\mathbb{C}}(\omega_{X_t})$ with a non-zero holomorphic 2-form ω_t on X_t, we obtain a holomorphic map

$$\lambda \colon B \to \Omega. \tag{6.5}$$

We call λ the *period map* of a complex analytic family π.

We study the differential of this map at $t = t_0$,

$$d\lambda_{t_0} \colon T_{t_0}(B) \to T_{\lambda(t_0)}(\Omega). \tag{6.6}$$

Here $T_x(M)$ denotes the holomorphic tangent space of a complex manifold M at $x \in M$.

Lemma 6.15. *There exists a natural isomorphism*

$$T_{\lambda(t_0)}(\Omega) \cong \mathrm{Hom}(H^{2,0}(X), H^{1,1}(X)).$$

Proof. We first show that for $\ell \in \mathbb{P}(L \otimes \mathbb{C})$ there exists a natural isomorphism

$$T_\ell(\mathbb{P}(L \otimes \mathbb{C})) \cong \mathrm{Hom}(\ell, L \otimes \mathbb{C}/\ell). \tag{6.7}$$

Here ℓ is considered a 1-dimensional subspace of $L \otimes \mathbb{C}$. Let $\Delta = \{s \in \mathbb{C} : |s| < \varepsilon\}$ and for $\theta \in T_\ell(\mathbb{P}(L \otimes \mathbb{C}))$, we can take a holomorphic map $\gamma \colon \Delta \to \mathbb{P}(L \otimes \mathbb{C})$ satisfying $\gamma(0) = \ell$ and $(\frac{d}{ds}\gamma)(0) = \theta$. We denote by ℓ_s the line in $L \otimes \mathbb{C}$ corresponding to $\gamma(s)$. For a given $x \in \ell$ we take a lifting of γ,

$$\tilde{\gamma} \colon \Delta \to L \otimes \mathbb{C}$$

satisfying $\tilde{\gamma}(0) = x$, and then define $h(\theta) \in \mathrm{Hom}(\ell, L \otimes \mathbb{C}/\ell)$ by

$$h(\theta)(x) = \left(\frac{d}{ds}\tilde{\gamma}\right)(0) \bmod \ell.$$

If we choose another lifting $\tilde{\gamma}_1$ of γ, then their difference can be written as

$$\tilde{\gamma}(s) - \tilde{\gamma}_1(s) = s \cdot u(s),$$

where $u \colon \Delta \to L \otimes \mathbb{C}$ is a holomorphic map with $u(s) \in \ell_s$. Then we have

$$\left(\frac{d}{ds}\tilde{\gamma}\right)(0) - \left(\frac{d}{ds}\tilde{\gamma}_1\right)(0) = u(0) \in \ell$$

and hence $h(\theta)(x)$ is independent of the lifting. If $h(\theta)$ is a zero map, then it follows that $\theta = 0$ and thus h is injective. Since the dimensions of both spaces in (6.7) coincide, h is isomorphic.

Next we assume that $\ell \in \Omega$. Then the lifting of γ as above satisfies the homogeneous equation $\langle\tilde{\gamma}(s), \tilde{\gamma}(s)\rangle = 0$. Therefore we have $\langle\tilde{\gamma}(0), (\frac{d}{ds}\tilde{\gamma})(0)\rangle = 0$, and in particular $h(\theta)(x) \in \ell^{\perp}$. Thus we have obtained an isomorphism

$$T_{\ell}(\Omega) \cong \mathrm{Hom}(\ell, \ell^{\perp}/\ell).$$

In the case that $\ell = \mathbb{C}\omega$ corresponds to a holomorphic 2-form on a $K3$ surface $X = X_{t_0}$, that is, $\ell \cong H^{2,0}(X)$, by considering the Hodge decomposition $H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$, we have $\ell^{\perp}/\ell \cong H^{1,1}(X)$ and thus have finished the proof. □

Let X be a $K3$ surface and let $\pi\colon \mathcal{X} \to B$ be a deformation of X. By Corollary 5.18, we may assume that π is a complete complex analytic family at t_0 and the Kodaira–Spencer map is bijective.

Theorem 6.16 (Local Torelli theorem). *The period map λ is isomorphic around a neighborhood $t_0 \in B$.*

Proof. For a tangent vector $\frac{\partial}{\partial t} \in T_{t_0}(B)$, we choose a holomorphic map $b\colon \Delta \to B$ satisfying $b(0) = t_0$, $(\frac{d}{ds}b)(0) = \frac{\partial}{\partial t}$. The tangent vector $d\lambda_{t_0}(\frac{\partial}{\partial t}) \in T_{\lambda(t_0)}(\Omega)$ corresponds to

$$\left(\frac{d}{ds}\omega_{b(s)}\right)(0) \in \mathrm{Hom}(H^{2,0}(X_{t_0}), H^{1,1}(X_{t_0})).$$

Since each fiber $X_t = \pi^{-1}(t)$ $(t \in B)$ is isomorphic to X as differentiable manifolds, we can consider holomorphic local coordinates $z = (z_1, z_2)$ of X as differentiable local coordinates of X_t. Moreover, we can take holomorphic local coordinates $w_1 = w_1(z,t)$, $w_2 = w_2(z,t)$ of a complex manifold X_t which is holomorphic with respect to t and satisfies $w_1(z, t_0) = z_1$, $w_2(z, t_0) = z_2$ (Kodaira, Nirenberg, Spencer [KNS]). Let ω_t be a non-zero holomorphic 2-form on X_t. Then we have

$$\omega_t = \frac{1}{2}\sum_{i,j=1}^{2} \psi_{ij}(w,t)\, dw_i(z,t) \wedge dw_j(z,t). \tag{6.8}$$

Here $\psi_{ij}(w,t)$ is a holomorphic function with respect to w_1, w_2, t.

With expression (6.8) in terms of local coordinates, we have

$$\left(\frac{d}{ds}\omega_{b(s)}\right)(0) = \sum_{i,j=1}^{2} \psi_{ij}(z, t_0)\bar{\partial}\left(\frac{\partial w_i(z,t)}{\partial t}\right)_{t=t_0} \wedge dz_j$$

modulo holomorphic 2-forms. On the other hand, the image $\rho_{t_0}(\frac{\partial}{\partial t}) \in H^1(X_{t_0}, \Theta_{X_{t_0}})$ of the Kodaira–Spencer map is given by

$$\sum_{i=1}^{2} \bar\partial \left(\frac{\partial w_i(z,t)}{\partial t} \right)_{t=t_0} \frac{\partial}{\partial z_i}$$

by Lemma 5.21. Its image under the isomorphism

$$H^1(X_{t_0}, \Theta_{X_{t_0}}) \cong H^1(X_{t_0}, \Omega^1_{X_{t_0}}) \cong H^{1,1}(X_{t_0})$$

given in (5.7) is nothing but

$$\sum_{i,j=1}^{2} \psi_{ij}(z,t_0) \bar\partial \left(\frac{\partial w_i(z,t)}{\partial t} \right)_{t=t_0} \wedge dz_j .$$

Thus the differential of the period map (6.6) is a composition of the Kodaira–Spencer map ρ_{t_0} and the isomorphism (5.7):

$$\begin{array}{ccc} T_{t_0}(B) & \xrightarrow{d\lambda_{t_0}} & T_{\lambda(t_0)}(\Omega) \\ \downarrow \rho_{t_0} & \nearrow & \\ H^1(X_{t_0}, \Theta_{X_{t_0}}). & & \end{array}$$

Since the Kodaira–Spencer map is an isomorphism, $d\lambda_{t_0}$ is also, and hence the assertion has been proved. □

Remark 6.17. The local Torelli theorem was given in Kodaira [Kod2].

6.3 The Torelli-type theorem for Kummer surfaces

6.3.1 Sixteen non-singular rational curves on the Kummer surface. A Kummer surface $X = \mathrm{Km}(A)$ associated with a complex torus $A = \mathbb{C}^2/\Gamma$ is the minimal resolution of the quotient surface $A/\{\pm 1_A\}$ of A by the automorphism -1_A of order 2. Recall that X contains 16 mutually disjoint non-singular rational curves $E_1, \ldots, E_{16}$. Let $\tilde\sigma \colon \tilde A \to A$ blow up 16 points $\frac{1}{2}\Gamma/\Gamma$ in A of order 2. Then $\tilde\pi \colon \tilde A \to X$ is the double covering branched along $E_1, \ldots, E_{16}$ (Section 4.3), and $\frac{1}{2}\sum_{i=1}^{16} E_i \in S_X$ by Proposition 3.9. Here S_X is the Néron–Severi lattice of X.

Now let X be a $K3$ surface, and assume that X contains 16 mutually disjoint non-singular rational curves $E_1, \ldots, E_{16}$. Here we do not assume that X is a Kummer surface. Put $I = \{1, \ldots, 16\}$ and define

$$Q_X = \left\{ K \subset I : \tfrac{1}{2} \textstyle\sum_{i \in K} E_i \in S_X \right\} . \tag{6.9}$$

We denote Q_X by Q for simplicity if there is no confusion. If X is a Kummer surface and $E_1, \ldots, E_{16}$ are as mentioned above, then $I \in Q$. By definition of Q, the following lemma holds.

Lemma 6.18. *For $K, K' \in Q$ define $K + K' = K \cup K' \setminus K \cap K'$ (symmetric difference). Then $K + K' \in Q$, that is, Q is closed under symmetric difference.*

Lemma 6.19. *If $K \in Q$, then the number $|K|$ of elements of K is 0, 8, or 16.*

Proof. Let $K \in Q$, $K \neq \emptyset$. It follows from Proposition 3.9 that there exists a double covering $\pi\colon \tilde{X} \to X$ branched along $\sum_{i \in K} E_i$. Let $\tilde{E}_i$ be the inverse image of E_i $(i \in K)$. Then

$$-4 = 2(E_i)^2 = (\pi^*(E_i))^2 = (2\tilde{E}_i)^2$$

and hence $\tilde{E}_i$ is a non-singular rational curve with the self-intersection number -1. Therefore there exists a holomorphic map $\sigma\colon \tilde{X} \to Y$ blowing down $\tilde{E}_i$ and Y is non-singular. By the argument in the proof of Theorem 4.21, we can prove that the canonical line bundle K_Y is trivial. Moreover, for the signature $(b^+(Y), b^-(Y))$ of $H^2(Y, \mathbb{R})$ we have

$$b^+(Y) = b^+(\tilde{X}) \geq b^+(X) = 3 > 2p_g(Y),$$

and hence $b_1(Y)$ is even (Theorem 3.5). Therefore, by the classification of surfaces, Y is a $K3$ surface or complex torus. For the Euler numbers we have

$$e(Y) = e(\tilde{X}) - |K| = 2e(X) - \sum_{i \in K} e(E_i) - |K| = 48 - 3|K|.$$

Since the Euler number of a complex torus or $K3$ surface is 0 or 24, respectively, we have obtained $|K| = 8, 16$. □

Corollary 6.20 (Characterization of the Kummer surface). *Let X be a $K3$ surface and assume that X contains 16 mutually disjoint non-singular rational curves $E_1, \ldots, E_{16}$. Moreover, assume*

$$\frac{1}{2} \sum_{i=1}^{16} E_i \in S_X. \tag{6.10}$$

Then there exists a unique complex torus A up to isomorphisms such that X is the Kummer surface associated with A and $E_1, \ldots, E_{16}$ are exceptional curves of the minimal resolution.

Proof. It follows from Lemma 6.19 and its proof that there exists a double covering $\pi\colon \tilde{X} \to X$ branched along $E_1, \ldots, E_{16}$ and $\tilde{X}$ contains 16 exceptional curves. By blowing down them, the obtained surface Y is a complex torus A. If we denote by ι the covering transformation of the double covering π, then ι^* acts on $H^1(A, \mathbb{Z})$ as -1

because $H^1(X,\mathbb{Z}) = 0$. By noting $A \cong H^0(A,\Omega^1_A)^*/H_1(A,\mathbb{Z})$ (Section 4.5), we have proved the assertion. □

Remark 6.21. In fact, it is known that assumption (6.10) in Corollary 6.20 is not necessary, that is, the condition $I \in Q$ is automatically satisfied (Nikulin [Ni2]).

6.3.2 Sixteen non-singular rational curves and an affine geometry.

Definition 6.22. Let V be an n-dimensional vector space over the field K. For each $a \in V$ we define the translation $t_a\colon V \to V$ by $t_a(x) = x + a$ $(x \in V)$. Thus we consider V an n-dimensional *affine space*. Usually an affine space is denoted by $\mathbb{A}^n$. An element in V is called a point of the affine space, and a k-dimensional subspace U and its translation $U + x$ $(x \in V)$ are called k-dimensional subspaces of the affine space. A 1-dimensional subspace of the affine space is called a *line*, a 2-dimensional subspace a *plane*, and an $(n-1)$-dimensional subspace a *hyperplane*. For more details of affine spaces we refer the reader to Kawada [Ka].

In the case of a Kummer surface, the set of 16 non-singular rational curves $\{E_i\}_{i\in I}$ bijectively corresponds to the set $\frac{1}{2}\Gamma/\Gamma$ of points of order 2 in a complex torus $A = \mathbb{C}^2/\Gamma$, and hence the index set I has the structure of a 4-dimensional affine space over $\mathbb{F}_2$. The hyperplanes in a 4-dimensional affine space over $\mathbb{F}_2$ are 3-dimensional subspaces U of the vector space and their translations $U + x$. Since the number of 3-dimensional vector subspaces is 15 and $U + x = U + y$ iff $x - y \in U$, the number of hyperplanes is 30.

Lemma 6.23. *Let X be a Kummer surface. Then Q consists of $\emptyset$, I, and hyperplanes. In particular, $|Q| = 2^5$.*

Proof. Since X is a Kummer surface, the assertion $I \in Q$ follows from Proposition 3.9. For a hyperplane H, the origin is contained in H or $I \setminus H$, and hence one of them is a 3-dimensional vector subspace. By Lemma 6.18, the conditions $H \in Q$ and $I \setminus H \in Q$ are equivalent, and hence it is enough to prove $I \setminus H \in Q$ assuming that H is a vector subspace. To do this, we will prove the existence of a $K3$ surface which is a double covering of X branched along $\sum_{i\in I\setminus H} E_i$. Let $X = \mathrm{Km}(A)$ where $A = \mathbb{C}^2/\Gamma$ is a complex torus. Let $\tilde{A}$ be the surface obtained by blowing up 16 points in A of order 2, and let

$$\tilde{\pi}\colon \tilde{A} \to X$$

be the double covering branched along 16 non-singular rational curves $\{E_i\}_{i\in I}$. For a 3-dimensional vector subspace $H \subset \frac{1}{2}\Gamma/\Gamma$, there exists a subgroup $\Gamma' \subset \Gamma$ with $\frac{1}{2}\Gamma'/\Gamma = H$. Let $A' = \mathbb{C}^2/\Gamma'$. Then the projection

$$\tilde{p}\colon A' \to A$$

is an unramified double covering and $\tilde{p}(\frac{1}{2}\Gamma'/\Gamma') = H$. Let $\hat{A}'$ be the surface obtained by blowing up 32 points $\tilde{p}^{-1}(\frac{1}{2}\Gamma/\Gamma)$ in A'. We denote by $\hat{\iota}$ the involution of $\hat{A}'$ induced by $-1_{A'}$ and define $\hat{Y} = \hat{A}'/\langle\hat{\iota}\rangle$. On $\hat{Y}$ there are 16 non-singular rational curves with the self-intersection number -2 corresponding to the elements $\frac{1}{2}\Gamma'/\Gamma' = \tilde{p}^{-1}(H)$ in A' of order 2 fixed by $\hat{\iota}$ and 8 non-singular rational curves with the self-intersection number -1 which are the images of the exceptional curves obtained by blowing up $\tilde{p}^{-1}(I \setminus H)$. The double covering $\tilde{p}$ induces a double covering

$$p\colon \hat{Y} \to X$$

which is ramified exactly along 8 non-singular rational curves with the self-intersection number -1. It now follows from Proposition 3.9 that $I \setminus H \in Q$. Now we denote by $\tilde{A}'$ the surface obtained by blowing up 16 points in A' of order 2 and by ι the automorphism of $\tilde{A}'$ induced from $-1_{A'}$. Then $Y = \tilde{A}'/\langle\iota\rangle$ is the Kummer surface associated with A' and Y is nothing but the surface obtained by blowing down 8 exceptional curves on $\hat{Y}$:

$$\begin{array}{ccccc} \hat{A}' & \longrightarrow & A' & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ \hat{Y} & \longrightarrow & Y & \longrightarrow & X. \end{array}$$

Conversely, let $K \in Q$ with $|K| = 8$. If necessary by considering $I \setminus K$, we may assume that K contains the origin. We have a $K3$ surface Y by blowing down the double covering

$$p\colon \hat{Y} \to X$$

corresponding to K, and have 16 mutually disjoint non-singular rational curves on Y, as the inverse images of $\{E_i\}_{i \in I \setminus K}$, whose sum is divisible by 2 in S_Y. Therefore Y is the Kummer surface associated with a complex torus A', and we recover the setting in the first half. Then K is the image of the points in A' of order 2 under the homomorphism and hence it is a vector subspace. □

Definition 6.24. Let $\langle E_1, \ldots, E_{16}\rangle$ be the sublattice of $H^2(X, \mathbb{Z})$ generated by classes of 16 non-singular rational curves $E_1, \ldots, E_{16}$. Theorem 1.32 implies that this sublattice is not primitive in $H^2(X, \mathbb{Z})$. Let Π be the primitive sublattice of $H^2(X, \mathbb{Z})$ which is an overlattice of $\langle E_1, \ldots, E_{16}\rangle$, that is,

$$\Pi = \left\{\sum_{i=1}^{16} a_i E_i \in H^2(X, \mathbb{Z}) : a_i \in \mathbb{Q}\right\} \subset S_X.$$

We define a map $\gamma\colon Q \to \Pi/\langle E_1, \ldots, E_{16}\rangle$ by

$$\gamma(K) = \frac{1}{2}\sum_{i \in K} E_i \bmod \langle E_1, \ldots, E_{16}\rangle \quad (K \in Q).$$

Lemma 6.25. *The map γ is bijective.*

Proof. If $\gamma(K) = \gamma(K')$, then $\gamma(K + K') = 0$. Hence $K + K' = \emptyset$ and the injectivity follows. On the other hand, for any

$$x = \sum_{i=1}^{16} a_i E_i \in \Pi \quad (a_i \in \mathbb{Q}),$$

the surjectivity follows from the fact $-2a_i = \langle x, E_i \rangle \in \mathbb{Z}$. $\square$

Let $X = \mathrm{Km}(A)$ be a Kummer surface. It follows from Lemmas 6.23, 6.25 that $|\Pi/\langle E_1, \ldots, E_{16}\rangle| = 2^5$, and hence

$$\det(\Pi) = \frac{2^{16}}{|\Pi/\langle E_1, \ldots, E_{16}\rangle|^2} = 2^6.$$

We denote by $\Pi^\perp$ the orthogonal complement of Π in $H^2(X, \mathbb{Z})$ (Section 1.1.1). By Theorem 1.32 and $\det(H^2(X, \mathbb{Z})) = -1$, we obtain

$$\det(\Pi^\perp) = -2^6.$$

Let $\tilde{\sigma}\colon \tilde{A} \to A$ be the blowing up of 16 points of order 2 in the complex torus A, $\tilde{\pi}\colon \tilde{A} \to X$ the double covering branched along 16 exceptional curves, and ι the covering transformation. Consider the homomorphism

$$q_* = \tilde{\pi}_* \circ \tilde{\sigma}^*\colon H^2(A, \mathbb{Z}) \to H^2(X, \mathbb{Z}). \tag{6.11}$$

Since $\tilde{\sigma}^*(x)$ is ι^*-invariant for $x \in H^2(A, \mathbb{Z})$, we have

$$\tilde{\pi}^* \tilde{\pi}_*(\tilde{\sigma}^*(x)) = \tilde{\sigma}^*(x) + \iota^*(\tilde{\sigma}^*(x)) = 2\tilde{\sigma}^*(x).$$

Hence for $x, y \in H^2(A, \mathbb{Z})$,

$$4\langle \tilde{\sigma}^*(x), \tilde{\sigma}^*(y)\rangle = \langle \tilde{\pi}^*\tilde{\pi}_*(\tilde{\sigma}^*(x)), \tilde{\pi}^*\tilde{\pi}_*(\tilde{\sigma}^*(y))\rangle = 2\langle \tilde{\pi}_*(\tilde{\sigma}^*(x)), \tilde{\pi}_*(\tilde{\sigma}^*(x))\rangle.$$

Thus we have obtained

$$\langle q_*(x), q_*(y)\rangle = 2\langle x, y\rangle \quad (\forall\, x, y \in H^2(A, \mathbb{Z})). \tag{6.12}$$

In particular, q_* is injective and its image is contained in $\Pi^\perp$. On the other hand, since $H^2(A, \mathbb{Z})$ is isomorphic to the even unimodular lattice $U^{\oplus 3}$ (formula (4.11)) and q_* multiplies the intersection form by 2, its image has the discriminant -2^6. Therefore $q_*(H^2(A, \mathbb{Z})) = \Pi^\perp$. We now conclude the following.

Corollary 6.26. *The map* $q_*\colon H^2(A,\mathbb{Z}) \to \Pi^\perp$ *is a group isomorphism satisfying* (6.12). *In particular,* $\Pi^\perp \cong U(2)^{\oplus 3}$ (*an isomorphism of lattices*) *and* $q_*(H^2(A,\mathbb{Z}))$ *is a primitive sublattice of* $H^2(X,\mathbb{Z})$.

Exercise 6.27. Let L be an even unimodular lattice of signature $(3,19)$. Show that, for any primitive embedding of Π into L, the orthogonal complement $\Pi^\perp$ is isomorphic to $U(2)^{\oplus 3}$.

The following two lemmas will be used in the proof of the uniqueness of an isomorphism in the Torelli-type theorem for Kummer surfaces.

Lemma 6.28. *Let* X, X' *be Kummer surfaces,* $\{E_i\}_{i\in I}$, $\{E'_{i'}\}_{i'\in I'}$ 16 *non-singular rational curves, and* ω_X, $\omega_{X'}$ *non-zero holomorphic* 2*-forms, respectively. Suppose that an isomorphism* $\phi\colon H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ *of lattices satisfies*

(i) ϕ *sends* $\{E_i\}_{i\in I}$ *to* $\{E'_{i'}\}_{i'\in I'}$,

(ii) $\phi(\omega_X) \in \mathbb{C}\omega_{X'}$.

Then ϕ *induces an affine map between the affine spaces* I, I'.

Proof. Condition (ii) implies that $\phi(S_X) = S_{X'}$. Hence $\phi(Q_X) = Q_{X'}$. By Lemma 6.23, ϕ sends a hyperplane in I to one in I'. If necessary by compositing it with a translation, we may assume that ϕ preserves the origins. Then ϕ preserves 3-dimensional vector subspaces and hence preserves their intersections, that is, 2-dimensional vector subspaces. Let $P = \{0, x, y, x+y\}$ be a 2-dimensional vector subspace. Then $\phi(P) = \{0, \phi(x), \phi(y), \phi(x+y)\}$ and hence $\phi(x+y) = \phi(x) + \phi(y)$, that is, ϕ is a linear map. □

Lemma 6.29. *Let* X *be a Kummer surface. Suppose that an isomorphism* $\phi\colon H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ *of lattices satisfies*

(i) ϕ *preserves the classes of* 16 *non-singular rational curves* $\{E_i\}_{i\in I}$ *and fixes at least one of them,*

(ii) $\phi|\Pi^\perp = 1_{\Pi^\perp}$.

Then ϕ *is the identity map.*

Proof. Since $\omega_X \in \Pi^\perp \otimes \mathbb{C}$, it follows from Lemma 6.28 that ϕ induces an affine transformation of the affine space I. Now, for any affine plane $P \subset I$, define

$$\delta_P = \frac{1}{2}\sum_{i\in P} E_i. \tag{6.13}$$

Recall that $K \in Q$ $(K \neq \emptyset, I)$ is a hyperplane of the affine space I. Since the number of points in the intersection of a hyperplane and a plane is 0, 2, or 4 and $E_i^2 = -2$, we have $\delta_P \in \Pi^*$. On the other hand, it follows from Theorem 1.32 that there exists an isomorphism of the discriminant quadratic forms

$$\Pi^*/\Pi \cong (\Pi^\perp)^*/\Pi^\perp,$$

and by condition (ii), ϕ acts trivially on Π^*/Π. Therefore we have $\delta_P \equiv \delta_{\phi(P)} \bmod \Pi$ for any affine plane P. This implies that $P = \phi(P)$ or $P \cap \phi(P) = \{\emptyset\}$ by Lemma 6.19.

Now we may assume that ϕ fixes E_0 corresponding to the origin 0 of I. If an affine plane P contains the origin 0 then $\phi(P)$ contains the origin too, and hence $\phi(P) = P$. Take any point $i \in I$ which is not the origin. Consider two affine planes P, P' intersecting two points $\{0, i\}$. Then we have $\phi(i) \in \phi(P) \cap \phi(P') = P \cap P' = \{0, i\}$ and hence $\phi(i) = i$. Thus the affine transformation ϕ is the identity map. Therefore ϕ acts trivially on the sublattice $\Pi \oplus \Pi^\perp$ of $H^2(X, \mathbb{Z})$ of finite index and hence ϕ itself acts trivially on $H^2(X, \mathbb{Z})$. □

6.3.3 The affine geometry and complex tori. Now let A be a 2-dimensional complex torus, I the set of points in A of order 2, $\tilde{A}$ the blowing up 16 points of order 2, and $X = \mathrm{Km}(A)$ the Kummer surface. Then we have the commutative diagram

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\tilde{\sigma}} & A \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ X & \xrightarrow{\sigma} & A/\{\pm 1_A\}. \end{array}$$

Here $\tilde{\pi}$, $\tilde{\sigma}$ are as before, π is the quotient map, and σ is the resolution of singularities. Recall that $I \cong H_1(A, \mathbb{Z}/2\mathbb{Z})$, $I^* \cong H^1(A, \mathbb{Z}/2\mathbb{Z})$, $H^2(A, \mathbb{Z}) = \wedge^2 H^1(A, \mathbb{Z})$.

Lemma 6.30. *For $u, v \in H^1(A, \mathbb{Z})$, we put*

$$u_2 = u \bmod 2, \quad v_2 = v \bmod 2 \in H^1(A, \mathbb{Z}/2\mathbb{Z}).$$

Assume that $u_2, v_2 \neq 0$, $u_2 \neq v_2$, and denote by P $(\subset I)$ the 2-dimensional affine subspace which is a translation of $\mathrm{Ker}(u_2) \cap \mathrm{Ker}(v_2)$. *Then*

$$q_*(u \wedge v) \equiv \sum_{i \in P} E_i \bmod 2.$$

Here q_* is the monomorphism given by (6.11).

Proof. It suffices to prove the assertion in the case that u, v are a part of a basis of $H^1(A, \mathbb{Z})$. By considering a deformation of complex structures on A, we may assume that A is the product $E \times F$ of two elliptic curves E, F, and u, v are the pullback of a basis of $H^1(F,\mathbb{Z})$ (see Example 4.24). Then $u \wedge v$ is represented by E, and P corresponds to the set of points in E of order 2. Moreover,

$$(\tilde{\sigma})^*(u \wedge v) = \tilde{E} + \sum_{i \in P} \tilde{E}_i.$$

Here $\tilde{E}_i \subset \tilde{A}$ is the exceptional curve over $i \in P$ and $\tilde{E}$ is the proper transform of E. Since $\tilde{E}$ is invariant under the action of the automorphism of $\tilde{A}$ induced from $-1_A = (-1_E, -1_F)$, we have

$$q_*(\tilde{E}) \in 2H^2(X,\mathbb{Z}),$$

and have finished the proof of the lemma. □

Consider the situation as in Lemma 6.28. Let

$$\tau\colon I = H_1(A,\mathbb{Z}/2\mathbb{Z}) \to I' = H_1(A',\mathbb{Z}/2\mathbb{Z})$$

be the affine map induced from the isomorphism $\phi\colon H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ of lattices. We may assume that τ is linear by composing it with a translation. Let $\tau^*\colon H^1(A',\mathbb{Z}/2\mathbb{Z}) \to H^1(A,\mathbb{Z}/2\mathbb{Z})$ be the dual map. It follows from Corollary 6.26 that ϕ induces an isomorphism $\psi\colon H^2(A,\mathbb{Z}) \to H^2(A',\mathbb{Z})$. We will show that ψ satisfies condition (ii) in the Torelli theorem for complex tori (Theorem 4.35). To do this, we put $\psi_2 = \psi \bmod 2$ and consider the isomorphism

$$\psi_2\colon H^2(A,\mathbb{Z}/2\mathbb{Z}) \to H^2(A',\mathbb{Z}/2\mathbb{Z}).$$

Then the following holds.

Lemma 6.31. $\psi_2 = (\tau^*)^{-1} \wedge (\tau^*)^{-1}$.

Proof. Let

$$q_2\colon H^2(A,\mathbb{Z}/2\mathbb{Z}) \to H^2(X,\mathbb{Z}/2\mathbb{Z})$$

be the map induced from the injection $q_*\colon H^2(A,\mathbb{Z}) \to H^2(X,\mathbb{Z})$. Since the image of q_* is a primitive sublattice (Corollary 6.26), q_2 is injective. Consider the commutative diagram

$$\begin{array}{ccc} H^2(A,\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\psi_2} & H^2(A',\mathbb{Z}/2\mathbb{Z}) \\ \downarrow q_2 & & \downarrow q_2' \\ H^2(X,\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\phi_2} & H^2(X',\mathbb{Z}/2\mathbb{Z}). \end{array}$$

It suffices to show that

$$\phi_2(q_2(u_2 \wedge v_2)) = q_2'((\tau^*)^{-1}(u_2) \wedge (\tau^*)^{-1}(v_2)) \tag{6.14}$$

for any $u_2, v_2 \in H^2(A, \mathbb{Z}/2\mathbb{Z})$. To do this, we may assume that $u_2, v_2 \neq 0$, $u_2 \neq v_2$. It follows from Lemma 6.30 that

$$q_2(u_2 \wedge v_2) \equiv \sum_{i \in P} E_i \bmod 2.$$

Here P is a translation of the 2-dimensional subspace $\mathrm{Ker}(u_2) \cap \mathrm{Ker}(v_2)$. If we set $\phi_2(P) = P'$, then we obtain

$$\phi_2(q_2(u_2 \wedge v_2)) \equiv \sum_{i \in P'} E'_i \bmod 2.$$

Let $u'_2 = (\tau^*)^{-1}(u_2)$, $v'_2 = (\tau^*)^{-1}(v_2) \in H^1(A', \mathbb{Z}/2\mathbb{Z})$. Then τ is induced from ϕ and hence P' is a translation of $\mathrm{Ker}(u'_2) \cap \mathrm{Ker}(v'_2) \subset I'$. Again by Lemma 6.30 we have

$$q'_2(u'_2 \wedge v'_2) \equiv \sum_{i \in P'} E'_i \bmod 2.$$

Thus we have proved (6.14). □

6.3.4 Torelli-type theorem for Kummer surfaces and its proof.

Theorem 6.32 (Torelli-type theorem for Kummer surfaces). *Let X, X' be $K3$ surfaces and assume that X is a Kummer surface. Suppose that an isomorphism $\phi\colon H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ of lattices satisfies the following two conditions:*

(a) $\phi(\omega_X) \in \mathbb{C}\omega_{X'}$,

(b) $\phi(D(X)) = D(X')$.

Then there exists a unique isomorphism $\varphi\colon X' \to X$ of complex manifolds with $\varphi^* = \phi$.

Proof. Let $E_1, \ldots, E_{16}$ be 16 non-singular rational curves on X. Condition (b) implies that ϕ preserves effective divisors and hence $\phi(E_i)$ is effective (Lemma 6.3). We show that $\phi(E_i)$ is irreducible. Assume that $\phi(E_i)$ is reducible and let

$$\phi(E_i) = \sum_j m_j C_j, \quad m_j \in \mathbb{N}$$

be the irreducible decomposition. Then ϕ^{-1} preserves the Kähler cone and hence, again by Lemma 6.3, in

$$E_i = \sum_j m_j \phi^{-1}(C_j),$$

$\phi^{-1}(C_j)$ is effective. This implies that $\dim H^0(X, \mathcal{O}_X(E_i)) \geq 2$, which contradicts $\dim H^0(X, \mathcal{O}_X(E_i)) = 1$ (Lemma 4.13). Thus $\phi(E_i)$ is an irreducible divisor with arithmetic genus 0, and hence is a non-singular rational curve. We now conclude that there exist 16 mutually disjoint non-singular rational curves $E'_i = \phi(E_i)$ on X' with the property

$$\frac{1}{2}\sum_{i=1}^{16} E'_i \in S_{X'},$$

and hence X' is a Kummer surface (Corollary 6.20). Let X' be the Kummer surface associated with a complex torus A'. Let Π' be the primitive sublattice in $H^2(X', \mathbb{Z})$ containing $E'_1, \ldots, E'_{16}$, and let $\Pi'^{\perp}$ be its orthogonal complement. Since ϕ induces an isomorphism from $\Pi^{\perp}$ to $\Pi'^{\perp}$, by Corollary 6.26 it induces an isomorphism $\psi\colon H^2(A, \mathbb{Z}) \to H^2(A', \mathbb{Z})$. By assumption (a), ψ preserves holomorphic 2-forms. Moreover, by Lemma 6.31 we confirm that it satisfies the assumption in the Torelli theorem for complex tori (Theorem 4.35). Therefore there exists a unique isomorphism $\tilde{\varphi}\colon A' \to A$ of complex tori with $\tilde{\varphi}^* = \psi$ up to $\pm 1_A$. Since $\tilde{\varphi}$ preserves points of order 2, it induces an isomorphism $\varphi\colon X' \to X$ which satisfies $\varphi^*|\Pi^{\perp} = \phi|\Pi^{\perp}$ by construction. Note that a translation of A by a point of order 2 acts on $H^2(A, \mathbb{Z})$ trivially. Hence we may assume that $\varphi(E'_1) = E_1$ by composing a translation if necessary. Then it follows from Lemma 6.29 that $\varphi^* = \phi$. Finally, we prove the uniqueness of φ. If an automorphism g of X acts trivially on $H^2(X, \mathbb{Z})$, then g preserves 16 non-singular rational curves and hence induces an automorphism $\tilde{g}$ of A. Since g^* acts on $\Pi^{\perp}$ trivially, $\tilde{g}^*$ acts on $H^2(A, \mathbb{Z})$ in the same way. Now the Torelli theorem for complex tori implies that $\tilde{g} = \pm 1_A$ and hence $g = 1_X$. Thus we have finished the proof of the uniqueness of φ. □

6.4 Density of the periods of Kummer surfaces

Let L be an even unimodular lattice of signature $(3, 19)$ and let Ω be the period domain of $K3$ surfaces given in (6.1). In this section we will show that the set of periods of special Kummer surfaces is everywhere dense in the period domain Ω.

For $\omega \in \Omega$ we denoted by $E(\omega)$ the subspace of $L \otimes \mathbb{R}$ generated by $\mathrm{Re}(\omega), \mathrm{Im}(\omega)$. Then $E(\omega)$ is a 2-dimensional positive definite real subspace and $\{\mathrm{Re}(\omega), \mathrm{Im}(\omega)\}$ is an oriented basis (relation (4.2)). Conversely, for a 2-dimensional oriented positive definite subspace E and an oriented basis x_E, y_E of E with $x_E^2 = y_E^2$, $\omega = x_E + \sqrt{-1}\,y_E$ is a point in Ω. Now we denote by $G_2^+(L)$ the set of 2-dimensional oriented positive definite subspaces of $L \otimes \mathbb{R}$. Then the map

$$\Omega \to G_2^+(L), \quad \omega \to E(\omega) \tag{6.15}$$

is bijective.

Recall that the Picard number of any $K3$ surface is at most 20. A $K3$ surface with Picard number 20 is called a *singular* $K3$ surface. If X is a singular $K3$ surface, then the transcendental lattice T_X (Definition 4.9) is an even positive definite lattice of rank 2. Moreover, if X is a Kummer surface, then

$$\Pi^{\perp} \cong U(2)^{\oplus 3}$$

(Corollary 6.26). Since $\Pi \subset S_X$, $T_X \subset \Pi^{\perp}$ and hence for any $x \in T_X$,

$$\langle x, x\rangle \equiv 0 \bmod 4.$$

The next theorem claims that the converse is also true.

Theorem 6.33. *Let T be a positive definite lattice of rank 2 satisfying*

$$\langle x, x\rangle \equiv 0 \bmod 4 \quad (\forall\, x \in T). \tag{6.16}$$

Then there exists a Kummer surface X with $T_X \cong T$.

Proof. Condition (6.16) implies that $T(1/2)$ is an even lattice. If we construct a complex torus A whose transcendental lattice T_A is isomorphic to $T(1/2)$, then the Kummer surface associated with A is the desired one (the transcendental lattice of a complex torus is defined in the same way as $K3$ surfaces; see Definition 4.9). It follows from Proposition 1.46 that $T(1/2)$ can be embedded in $U^{\oplus 3}$ primitively. Let Γ be a free abelian group of rank 4 and let us embed $T(1/2)$ into $\wedge^2\Gamma^* \cong U^{\oplus 3}$. Let x, y be an oriented orthogonal basis of $T(1/2)\otimes\mathbb{R}$ with $x^2 = y^2$. Put $\omega = x + \sqrt{-1}y$. Then $\mathbb{C}\omega \subset \wedge^2\Gamma^*_{\mathbb{C}}$ satisfies the Riemann condition (4.1) and hence it is a 1-dimensional isotropic subspace in the Grassmann manifold $G(2, \Gamma^*_{\mathbb{C}})$. Therefore there exist $\eta_1, \eta_2 \in \Gamma^*_{\mathbb{C}}$ satisfying $\omega = \eta_1 \wedge \eta_2$. Let H be a 2-dimensional subspace of $\Gamma^*_{\mathbb{C}}$ generated by η_1, η_2. Then we have $\wedge^2 H = \mathbb{C}\omega$. It follows from the Riemann condition $\langle\omega, \bar{\omega}\rangle > 0$ in (4.1) that $\eta_1 \wedge \eta_2 \wedge \bar{\eta}_1 \wedge \bar{\eta}_2 \neq 0$. Therefore we have $H \cap \bar{H} = \{0\}$ and hence $\Gamma^*_{\mathbb{C}} = H \oplus \bar{H}$. Again by the condition $\langle\omega, \bar{\omega}\rangle > 0$, the map

$$\Gamma \to \mathbb{C}^2, \quad \gamma \to (\eta_1(\gamma), \eta_2(\gamma))$$

is an embedding. Then the complex torus $A = \mathbb{C}^2/\Gamma$ satisfies the property $H_1(A, \mathbb{Z}) \cong \Gamma$ and the transcendental lattice T_A coincides with $T(1/2)$. □

We prepare a lemma for the proof of the density.

Lemma 6.34. *Let m, n be natural numbers and let M be a lattice. Suppose that the set*

$$\mathcal{R} = \{\mathbb{R}e \in \mathbb{P}(M \otimes \mathbb{R}) : e \text{ is a primitive element in } M \text{ with } \langle e, e\rangle \equiv m \bmod n\}$$

is not empty. Then $\mathcal{R}$ is a dense subset in $\mathbb{P}(M \otimes \mathbb{R})$.

Proof. By the assumption, M contains a primitive element e_0 satisfying $\langle e_0, e_0\rangle \equiv m \bmod n$. Let V be a non-empty open subset of $\mathbb{P}(M \otimes \mathbb{R})$. Let $\mathbb{R}e \in V$ be a line generated by a primitive element e of M. If $e = \pm e_0$, then $\mathbb{R}e \in \mathcal{R} \cap V$. Now we assume $e \neq \pm e_0$ and will show the existence of an element of $\mathcal{R}$ contained in V. Consider the primitive sublattice $M' = M \cap (\mathbb{Q}e + \mathbb{Q}e_0)$ in M of rank 2. Since e is primitive, we can take an element $f \in M'$ such that $\{e, f\}$ is a basis of M'. Put $e_0 = ae + bf$ $(a, b \in \mathbb{Z})$. Since e_0 is primitive, $e_0 = \pm f$ (i.e., $a = 0, b = \pm 1$) or a and b are coprime. Therefore, for any natural number N, $e_N = e_0 + Nbe = (a + Nb)e + bf$ is primitive too. If N is a multiple of n, then

$$\langle e_N, e_N\rangle \equiv \langle e_0, e_0\rangle \equiv m \bmod n.$$

Moreover, if we take a sufficiently large N, then we have

$$\mathbb{R}e_N = \mathbb{R}\left(e + \frac{1}{Nb}e_0\right) \in V,$$

and hence $e_N \in V \cap \mathcal{R}$. □

Lemma 6.35. *The set of 2-dimensional subspaces in $L \otimes \mathbb{R}$ generated by lattices of rank 2 satisfying condition (6.16) is dense in $G_2^+(L)$.*

Proof. We show that for any given $P_0 \in G_2^+(L)$ there exists a lattice T of rank 2 satisfying condition (6.16) such that $P = T \otimes \mathbb{R} \in G_2^+(L)$ is sufficiently closed to P_0. Let $\{e_1^0, e_2^0\}$ be an orthogonal basis of P_0. Fix a direct decomposition $L = U \oplus U \oplus U \oplus E_8 \oplus E_8$. Note that a direct summand U contains an element of norm 4 (e.g., let e, f be a basis of U with $e^2 = f^2 = 0$, $\langle e, f\rangle = 1$; then $e + 2f$ has norm 4). It follows from Lemma 6.34 that there exists a primitive element $e_1 \in L$ satisfying $\langle e_1, e_1\rangle \equiv 4 \bmod 8$ and $\mathbb{R}e_1$ is sufficiently closed to $\mathbb{R}e_1^0$. Let $\langle e_1, e_1\rangle = 2m$. It follows from Lemma 1.45 that any element in L of norm $2m$ can be sent to an element in the first summand U under the action of $\mathrm{O}(L)$. This implies that the orthogonal complement M of e_1 in L is isomorphic to $U \oplus U \oplus E_8 \oplus E_8 \oplus \langle -2m\rangle$, and in particular M contains a primitive element of norm 64 (e.g., $e + 32f$). Again by Lemma 6.34, there exists a primitive element $e_2 \in M$ satisfying $\langle e_2, e_2\rangle \equiv 0 \bmod 64$ and $\mathbb{R}e_2$ is sufficiently closed to $\mathbb{R}e_2^0$. By the choice of e_1 and e_2, they are perpendicular to each other and the norms of e_1^0, e_2^0 are positive, and hence e_1, e_2 generate a positive definite subspace P and P is sufficiently closed to P_0.

It suffices to prove that $T = P \cap L$ satisfies condition (6.16). For any $x \in T$, $\langle e_1, e_1\rangle x - \langle e_1, x\rangle e_1 \in T$ is perpendicular to e_1, and hence it is a multiple of e_2. Therefore, by the fact that e_2 is primitive in M, there exists $n \in \mathbb{Z}$ with

$$\langle e_1, e_1\rangle x = \langle e_1, x\rangle e_1 + n e_2.$$

By taking the norms of both sides, we have

$$\langle e_1, e_1\rangle^2 \langle x, x\rangle = \langle e_1, x\rangle^2 \langle e_1, e_1\rangle + n^2 \langle e_2, e_2\rangle. \tag{6.17}$$

Let $\langle e_1, x\rangle = 2^a \cdot k$ ($a \geq 0$ and k is odd). By the choice of e_1, e_2, the 2-power of the first term of the right-hand side of (6.17) is exactly 2^{2+2a} and the second term is divisible by at least 2^6. Hence the 2-power of the right-hand side is at least 2^6 or is of the form $2^{2\ell}$ ($\ell \in \mathbb{N}$). On the other hand, if $\langle x, x\rangle = 2^b \cdot k'$ ($b \geq 1$ and k' is odd), then the 2-power of the left-hand side is 2^{4+b}. By comparing both sides, we can prove that $\langle x, x\rangle$ is divisible by 4. □

For $\omega \in \Omega$, define

$$S_\omega = \{x \in L : \langle x, \omega\rangle = 0\}, \quad T_\omega = S_\omega^\perp.$$

By Lemma 6.35 we have the following theorem.

Theorem 6.36. *Let $\mathcal{S}$ be the subset of Ω consisting of ω satisfying the following conditions*:

(i) $\mathrm{rank}(T_\omega) = 2$.

(ii) $x^2 \equiv 0 \bmod 4$ ($\forall\, x \in T_\omega$).

Then $\mathcal{S}$ is dense in Ω.

Remark 6.37. Let $x \in L$ be a primitive element with $x^2 \equiv 0 \bmod 4$. Then there is a dense subset in $L \otimes \mathbb{R}$ consisting of elements $y \in L$ satisfying the following condition: $M = (\mathbb{Q} \cdot x + \mathbb{Q} \cdot y) \cap L$ is a primitive sublattice in L of rank 2 with

$$z^2 \equiv 0 \bmod 4 \quad (\forall\, z \in M).$$

The proof of this fact is to set $e_1^0 = x$ and choose e_2^0 as any element in $L \otimes \mathbb{R}$ perpendicular to e_1 in the proof of Lemma 6.35.

Combining Theorems 6.33 and 6.36 we have the following.

Corollary 6.38. *The set of periods of marked Kummer surfaces is dense in Ω.*

Corollary 6.39. *Any $K3$ surfaces are deformation equivalent.*

Proof. By the local Torelli theorem (Theorem 6.16) and Corollary 6.38, any $K3$ surface can be deformed to a Kummer surface. Since any complex tori are deformation equivalent, any Kummer surfaces are the same. Thus we have obtained the assertion. □

Corollary 6.40. *A $K3$ surface is simply connected*

Proof. It follows from Lefschetz hyperplane theorem (Theorem 3.8) that a non-singular quartic surface in $\mathbb{P}^3$ is simply connected. The assertion now follows from Corollary 6.39. □

Remark 6.41. Corollary 6.39 is due to Kodaira [Kod2]. Kodaira proved Corollary 6.39 to show that the periods of $K3$ surfaces with the structure of an elliptic fibration are dense in Ω and such $K3$ surfaces are deformation equivalent. In the following we introduce Kodaira's argument. First, the next lemma follows from Theorem 6.36.

Lemma 6.42. *The set $\Omega \cap \mathbb{P}(L \otimes \mathbb{Q}(\sqrt{-1}))$ is dense in Ω.*

Lemma 6.43. *For $\mu \in \Omega \cap \mathbb{P}(L \otimes \mathbb{Q}(\sqrt{-1}))$, there exists $m \in \mathbb{P}(L \otimes \mathbb{Q})$ satisfying*

$$\langle \mu, m \rangle = \langle m, m \rangle = 0.$$

Proof. Let $\mu = r + \sqrt{-1}\, s$ $(r, s \in L \otimes \mathbb{Q})$. Since $\mu \in \Omega$ we have

$$\langle r, r \rangle = \langle s, s \rangle > 0, \quad \langle r, s \rangle = 0.$$

The orthogonal complement of the sublattice $(\mathbb{Q}r + \mathbb{Q}s) \cap L$ in L has signature $(1, 19)$, and hence the existence of m follows from Proposition 1.24. □

Now we define

$$\mathcal{E} = \{\omega \in \Omega : \exists\, m \in L,\ \langle \omega, m \rangle = \langle m, m \rangle = 0,\ \langle \omega, n \rangle \neq 0\ \forall\, n \in L,\ n \notin \mathbb{Q}m\}.$$

Lemma 6.44. *The set $\mathcal{E}$ is dense in Ω.*

Proof. For $m \in L$, put

$$\Omega_m = \{\omega \in \Omega : \langle \omega, m \rangle = 0\}.$$

For any open subset U in Ω, there exists $\omega \in U \cap \mathbb{P}(L \otimes \mathbb{Q}(\sqrt{-1}))$ by Lemma 6.42. Moreover, by Lemma 6.43, there exists $m \in L$ satisfying $\langle m, m \rangle = 0$ and $\omega \in \Omega_m$. Hence $U \cap \Omega_m$ is a non-empty open subset of Ω_m. On the other hand, $\Omega_m \cap \Omega_n$ is a hyperplane of Ω_m for $n \neq m \in L$. Therefore

$$U \cap \left(\Omega_m \Big\backslash \sum_{n \neq m} \Omega_n \right)$$

is not empty and the assertion follows. □

We consider a geometric meaning of a marked $K3$ surface (X, α_X) such that the period ω_X satisfies the condition $\alpha_X(\omega_X) \in \mathcal{E}$. First, we may assume that m is primitive in L. Since $\langle \alpha_X(\omega_X), m \rangle = 0$, there is a primitive element e in S_X with $\alpha_X(e) = m$. Moreover, the condition $\alpha_X(\omega_X) \in \mathcal{E}$ implies that any element in S_X perpendicular to ω_X is a multiple of e. That is, $S_X = \mathbb{Z}e$. Since $e^2 = 0$, X is not algebraic (X is a $K3$ surface corresponding to case (2) in Proposition 4.11). Consider a line bundle L on X with $c_1(L) = e$. It follows from the proof of Lemma 4.16 that

$$\dim H^0(X, \mathcal{O}_X(L)) + \dim H^0(X, \mathcal{O}_X(-L)) \geq 2.$$

If necessary by replacing L by $-L$, we may assume that $\dim H^0(X, \mathcal{O}_X(L)) > 0$. Since $S_X = \mathbb{Z}e$, we have $\dim H^0(X, \mathcal{O}_X(L)) = 2$. The linear system $|L|$ has no fixed components and no base points by $e^2 = 0$. Thus we have a holomorphic map

$$\Phi_{|L|} \colon X \longrightarrow \mathbb{P}^1.$$

Its general fiber is an elliptic curve and any singular fiber is irreducible, that is, of type I_1 or II, because there are no curves C with $C^2 \neq 0$ (we will reconsider the structure of an elliptic fibration on $K3$ surfaces in Section 9.2.2).

Lemma 6.44 claims that the periods of marked $K3$ surfaces X with such an elliptic fibration are dense in Ω. It follows from the local Torelli theorem (Theorem 6.16) that any $K3$ surface can be deformed to such a $K3$ surface X. Note that X is not algebraic and hence the elliptic fibration has no sections. It is known that X can be deformed to a $K3$ surface belonging to the family $\mathcal{Y}$ given in Example 5.10 (Kodaira [Kod2]). Thus Corollary 6.39 follows.

Remark 6.45. Recall that a $K3$ surface with Picard number 20 is called a singular $K3$ surface. It is known that the set of isomorphism classes of singular $K3$ surfaces bijectively corresponds to the quotient $\mathcal{Q}/\mathrm{SL}(2,\mathbb{Z})$ of the set $\mathcal{Q}$ of even positive definite lattices of rank 2 by $\mathrm{SL}(2,\mathbb{Z})$ by sending a singular $K3$ surface to its transcendental lattice. For

$$T = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \in \mathcal{Q} \quad (a, b, c \in \mathbb{Z},\ a, c > 0,\ b^2 - 4ac < 0),$$

by defining

$$\tau_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad \tau_2 = \frac{b + \sqrt{b^2 - 4ac}}{2},$$

we have an elliptic curve

$$E_i = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau_i \quad (i = 1, 2).$$

Then it is known that a singular $K3$ surface X can be constructed as a double cover of the Kummer surface $\mathrm{Km}(A)$ associated with $A = E_1 \times E_2$. The relation of the transcendental lattices

$$T_X(2) \cong T_{\mathrm{Km}(A)} \cong T_A(2) \cong T(2)$$

implies $T_X \cong T$ and hence we obtain the surjectivity of the above correspondence. The injectivity follows from the Torelli-type theorem for $K3$ surfaces. This correspondence is due to Shioda, Inose [SI].

6.5 Behavior of the Kähler cones under a deformation

We consider a deformation family $\pi\colon \mathcal{X} \to B$ of $K3$ surfaces. Assume that π is complete and the Kodaira–Spencer map is bijective. Moreover, assume that the base B is contractible and let $X_t = \pi^{-1}(t)$ for $t \in B$ and $X = X_0$ $(0 \in B)$. We fix an isomorphism

$$\alpha\colon R^2\pi_*(\mathbb{Z}) \cong L,$$

where L is the constant sheaf over B, and by identifying by this isomorphism we have the inclusion

$$D(X_t) \subset P^+(X_t) \subset H^{1,1}(X_t, \mathbb{R}) \subset L \otimes \mathbb{R}.$$

In other words, there are subspaces $H^{1,1}(X_t, \mathbb{R})$ in the fixed space $L \otimes \mathbb{R}$ parametrized by B continuously. In this section we show that the union of Kähler cones

$$\bigcup_{t\in B} D(X_t) \subset \bigcup_{t \in B} H^{1,1}(X_t, \mathbb{R})$$

is an open subset.

First of all, we set

$$\widetilde{\Omega} = \{(\omega, \kappa) \in \Omega \times (L \otimes \mathbb{R}) : \langle \omega, \kappa \rangle = 0,\ \langle \kappa, \kappa \rangle > 0\}. \tag{6.18}$$

Let

$$\Delta = \{\delta \in L : \langle \delta, \delta \rangle = -2\}, \quad \Delta_\omega = \{\delta \in \Delta : \langle \delta, \omega \rangle = 0\} \quad (\omega \in \Omega) \tag{6.19}$$

and let $W(L)$ be the subgroup of $\mathrm{O}(L)$ generated by $\{s_\delta : \delta \in \Delta\}$. Moreover, we define

$$\widetilde{\Omega}^\circ = \left\{(\omega, \kappa) \in \widetilde{\Omega} : \langle r, \kappa \rangle \neq 0\ \forall\, r \in \Delta_\omega\right\}. \tag{6.20}$$

Recall that for $\omega \in \Omega$ we denote by $E(\omega)$ the oriented subspace with a basis $\{\mathrm{Re}(\omega), \mathrm{Im}(\omega)\}$. For $(\omega, \kappa) \in \widetilde{\Omega}$ we denote by $E(\omega, \kappa)$ the 3-dimensional oriented subspace in $L \otimes \mathbb{R}$ generated by a basis $\{\mathrm{Re}(\omega), \mathrm{Im}(\omega), \kappa\}$. Note that $E(\omega, \kappa)$ is positive definite. Let $G_3^+(L \otimes \mathbb{R})$ be the Grassmann manifold consisting of 3-dimensional oriented subspaces in $L \otimes \mathbb{R}$. Then the following holds.

Lemma 6.46. *The set*

$$\mathcal{F} = \left\{E \in G_3^+(L \otimes \mathbb{R}) : \exists\, \delta \in \Delta,\ \langle E, \delta\rangle = 0\right\}$$

is a closed set in $G_3^+(L \otimes \mathbb{R})$.

Proof. We show that the complement $G_3^+(L \otimes \mathbb{R}) \setminus \mathcal{F}$ is open. The orthogonal group $\mathrm{O}(L \otimes \mathbb{R})$ acts on $G_3^+(L \otimes \mathbb{R})$ transitively and the isotropy subgroup of $E \in G_3^+(L \otimes \mathbb{R})$ is a subgroup of $\mathrm{O}(E) \times \mathrm{O}(E^\perp)$. Here E is positive definite and its orthogonal complement $E^\perp$ is negative definite, and hence the orthogonal groups of them are compact. This implies that the action of $\mathrm{O}(L \otimes \mathbb{R})$ on $G_3^+(L \otimes \mathbb{R})$ is proper. Therefore the action of a discrete subgroup $W(L)$ is also proper. Thus the action of $W(L)$ is properly discontinuous (Definition 2.4). Note that E is fixed by a reflection s_δ if and only if E is perpendicular to δ. Hence for any E in $G_3^+(L \otimes \mathbb{R}) \setminus \mathcal{F}$ there exists a neighborhood $U \subset G_3^+(L \otimes \mathbb{R})$ of E such that $w(U) \cap U = \emptyset$ for any $w \in W(L)$. (There exists a neighborhood V of E such that the number of w with $w(V) \cap V \neq \emptyset$ is finite. Let $w_1, \ldots, w_k$ be such a w. Since $w_i(E) \neq E$, there exists a neighborhood V_i of E satisfying $w_i(V_i) \cap V_i = \emptyset$. Then we can take $U = V \cap V_1 \cap \cdots \cap V_k$ as a neighborhood of E.) Thus we have $U \subset G_3^+(L \otimes \mathbb{R}) \setminus \mathcal{F}$. □

The condition $(\omega, \kappa) \in \widetilde{\Omega} \setminus \widetilde{\Omega}^\circ$ is equivalent to $E(\omega, \kappa) \in \mathcal{F}$. Therefore we have the following.

Corollary 6.47. *$\widetilde{\Omega}^\circ$ is an open set in $\widetilde{\Omega}$.*

Now we return to the initial situation. We may assume that B is an open set in Ω by the local isomorphism of the period map (Theorem 6.16). Then

$$\bigcup_{t \in B} P^+(X_t)$$

is a connected component of $\widetilde{\Omega} \cap (B \times L \otimes \mathbb{R})$ and an open subset of $\widetilde{\Omega}$. Consider the union of Kähler cones

$$\mathcal{D}(\mathcal{X}) = \bigcup_{t \in B} D(X_t) \subset \widetilde{\Omega}.$$

Lemma 6.48. *$\mathcal{D}(\mathcal{X})$ is an open set in $\widetilde{\Omega}$.*

Proof. We fix a point $x_0 \in D(X_0)$, and take a Kähler class $\kappa_0 \in D(X_0)$. Then the segment $[x_0\ \kappa_0]$ connecting x_0 and κ_0 is contained in $D(X_0)$. It follows from

Corollary 6.47 that there exists an open neighborhood $\mathcal{K}$ of the segment $[x_0\ \kappa_0]$ in $\widetilde{\Omega}$ satisfying

$$\mathcal{K} \subset \widetilde{\Omega}^\circ.$$

Then we may assume that $\mathcal{K} \cap H^{1,1}(X_0, \mathbb{R}) \subset D(X_0)$. Since $\bigcup_{t\in B} H^{1,1}(X_t, \mathbb{R})$ is locally homeomorphic to the product $B \times \mathbb{R}^{20}$, we may assume that $\mathcal{K} \cap H^{1,1}(X_t, \mathbb{R})$ is connected and

$$\mathcal{K} \cap H^{1,1}(X_t, \mathbb{R}) \subset P^+(X_t)$$

for any $t \in B$. On the other hand, since a small deformation of a Kähler class is a Kähler class (Kodaira, Spencer [KS2, III, Thm. 15]), we may assume that $\mathcal{K} \cap H^{1,1}(X_t, \mathbb{R})$ $(\forall\, t \in B)$ contains a Kähler class for a sufficiently small B. Thus we can choose $\mathcal{K}$ satisfying

$$\mathcal{K} \cap H^{1,1}(X_t, \mathbb{R}) \subset D(X_t)$$

for any $t \in B$. Therefore $\mathcal{K} \subset \mathcal{D}(\mathcal{X})$ and the proof is finished. □

Now we consider the situation as in Theorem 6.1, that is, let X, X' be $K3$ surfaces and let $\phi\colon H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ be an isomorphism of lattices preserving the periods and the Kähler cones. Let

$$\pi\colon \mathcal{X} \to B, \quad \pi'\colon \mathcal{X}' \to B'$$

be deformation families of X, X', respectively such that $X = \pi^{-1}(0)$, $X' = \pi'^{-1}(0')$ $(0 \in B, 0' \in B')$ and B, B' are contractible. Moreover, assume that π, π' are complete and their Kodaira–Spencer maps are bijective. We consider an isomorphism

$$\alpha\colon R^2\pi'_*(\mathbb{Z}) \cong L$$

with a constant sheaf L over B' and an extension

$$\Phi\colon R^2\pi_*(\mathbb{Z}) \cong R^2\pi'_*(\mathbb{Z})$$

of ϕ. By using markings α, $\alpha \circ \Phi$, we have period maps

$$\lambda\colon B \to \Omega, \quad \lambda'\colon B' \to \Omega.$$

We may assume that $B = B'$, $0 = 0'$ by the local isomorphism of the period map (Theorem 6.16). We set $X_t = \pi^{-1}(t)$, $X'_t = \pi'^{-1}(t)$ for $t \in B$. We denote by ϕ_t the isomorphism $H^2(X_t, \mathbb{Z}) \to H^2(X'_t, \mathbb{Z})$ of lattices induced from Φ. Then we have the following.

Theorem 6.49. *If t is sufficiently close to 0, then ϕ_t sends the Kähler cone of X_t to that of X'_t.*

Proof. The isomorphism Φ induces a homeomorphism

$$\bigcup_{t\in B} H^{1,1}(X_t,\mathbb{R}) \to \bigcup_{t\in B} H^{1,1}(X'_t,\mathbb{R})$$

over B. Thus the assertion follows from Lemma 6.48. □

6.6 Proof of the Torelli-type theorem for $K3$ surfaces

6.6.1 Limit of isomorphisms between $K3$ surfaces. The next result is key in the proof of the Torelli-type theorem.

Theorem 6.50. *We assume the same situation as in Theorem 6.49. Suppose that there exists a dense subset K in B such that ϕ_t $(t \in K)$ is induced from an isomorphism $\varphi_t : X'_t \to X_t$ of complex manifolds. Then there exists an isomorphism $\varphi: X' \to X$ with $\varphi^* = \phi$.*

Proof. We will show this theorem by dividing the proof into several lemmas. First, we use the next result due to Bishop [Bi].

Proposition 6.51. *Suppose that a sequence of points $t_1, t_2, \ldots \in K$ converges to the origin 0 and $\Gamma_i \subset X_{t_i} \times X'_{t_i}$ is the graph of an isomorphism φ_{t_i}. Suppose that the volumes of the Γ_i with respect to a hermitian metric on $X \times X'$ are bounded. Then, if necessary by choosing a subsequence of $\{t_i\}$, $\{\Gamma_i\}$ converges to a 2-dimensional complex analytic subset Γ_0 in $X \times X'$.*

In the proof of the following lemma we use essentially the Kählerness of $K3$ surfaces.

Lemma 6.52. *The volumes of Γ_i are bounded.*

Proof. By taking B sufficiently small, we may assume that Kähler metrics of X_t, X'_t are continuous in $t \in B$ (Kodaira, Spencer [KS2, III, Thm. 15]). Let κ_t, κ'_t be the associated Kähler forms. Then κ_t, κ'_t are continuous in $t \in B$. Let p_1, p_2 be the projections from $X_t \times X'_t$ to the first and the second factors respectively. Then the volume $\mathrm{vol}(\Gamma_i)$ of Γ_i is given by

$$\mathrm{vol}(\Gamma_i) = \int_{\Gamma_i} (p_1^*(\kappa_{t_i}) + p_2^*(\kappa'_{t_i}))^2.$$

Since Γ_i is the image of the map

$$\varphi_{t_i} \times 1_{X'_{t_i}} : X'_{t_i} \to X_{t_i} \times X'_{t_i},$$

we have

$$\mathrm{vol}(\Gamma_i) = \int_{X'_{t_i}} ((\varphi_{t_i})^* \kappa_{t_i} + \kappa'_{t_i})^2 = \int_{X_{t_i}} (\kappa_{t_i})^2 + \int_{X'_{t_i}} (\kappa'_{t_i})^2 + 2 \int_{X'_{t_i}} \varphi^*_{t_i}(\kappa_{t_i}) \wedge \kappa'_{t_i}.$$

On the right-hand side of this equation, the first and the second terms are bounded around the origin 0 because the integral functions are continuous. The boundedness of the third term is non-trivial because φ_{t_i} is defined only on K. Now we use the Kählerness of the metrics. That is, the Kähler forms are closed and hence we can consider it a cohomology class. Let $[\kappa]$ be the cohomology class of the Kähler form κ. Then we have

$$\int_{X'_{t_i}} \varphi^*_{t_i}(\kappa_{t_i}) \wedge \kappa'_{t_i} = [\varphi^*_{t_i}(\kappa_{t_i})] \cdot [\kappa'_{t_i}] = [\phi_{t_i}(\kappa_{t_i})] \cdot [\kappa'_{t_i}]$$

which is continuous in $t \in B$ and hence is bounded around 0. Thus we have finished the proof of Lemma 6.52. □

It follows from Lemma 6.52 and Proposition 6.51 that there exists a limit Γ_0 of the graphs Γ_i of φ_{t_i} as a complex analytic set whose cohomology class coincides with $[\phi] \in H^4(X \times X', \mathbb{Z})$.

Lemma 6.53. *In the above situation,*

$$\Gamma_0 = \Delta_0 + \sum a_{ij} C_i \times C'_j, \quad a_{ij} \in \mathbb{Z},\ a_{ij} \geq 0.$$

Here Δ_0 is the graph of an isomorphism between X and X', and $C_i \subset X$, $C'_j \subset X'$ are irreducible curves.

Proof. Let $p\colon X \times X' \to X$, $p'\colon X \times X' \to X'$ be the respective projections. The cohomology class $z \in H^4(X \times X', \mathbb{Z})$ induces a linear map

$$z^*\colon H^*(X, \mathbb{Z}) \to H^*(X', \mathbb{Z})$$

as follows. First, for $x \in H^i(X, \mathbb{Z})$, taking the cup product of $p^*(x) \in H^i(X \times X', \mathbb{Z})$ and z, we have $\langle p^*(x), z \rangle \in H^{i+4}(X \times X', \mathbb{Z})$. Then the image under the Gysin map

$$p'_*\colon H^{i+4}(X \times X', \mathbb{Z}) \to H^i(X', \mathbb{Z})$$

is $z^*(x)$.

Now any irreducible component Z of Γ_0 is one of the following:

(a) $p(Z) = X$ and $p'(Z) = X'$.

(b) Both $p(Z)$ and $p'(Z)$ are curves.

(c) $p(Z) = X$ and $p'(Z)$ is a point.

(c′) $p(Z)$ is a point and $p'(Z) = X'$.

(d) $p(Z) = X$ and $p'(Z)$ is a curve.

(d′) $p(Z)$ is a curve and $p'(Z) = X'$.

A 2-dimensional analytic subset $Z \subset X \times X'$ is said to be of type (p, q) if Z satisfies $p_*(Z) = p \cdot 1 \in H^0(X, \mathbb{Z})$, $p'_*(Z) = q \cdot 1' \in H^0(X', \mathbb{Z})$. Here 1, $1'$ are generators of $H^0(X, \mathbb{Z})$, $H^0(X', \mathbb{Z})$, respectively. Since the graph Γ_i of an isomorphism is of type $(1, 1)$, its limit Γ_0 is also of type $(1, 1)$. In the case that Z is as in the above (a), . . . , (d′), only case (b) is of type $(0, 0)$. Moreover, except for case (a), the map $[Z]^*(x) = p'_*(\langle p^*(x), [Z]\rangle)$ factors to the cohomology group of a curve or a point, and hence it maps $H^{2,0}(X)$ to 0. Since Γ_0^* induces an isomorphism from $H^{2,0}(X)$ to $H^{2,0}(X')$, at least one component of type (a) appears in Γ_0. Since Γ_0 is of type $(1, 1)$, a component of type (a) is unique and it is of type $(1, 1)$. Let Δ_0 be such a component. It follows that irreducible components of types (c), (c′), (d), (d′) do not appear. Moreover, the projections from Δ_0 to X, X' have degree 1 and hence they are bimeromorphic. Finally, $K3$ surfaces are minimal, and hence any bimeromorphic map between $K3$ surfaces is biholomorphic. Thus Δ_0 is the graph of an isomorphism between X and X'. □

Lemma 6.54. *In Lemma 6.53,* $a_{ij} = 0$.

Proof. Identifying $X = X'$ under the isomorphism Δ_0, we may assume that Δ_0 is the graph of the identity map. If $\kappa \in H^2(X, \mathbb{R})$ is a Kähler class, then in the equation

$$\phi(\kappa) = [\Gamma_0]^*(\kappa) = \kappa + \sum_{i,j} a_{ij} \langle C_i, \kappa\rangle C'_j,$$

we have $\langle C_i, \kappa\rangle > 0$. Since ϕ preserves the cup product, $\langle \phi(\kappa), \phi(\kappa)\rangle - \langle \kappa, \kappa\rangle = 0$ and hence it follows that

$$\sum_{i,j} a_{ij} \langle C_i, \kappa\rangle \langle C'_j, \phi(\kappa) + \kappa\rangle = \langle \phi(\kappa) - \kappa, \phi(\kappa) + \kappa\rangle = 0.$$

Since $\phi(\kappa) + \kappa$ is a Kähler class, we have $\langle C'_j, \phi(\kappa) + \kappa\rangle > 0$. Therefore $a_{ij} = 0$. □

Thus we have finished the proof of Theorem 6.50. □

Remark 6.55. In this book we use the fact that $K3$ surfaces are Kähler (Siu's theorem) without proof. The proof of the Kählerness of any $K3$ surface due to Siu [Si] is as follows. Assume that X is a Kähler $K3$ surface and X' any $K3$ surface in

Theorem 6.50. He showed that there exists a metric such that Lemma 6.52 holds. Then the rest of the proof implies Theorem 6.50. Thus X' is isomorphic to the Kähler $K3$ surface X and hence X' is also Kähler.

6.6.2 Automorphisms of a $K3$ surface acting trivially on the cohomology group.

The uniqueness in Theorem 6.1 follows from the next theorem.

Theorem 6.56. *Let X be a $K3$ surface and let g be an automorphism of X acting trivially on $H^2(X,\mathbb{Z})$. Then g is the identity map.*

Proof. We first show that g is of finite order. The automorphism group $\mathrm{Aut}(X)$ of X is a complex Lie group and its Lie algebra is isomorphic to $H^0(X,\Theta_X)$. By Lemma 5.17, $H^0(X,\Theta_X) = 0$ and hence $\mathrm{Aut}(X)$ is a discrete group. Let $G \subset \mathrm{Aut}(X)$ be a subgroup of automorphisms acting trivially on $H^2(X,\mathbb{Z})$. It follows from Theorem 6.50 that G is compact. Thus we have proved that G is a finite group.

Now let $g \neq 1_X$ and let n be the order of g. Assume that $p \in X$ is a fixed point of g. Then the action of g on the tangent space $T_p(X)$ is non-trivial. This is because the action of g can be linearized for suitable coordinates around p as shown in Section 5.1 by using formula (5.4). On the other hand, since g preserves a holomorphic 2-form and $\Omega^2_{X,p} \cong \wedge^2 T_p(X)^*$, the determinant of the action of g on $T_p(X)$ is 1. Since g has finite order, the action of g on $T_p(X)$ can be diagonalized and is given by $\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{smallmatrix}\right)$. Here ϵ is a primitive nth root of unity. In particular, any fixed points of g are isolated if they exist, and hence they are finite. On the other hand, it follows from the Lefschetz fixed point formula (e.g., Ueno [U]) that the number of fixed points of g coincides with

$$\sum_{i=0}^{4} (-1)^i \operatorname{trace} g^* | H^i(X,\mathbb{C}),$$

and is equal to 24.

Now consider the quotient surface $Y = X/\langle g \rangle$. We can see that Y has a rational double point of type A_{n-1} (see Remark 4.22) at the images of the fixed points of g as follows. Let (x, y) be local coordinates around p with $g^*(x) = \epsilon x$, $g^*(y) = \epsilon^{-1} y$. Let $u = x^n$, $v = y^n$, $w = xy$. Then Y is locally given by $uv = w^n$ around the image of p. The origin is a singularity of Y which is called a rational double point of type A_{n-1} (see Remark 4.22). Let $f : Y' \to Y$ be the minimal resolution of singularities. Since g acts on $H^2(X,\mathbb{Z})$ trivially, in particular it fixes a holomorphic 2-form. Therefore it induces a nowhere-vanishing holomorphic 2-form on an open subset of Y deleting the singularities. The property of rational double points (see Remark 4.22) implies that this holomorphic 2-form can be extended to a non-zero holomorphic 2-form on Y'. Therefore Y' has trivial canonical line bundle and by the classification of surfaces it is a complex torus, a $K3$ surface, or a Kodaira surface. On the other hand, there exists at least one exceptional curve over each singular point which gives an independent class in $H^2(Y', \mathbb{Z})$. There are 24 fixed points and hence rank $H^2(Y', \mathbb{Z}) \geq 24$, which contradicts the fact that the 2nd Betti number of a complex torus, a $K3$ surface, or a Kodaira surface is at most 22. □

Remark 6.57. Theorem 6.56 was first proved in the algebraic case by Piatetskii-Shapiro, Shafarevich [PS]. In the latter half of the proof of Theorem 6.56, one can use the holomorphic Lefschetz fixed point formula instead of considering the quotient surface $X/\langle g\rangle$ (Barth, Hulek, Peters, Van de Ven [BHPV, Prop. 11.3]).

6.6.3 Proof of the Torelli-type theorem for $K3$ surfaces. We give a proof of Theorem 6.1 (the Torelli-type theorem for $K3$ surfaces).

Proof. Consider the same situation as in Theorem 6.49. Then ϕ_t preserves holomorphic 2-forms. Moreover, Theorem 6.49 implies that it preserves Kähler cones. On the other hand, it follows from Corollary 6.38 that the periods of Kummer surfaces are dense in Ω. Therefore there exists a sequence $\{t_n\} \subset B$ converging to the origin 0 such that $\lambda(t_n)$ are periods of Kummer surfaces. It follows from the Torelli-type theorem for Kummer surfaces (Theorem 6.32) that there exist isomorphisms

$$\varphi_{t_n} : X'_{t_n} \longrightarrow X_{t_n}$$

with $\varphi^*_{t_n} = \phi_{t_n}$. Now by applying Theorem 6.50, we have obtained an isomorphism $\varphi\colon X' \to X$ satisfying $\varphi^* = \phi$. Finally, the uniqueness of φ follows from Theorem 6.56 and the proof of the Torelli-type theorem has been finished. □

Remark 6.58. The references for the proof of the Torelli-type theorem in this chapter are not only the original article Burns, Rapoport [BR], but also Barth, Hulek, Peters, Van de Ven [BHPV], Beauville [Be3].

In the following we give a rough argument of the proof of the Torelli-type theorem for polarized $K3$ surfaces due to Piatetskii-Shapiro, Shafarevich [PS]. In this case one can construct a family of marked polarized $K3$ surfaces (X, H, α_X) of degree $2d$.

Consider a polarized $K3$ surface X of degree $2d$. If necessary by replacing H by mH, we may assume that H is the hyperplane section of $X \subset \mathbb{P}^n$. Here $n = d + 1$. Then there exists an open set $\mathfrak{M}$ of the Hilbert scheme with the Hilbert polynomial $P(k) = \chi(\mathcal{O}(kH))$ and a family of non-singular $K3$ surfaces of degree $2d$,

$$\mathcal{Z} \subset \mathbb{P}^n \times \mathfrak{M} \xrightarrow{\pi} \mathfrak{M}. \tag{6.21}$$

For $x \in \mathfrak{M}$ we denote by Z_x the fiber $\pi^{-1}(x) \subset \mathbb{P}^n$. Let $\mathcal{N}_{\mathbb{P}^n/Z_x}$ be the normal bundle of Z_x; then $\mathfrak{M}$ is non-singular at x iff $H^1(Z_x, \mathcal{N}_{\mathbb{P}^n/Z_x}) = 0$. In our case, the fact that

Z_x is a $K3$ surface implies that $H^1(Z_x, \mathcal{N}_{\mathbb{P}^n/Z_x}) = 0$ and hence the smoothness of $\mathfrak{M}$ follows. If necessary by considering a connected component of $\mathfrak{M}$, we may assume that $\mathfrak{M}$ is connected. Let $\widetilde{\mathfrak{M}}$ be the universal covering of $\mathfrak{M}$. Then the pullback $\mathcal{X} \to \widetilde{\mathfrak{M}}$ of the family $\mathcal{Z} \to \mathfrak{M}$ is a family of marked $K3$ surfaces. Thus we have the following.

Theorem 6.59. *There exist a complex manifold $\mathcal{X}$, $\widetilde{\mathfrak{M}}$, and a holomorphic map $\pi\colon \mathcal{X} \to \widetilde{\mathfrak{M}}$ satisfying the following conditions*:

(1) *For $t \in \widetilde{\mathfrak{M}}$, the fiber $\pi^{-1}(t)$ is a marked polarized $K3$ surface of degree $2d$.*

(2) *Any marked polarized $K3$ surface of degree $2d$ appears as a fiber of π.*

(3) $\dim \widetilde{\mathfrak{M}} = 19$ *and the period map $\tilde{\lambda}_{2d}\colon \widetilde{\mathfrak{M}} \to \Omega_{2d}$ associated with π is locally isomorphic.*

It is enough to prove the injectivity of the period map $\tilde{\lambda}_{2d}$. Density of the periods of Kummer surfaces (Corollary 6.38) also holds in this situation. In fact, Lemma 6.35 is key in the proof of Theorem 6.36, and in its proof we use the fact that $U \oplus U$ is a direct factor of L. This property holds in the case L_{2d}. Hence it follows from the local isomorphism of $\tilde{\lambda}_{2d}$ and the Torelli-type theorem for Kummer surfaces (Theorem 6.32) that $\tilde{\lambda}_{2d}$ is injective over a dense subset in Ω_{2d}. This and the next lemma imply the injectivity of $\tilde{\lambda}_{2d}$.

Lemma 6.60. *Let U, V be complex manifolds and let $f\colon U \to V$ be a locally isomorphic holomorphic map. Suppose that Z is a dense subset in V and the inverse image $f^{-1}(z)$ of each point $z \in Z$ is one point. Then f is injective.*

Exercise 6.61. Prove Lemma 6.60.

7

Surjectivity of the period map of $K3$ surfaces

We introduce a proof of the surjectivity of the period map of $K3$ surfaces. For a marked $K3$ surface (X, α_X), its holomorphic 2-form ω_X and a Kähler class κ_X determine a 3-dimensional positive definite subspace $E(\omega_X, \kappa_X)$ in $L \otimes \mathbb{R}$. Then for any decomposition $E(\omega_X, \kappa_X) = E \oplus \mathbb{R}\kappa$ of $E(\omega_X, \kappa_X)$, there exists a marked $K3$ surface Y such that κ is a Kähler class of Y and E coincides with the subspace $E(\omega_Y)$ determined by the period ω_Y of Y. The proof depends on the Calabi conjecture solved by S. T. Yau. This fact and the density of periods of $K3$ surfaces imply the surjectivity. Finally, we introduce an outline of the proof of the surjectivity of the period map of projective $K3$ surfaces.

7.1 The period map of marked Kähler $K3$ surfaces

For $\omega \in \Omega$, we define

$$H^{1,1}_\omega = \{x \in L \otimes \mathbb{R} : \langle \omega, x \rangle = 0\}, \quad \Delta_\omega = \{\delta \in L : \langle \delta, \delta \rangle = -2, \langle \delta, \omega \rangle = 0\}.$$

Let P^+_ω be a connected component of $\{x \in H^{1,1}_\omega : \langle x, x \rangle > 0\}$, W_ω the reflection group generated by reflections $\{s_\delta : \delta \in \Delta_\omega\}$, and $H_\delta = \{x \in P^+_\omega : \langle x, \delta \rangle = 0\}$. Recall that each connected component of

$$P^+_\omega \setminus \bigcup_{\delta \in \Delta_\omega} H_\delta$$

is a fundamental domain of W_ω with respect to the action on P^+_ω (Theorem 2.9).

Definition 7.1. A triplet (X, α_X, κ_X) is said to be a *marked Kähler $K3$ surface* where X is a $K3$ surface, $\alpha_X : H^2(X, \mathbb{Z}) \to L$ is an isomorphism of lattices and a Kähler class $\kappa_X \in D(X)$ of X. We denote by $\widetilde{\mathcal{M}}$ the set of isomorphism classes of marked Kähler $K3$ surfaces. Any marked Kähler $K3$ surface (X, α_X, κ_X) defines

$$(\alpha_X(\omega_X), \alpha_X(\kappa_X)) \in \widetilde{\Omega}^\circ.$$

Here we recall that

$$\widetilde{\Omega}^\circ = \left\{(\omega, \kappa) \in \widetilde{\Omega} : \langle r, \kappa \rangle \neq 0 \text{ for any } r \in \Delta_\omega\right\}$$

(see (6.20)). Thus we have obtained the *period map for marked Kähler $K3$ surfaces*

$$\tilde{\lambda}\colon \widetilde{\mathcal{M}} \to \widetilde{\Omega}^{\circ} \tag{7.1}$$

as a refinement of the period map λ given in Definition 6.8.

The injectivity of $\tilde{\lambda}$ follows from the Torelli-type theorem for $K3$ surfaces (Theorem 6.1). To prove the surjectivity of λ, it suffices to prove the surjectivity of the period map $\tilde{\lambda}$.

7.2 Surjectivity of the period map of $K3$ surfaces

Recall that for $(\omega, \kappa) \in \widetilde{\Omega}^{\circ}$, we denote by $E(\omega)$ the subspace of $L \otimes \mathbb{R}$ generated by $\mathrm{Re}(\omega)$, $\mathrm{Im}(\omega)$. It follows from (4.2) that $E(\omega)$ is a positive definite 2-dimensional subspace. Since κ is perpendicular to ω, $E(\omega) \oplus \mathbb{R}\kappa$ defines a positive definite 3-dimensional subspace $E(\omega, \kappa)$ in $L \otimes \mathbb{R}$ and $\{\mathrm{Re}(\omega), \mathrm{Im}(\omega), \kappa\}$ is its orthogonal basis. Let $G_3^+(L \otimes \mathbb{R})$ be the set of all 3-dimensional oriented positive definite subspaces in $L \otimes \mathbb{R}$. Then we obtain a map

$$\pi\colon \widetilde{\Omega}^{\circ} \to G_3^+(L \otimes \mathbb{R}), \quad (\omega, \kappa) \to E(\omega, \kappa). \tag{7.2}$$

Now let (X, α_X, κ_X) be a marked Kähler $K3$ surface and define

$$E(\omega_X, \kappa_X) = \pi(\alpha_X(\omega_X), \alpha_X(\kappa_X)) \in G_3^+(L \otimes \mathbb{R}).$$

If $\omega \in E(\omega_X, \kappa_X) \otimes \mathbb{C}$ is a vector satisfying $\langle \omega, \omega \rangle = 0$, $\langle \omega, \bar{\omega} \rangle > 0$, then $E(\omega)$ is positive definite and $\omega \in \Omega$. Assume that the orthogonal complement of $E(\omega)$ in $E(\omega_X, \kappa_X)$ is generated by κ. Since $E(\omega_X, \kappa_X)$ corresponds to the period of a marked Kähler $K3$ surface and a Kähler class, there are no vectors δ in L of norm -2 perpendicular to $E(\omega_X, \kappa_X)$. This implies that $(\omega, \kappa) \in \widetilde{\Omega}^{\circ}$. Then

$$(\omega, \kappa) \in \mathrm{Im}(\tilde{\lambda}),$$

that is, the following theorem holds.

Theorem 7.2. *For $(\omega, \kappa) \in \widetilde{\Omega}^{\circ}$, if $(\omega, \kappa) \in \mathrm{Im}(\tilde{\lambda})$, then we have $\pi^{-1}(\pi(\omega, \kappa)) \subset \mathrm{Im}(\tilde{\lambda})$. In other words, the image of $\tilde{\lambda}$ is the union of fibers of π given in formula (7.2).*

The proof of this theorem depends on a result by Yau on the Calabi conjecture. For the proof we refer the reader to Todorov [To], Beauville [Be3], Barth, Hulek, Peters, Van de Ven [BHPV], Namikawa [Na1].

By using Theorem 7.2, we will complete the proof of the surjectivity. We divide it into three steps.

Lemma 7.3. *Let* $(\omega, \kappa) \in \widetilde{\Omega}^\circ$. *If* $E(\omega, \kappa) \cap L$ *contains a primitive sublattice* M *of rank* 2 *satisfying*

$$\langle x, x \rangle \equiv 0 \bmod 4 \quad (\forall\, x \in M), \tag{7.3}$$

then $(\omega, \kappa) \in \mathrm{Im}(\tilde{\lambda})$.

Proof. By Theorem 7.2 it suffices to prove the assertion under the assumption $M \subset E(\omega)$. Then it follows from Theorem 6.33 that there exists a marked $K3$ surface (X, α_X) with $\alpha_X(\omega_X) = \omega$. If necessary by replacing α_X by the composition $w \circ \alpha_X$ with some w in $W(X)$, we may assume that $\alpha_X^{-1}(\kappa) \in D(X)$. An element in the Kähler cone $D(X)$ defined over $\mathbb{Q}$ is an ample class and hence a Kähler class. In our case, since $E(\omega)$ is defined over $\mathbb{Q}$, $H^{1,1}(X, \mathbb{R})$ is defined over $\mathbb{Q}$ and hence the set of elements in $D(X)$ defined over $\mathbb{Q}$ is dense. Therefore the set of Kähler classes is dense in the Kähler cone. On the other hand, the set of Kähler classes is a convex set because their constant multiple and their sum are Kähler classes. Thus any element in $D(X)$ is a Kähler class and hence $\alpha_X^{-1}(\kappa)$ is. Hence we have obtained $(\omega, \kappa) \in \mathrm{Im}(\tilde{\lambda})$. □

Lemma 7.4. *Let* $(\omega, \kappa) \in \widetilde{\Omega}^\circ$. *If* $E(\omega, \kappa) \cap L$ *contains a primitive element* x *satisfying*

$$\langle x, x \rangle \equiv 0 \bmod 4, \tag{7.4}$$

then $(\omega, \kappa) \in \mathrm{Im}(\tilde{\lambda})$.

Proof. By Theorem 7.2 it suffices to prove the assertion under the assumption $x \in E(\omega)$. Let $D \subset P_\omega^+$ be a fundamental domain of W_ω and let $\kappa \in D$. By Remark 6.37, there is a dense subset of $L \otimes \mathbb{R}$ consisting of elements $y \in L$ such that $M = \mathbb{Z}x + \mathbb{Z}y$ is a primitive sublattice of L of rank 2 satisfying assumption (7.3). If $y \in E(\omega)$, then the assertion follows from Lemma 7.3. Now assume $y \notin E(\omega)$. Then if we denote by η the projection of y into D, $E(\omega, \eta)$ contains M and moreover such η exist in D densely. It follows from Lemma 7.3 that there exists a marked Kähler $K3$ surface $(X_\eta, \alpha_{X_\eta}, \kappa_\eta)$ satisfying

$$\tilde{\lambda}(X_\eta, \alpha_{X_\eta}, \kappa_\eta) = (\omega, \eta).$$

It follows from the Torelli-type theorem of $K3$ surfaces (Corollary 6.2) that X_η is independent of η and isomorphic to a $K3$ surface X. Therefore $(\{\omega\} \times D) \cap \mathrm{Im}(\tilde{\lambda})$ are dense and a convex cone in $\{\omega\} \times D$, and hence we have obtained $\{\omega\} \times D \subset \mathrm{Im}(\tilde{\lambda})$. □

Theorem 7.5. $\tilde{\lambda}$ *is surjective.*

Proof. Let $(\omega, \kappa) \in \widetilde{\Omega}^\circ$. Let $D \subset P_\omega^+$ be a fundamental domain of W_ω with $\kappa \in D$. If $E(\omega)$ contains a primitive element x satisfying assumption (7.4), then the assertion follows from Lemma 7.4. Otherwise, it follows from Lemma 6.34 that there exist primitive elements x of L satisfying assumption (7.4) in $D \times E(\omega)$ densely. Let η

be the projection of x into D. Then $x \in E(\omega,\eta)$ and such η exist in D densely. It follows from Lemma 7.4 that $(\omega,\eta) \in \mathrm{Im}(\tilde{\lambda})$. Hence $(\{\omega\} \times D) \cap \mathrm{Im}(\tilde{\lambda})$ is dense and a convex cone in $\{\omega\} \times D$, and we have thus obtained $\{\omega\} \times D \subset \mathrm{Im}(\tilde{\lambda})$. □

Remark 7.6. The surjectivity of the period map (Theorem 7.5) is due to Todorov [To]. For the above proof we referred to Looijenga [Lo].

Remark 7.7. For each point ω in the period domain Ω_{2d} of polarized $K3$ surfaces of degree $2d$, it follows from Theorem 7.5 that there exists a marked polarized $K3$ surface (X, H, α_X) with $\alpha_X(\omega_X) = \omega$. As mentioned before (see Theorem 6.12), $H = \alpha_X^{-1}(h)$ is not necessarily ample. Thus it is necessary to allow the projective model of X by the linear system $|mH|$ with rational double points.

7.3 Outline of a proof of the surjectivity of the period map of projective $K3$ surfaces

In the following we introduce a sketch of a proof for the surjectivity of the period map of projective $K3$ surfaces. First of all, we prepare a theory of degenerations of $K3$ surfaces. Let

$$\Delta = \{t \in \mathbb{C} : |t| < \varepsilon\}, \quad \Delta^* = \{t \in \mathbb{C} : 0 < |t| < \varepsilon\}.$$

A holomorphic map

$$\pi : \mathcal{X} \to \Delta$$

is called a *semi-stable degeneration* of $K3$ surfaces if the following conditions are satisfied:

(1) $\mathcal{X}$ is a Kähler manifold and π is a proper and flat holomorphic map.

(2) X_t $(\forall\, t \in \Delta^*)$ is a non-singular $K3$ surface.

(3) Let $X_0 = \sum_{i=1}^{k} S_i$ be the irreducible decomposition of X_0. Then S_i is a reduced and non-singular surface, and S_i and S_j meet transversely if $i \neq j$.

In this situation the following theorem is essential.

Theorem 7.8. *Let $\pi : \mathcal{X} \to \Delta$ be a semi-stable degeneration of $K3$ surfaces. Then the following commutative diagram exists:*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\varphi} & \mathcal{X}' \\ \downarrow \pi & & \downarrow \pi' \\ \Delta & = & \Delta, \end{array}$$

where $\pi'\colon \mathcal{X}' \to \Delta$ *is a semi-stable degeneration of* $K3$ *surfaces and the canonical line bundle* $K_{\mathcal{X}'}$ *is trivial, and* φ *is a bimeromorphic map which is isomorphic over* Δ^*.

In the following we assume that a semi-stable degeneration $\pi\colon \mathcal{X} \to \Delta$ has trivial $K_{\mathcal{X}}$. It follows from $K_{\mathcal{X}} = 0$ and the adjunction formula that $\omega_{X_0} = (K_{\mathcal{X}} + X_0)|X_0 = 0$. Here ω_{X_0} is the dualizing sheaf of X_0. Let $X_0 = \sum_{i=1}^{k} S_i$ be the irreducible decomposition. Again by the adjunction formula, we have

$$K_{S_i} = S_i|S_i = -\sum_{j\neq i} S_j|S_i = -\sum_{j\neq i} D_{ij}.$$

Here $D_{ij} = S_i \cap S_j$ is a non-singular curve by the assumption. Since X_0 is connected, S_i is a ruled surface if X_0 is reducible. We define the dual graph Σ of X_0 as follows. A vertex of the graph (resp. an edge, a face) represents an irreducible component of X_0 (resp. an irreducible curve, a point at which 3 irreducible components meet). When two irreducible components meet, the corresponding two vertices are joined by the edge corresponding to the irreducible curve appearing as the intersection of two components. We denote the obtained graph by Σ. On the other hand, it is known that there exists a representation

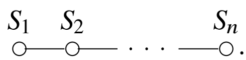
$$\phi\colon \pi_1(\Delta^*) \to \mathrm{GL}(H^2(X_t, \mathbb{Z}))$$

of the fundamental group called the monodromy representation. For a generator γ of $\pi_1(\Delta^*)$, we set $T = \phi(\gamma)$, $N = \log(T) = (T - I) - (T - I)^2/2$. Then it is known that N is a nilpotent matrix (Griffiths [G, §4]).

Theorem 7.9. X_0 *is one of the following*:

(I) X_0 *is a non-singular* $K3$ *surface and* $N = 0$.

(II) X_0 *decomposes as* $X_0 = S_1 + S_2 + \cdots + S_n$ $(n \geq 2)$. *Here* S_i *meets exactly* S_{i-1}, S_{i+1} *and* $E_i = S_{i-1} \cap S_i$ *is a non-singular elliptic curve except that* S_1, S_n *meet only* S_2, S_{n-1} *respectively. Moreover,* S_1, S_n *are rational surfaces and other* S_i *are ruled surfaces over the elliptic curve* E_i. *The dual graph* Σ *is given as*



In this case $N \neq 0$, $N^2 = 0$.

(III) *In the decomposition* $X_0 = \sum_{i=1}^{k} S_i$, *all irreducible components* S_i *are rational surfaces and the dual graph is a simplicial decomposition of a sphere. In this case* $N^2 \neq 0$, $N^3 = 0$.

Now we return to the proof of the surjectivity. The problem is to prove the surjectivity of the period map mentioned in Theorem 6.59. Let

$$\mathcal{Z} \subset \mathbb{P}^n \times \mathfrak{M} \to \mathfrak{M} \tag{7.5}$$

be a family of non-singular $K3$ surfaces of degree $2d$ given in (6.21). Here $\mathfrak{M}$ is a quasi-projective manifold. The period map

$$\lambda_{2d} \colon \mathfrak{M} \to \Omega_{2d}/\Gamma_{2d} \tag{7.6}$$

associated with the family (7.5) is induced from $\tilde{\lambda}_{2d}$ (this λ_{2d} is different from the λ_{2d} given in Section 6.1, which is the one defined on the quotient $\mathcal{M}_{2d}$ of $\mathfrak{M}$ by the projective transformation group $\mathrm{PGL}(n,\mathbb{C})$). Since λ_{2d} is locally isomorphic (Theorem 6.59), $\lambda_{2d}(\mathfrak{M})$ is an open and dense subset of Ω_{2d}/Γ_{2d}. Let

$$\begin{array}{ccc} \mathcal{Z} & \subset & \overline{\mathcal{Z}} \\ \downarrow & & \downarrow \\ \mathfrak{M} & \subset & \overline{\mathfrak{M}} \end{array}$$

be a compactification of the family $\mathcal{Z} \to \mathfrak{M}$. Here $\overline{\mathcal{Z}}$, $\overline{\mathfrak{M}}$ is a projective variety and we may assume that $\overline{\mathcal{Z}} \setminus \mathcal{Z}$ and $\overline{\mathfrak{M}} \setminus \mathfrak{M}$ are normal crossing divisors by Hironaka's resolution theorem. Consider the Baily–Borel compactification $\overline{\Omega_{2d}/\Gamma_{2d}}$ of Ω_{2d}/Γ_{2d}. It follows from the property (Remark 5.6) of the compactification mentioned in Section 5.1.2 that the period map λ_{2d} can be extended to a holomorphic map

$$\bar{\lambda}_{2d} \colon \overline{\mathfrak{M}} \to \overline{\Omega_{2d}/\Gamma_{2d}}.$$

Both $\overline{\mathfrak{M}}$ and $\overline{\Omega_{2d}/\Gamma_{2d}}$ are compact and $\lambda_{2d}(\mathfrak{M})$ is dense in $\overline{\Omega_{2d}/\Gamma_{2d}}$, and hence $\bar{\lambda}_{2d}$ is surjective. We denote by $[\omega]$ the image of a point $\omega \in \Omega_{2d}$ to Ω_{2d}/Γ_{2d}. Let x in $\lambda_{2d}^{-1}([\omega])$ and take a non-singular curve $C \subset \overline{\mathfrak{M}}$ passing through x and satisfying $C \cap \mathfrak{M} \neq \emptyset$. By restricting $\bar{\lambda}_{2d}$ to C, we obtain a holomorphic map

$$\bar{\lambda}_C \colon C \to \overline{\Omega_{2d}/\Gamma_{2d}}.$$

Let $\Delta = \{t \in \mathbb{C} : |t| < \varepsilon\} \subset C$ be a neighborhood of x. We assume that x is given by $t = 0$. We now have a family

$$\pi \colon \mathcal{X} \to \Delta$$

induced from $\overline{\mathcal{Z}} \to \overline{\mathfrak{M}}$. We choose Δ such that any fiber $X_t = \pi^{-1}(t)$ ($\forall\, t \in \Delta^* = \{t \in \mathbb{C} : 0 < |t| < \varepsilon\}$) is a non-singular $K3$ surface. By Hironaka's theory of resolution of singularities, we may assume that X_0 is a normal crossing divisor. Moreover, by Mumford's semi-stable reduction theorem (Kempf, Knudsen, Mumford, Saint-Donat

[KKMS]), if necessary by taking a base change $\Delta' \to \Delta$, $s \to t = s^m$, we may assume that $\pi\colon \mathcal{X} \to \Delta$ is a semi-stable degeneration of $K3$ surfaces. By restricting the period map $\bar{\lambda}_C$, we have a holomorphic map

$$\lambda_{\Delta^*}\colon \Delta^* \to \Omega_{2d}/\Gamma_{2d}.$$

Since $\bar{\lambda}_C(t) = [\omega] \in \Omega_{2d}/\Gamma_{2d}$, λ_{Δ^*} can be extended to a holomorphic map

$$\lambda_{\Delta}\colon \Delta \to \Omega_{2d}/\Gamma_{2d}.$$

It is known that if λ_{Δ^*} can be extended to a map from Δ to Ω_{2d}/Γ_{2d}, then T is of finite order (Griffiths [G, Thm. 4.11 of Remark]). Now by Theorem 7.8, we can change the family $\pi\colon \mathcal{X} \to \Delta$, without changing the monodromy, to the one satisfying $K_{\mathcal{X}} = 0$. Then Theorem 7.9 implies that X_0 is non-singular and hence ω is the period of a non-singular $K3$ surface. Finally, it is shown that $\alpha_{X_0}^{-1}(h)$ is represented by a divisor H_0 on X_0 such that (X_0, H_0) is a polarized $K3$ surface in the sense of Theorem 6.12. Thus we have finished the outline of the proof of the surjectivity.

Remark 7.10. Theorems 7.8, 7.9, and the proof of the surjectivity are due to Kulikov [Ku1], [Ku2]. Kulikov assumed the projectivity of π in Theorem 7.8. Later Persson, Pinkham [PP] gave a proof of this theorem without the assumption on the projectivity.

Remark 7.11. In Section 3.3 we discussed the singular fibers of elliptic surfaces. A singular fiber of type I_n $(n \geq 0)$ is the case that the fiber is reduced and a normal crossing. The case of type I_0 corresponds to case (I) in Theorem 7.9, and that of type I_n $(n \geq 2)$ corresponds to cases (II), (III).

Remark 7.12. The variety Ω_{2d}/Γ_{2d} is called the moduli space of polarized $K3$ surfaces of degree $2d$ which corresponds to $H^+/\mathrm{SL}(2, \mathbb{Z})$ in the case of elliptic curves. It is known that this space is *unirational* for small d, in particular, its Kodaira dimension is $-\infty$ (Mukai [Muk2]). Here an algebraic variety is said to be unirational if there exists a dominant rational map from a projective space to this variety. On the other hand, it has recently been proved that if the degree is sufficiently large, the Kodaira dimension coincides with the dimension of the moduli space, that is, the moduli space is of *general type* (Gritsenko, Hulek, Sankaran [GHS]). In the latter case, the proof depends on the theory of automorphic forms on bounded symmetric domains of type IV (Borcherds [Bor2], [Bor4]).

8

Application of the Torelli-type theorem to automorphisms

As an application we study automorphisms of $K3$ surfaces. First of all, we give a description of the automorphism group of a projective $K3$ surface in terms of the Néron–Severi lattice and its reflection group. Next we show that the action of any automorphism of a projective $K3$ surface on the transcendental lattice is finite and cyclic. As a corollary we show that the automorphism group of a general non-singular quartic surface is trivial. Moreover, we give a necessary and sufficient condition that a finite group can act on a projective $K3$ surface as an automorphism. Finally, we introduce the classification of automorphisms of $K3$ surfaces of order 2.

8.1 Automorphism group of a projective $K3$ surface

As an application of the Torelli-type theorem we study the structure of the automorphism group $\mathrm{Aut}(X)$ of a projective $K3$ surface X. Let S_X be the Néron–Severi lattice of X, $\mathrm{O}(S_X)$ the orthogonal group, and $A(X)$ the ample cone (see Remark 4.19 for the ample cone). We define

$$\mathrm{Aut}(A(X)) = \{\phi \in \mathrm{O}(S_X) : \phi(A(X)) = A(X)\}.$$

Then by Corollary 2.16 we have

$$\mathrm{Aut}(A(X)) \cong \mathrm{O}(S_X)/\{\pm 1\} \cdot W(X).$$

It follows from the Torelli-type theorem that $\mathrm{Aut}(X)$ is a subgroup of the orthogonal group of $H^2(X, \mathbb{Z})$ preserving the period and the ample cone. We have a homomorphism $\mathrm{Aut}(X) \to \mathrm{O}(S_X)$ because $\mathrm{Aut}(X)$ preserves the period. Moreover, it preserves the ample cone and hence we obtain a homomorphism

$$\rho\colon\ \mathrm{Aut}(X) \to \mathrm{Aut}(A(X)). \tag{8.1}$$

Theorem 8.1. *The kernel and the cokernel of ρ are finite.*

Proof. We may assume that X is embedded in a projective space $\mathbb{P}^N$ and let H be a hyperplane section. The kernel $\mathrm{Ker}(\rho)$ acts on S_X trivially and hence fixes H.

Therefore $\mathrm{Ker}(\rho)$ is an algebraic group of projective transformations of $\mathbb{P}^N$ and is also a discrete group (see the proof of Theorem 6.56), and thus it is a finite group. Next consider

$$G = \mathrm{Ker}\{\mathrm{O}(S_X) \to \mathrm{O}(q_{S_X})\} \cap \mathrm{Aut}(A(X)).$$

Since $\mathrm{O}(q_{S_X})$ is a finite group, G is a subgroup of $\mathrm{Aut}(A(X))$ of finite index. Since G is a subgroup of $\mathrm{O}(S_X)$ acting trivially on $A_{S_X} = S_X^*/S_X$, if we define the action of G on T_X as the identity, then the action of G can be extended to that on $H^2(X,\mathbb{Z})$ by Corollary 1.33. By definition, G preserves a holomorphic 2-form $\omega_X \in T_X \otimes \mathbb{C}$ and the ample cone. Therefore it follows from the Torelli-type theorem for $K3$ surfaces (Theorem 6.1) that G is realized as automorphisms of X. □

Corollary 8.2. *The automorphism group* $\mathrm{Aut}(X)$ *is finite if and only if* $W(X)$ *is of finite index in* $\mathrm{O}(S_X)$.

Exercise 8.3. Give an example of a Néron–Severi lattice S_X such that $\mathrm{Aut}(X)$ is finite.

Remark 8.4. Theorem 8.1 is due to Piatetskii-Shapiro, Shafarevich [PS]. In [PS, Sect. 7] they give an example, due to F. Severi, of a $K3$ surface whose automorphism group is infinite.

On the other hand, the classification of Néron–Severi lattices of $K3$ surfaces X with finite automorphism group $\mathrm{Aut}(X)$ is given by Nikulin [Ni5], [Ni7]. However, the case of rank 2 is given by Piatetskii-Shapiro, Shafarevich [PS, §7] and the case of rank 4 is due to Vinberg [V2].

8.2 Action of the automorphism group on the transcendental lattice

Let X be a $K3$ surface and let ω_X be a non-zero holomorphic 2-form on X. Here we do not assume that X is projective. Since any automorphism of X preserves holomorphic 2-forms, we have a non-zero constant $\gamma(g) \in \mathbb{C}^*$ by

$$g^*(\omega_X) = \gamma(g) \cdot \omega_X \tag{8.2}$$

for each $g \in \mathrm{Aut}(X)$. Thus we have a homomorphism

$$\gamma\colon \ \mathrm{Aut}(X) \to \mathbb{C}^*. \tag{8.3}$$

Definition 8.5. An automorphism $g \in \mathrm{Aut}(X)$ is called *symplectic* if $g \in \mathrm{Ker}(\gamma)$. A subgroup G of $\mathrm{Aut}(X)$ is called symplectic if any element of G is symplectic.

Exercise 8.6. Consider the Kummer surface in Example 4.24 and the structure of an elliptic fibration

$$\pi\colon \operatorname{Km}(E \times F) \to \mathbb{P}^1.$$

Recall that π has 4 sections. Fix one section s_0 and consider the intersection of s_0 and a non-singular fiber as the origin of this elliptic curve. Then another section s defines a translation of the elliptic curve which induces a birational automorphism t of the Kummer surface. Show that t is a symplectic automorphism.

A symplectic automorphism and a non-symplectic one have different properties. For example, the following hold.

Proposition 8.7. *Let G be a finite group of automorphisms of a $K3$ surface X:*

(1) *If G is symplectic, then the minimal model of the quotient surface X/G is a $K3$ surface.*

(2) *If G is not symplectic, then X/G is a rational or an Enriques surface.*

Proof. (1) Let Y be the non-singular minimal model of X/G. For $x \in X$, denote by G_x the stabilizer subgroup of x. If G is symplectic, then the action of G_x on the tangent space at x is isomorphic to the natural action of a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$ (e.g., the proof of Theorem 6.56). As mentioned in Remark 4.22, the singularities of X/G are rational double points. It follows from the property of rational double points (Remark 4.22) that a G-invariant holomorphic 2-form on X induces a nowhere-vanishing holomorphic 2-form on Y and hence K_Y is trivial. The condition $b_1(X) = 0$ implies $b_1(Y) = 0$, and hence Y is a $K3$ surface.

(2) In this case, G-invariant holomorphic 2-forms are only 0. Therefore if Y is the minimal non-singular model of X/G, then $p_g(Y) = 0$. Since a $K3$ surface is Kähler, Y is Kähler, and the assertion now follows from the classification of surfaces (Section 3.2). □

Example 8.8. In the proof of Lemma 6.23 the double covering (rational map) $Y \to X$ between $K3$ surfaces is obtained by composing the quotient map of a symplectic automorphism of order 2 and the resolution of singularities. On the other hand, in the case of the $K3$ surface obtained as the double covering of $\mathbb{P}^2$ branched along a smooth sextic curve (Example 4.1), its covering transformation is a non-symplectic automorphism of order 2.

In the following we study the properties of non-symplectic automorphisms in the case that X is projective. Let X be a projective $K3$ surface. The automorphism group $\mathrm{Aut}(X)$ preserves the Néron–Severi lattice S_X and hence acts on its orthogonal complement T_X the transcendental lattice. Now we study the action of $\mathrm{Aut}(X)$ on T_X. Let r be the rank of S_X. Since X is projective, the signature of S_X is given by

$$(1, r-1) \quad (1 \le r \le 20)$$

(Proposition 4.11). It follows that the signature of T_X is $(2, 20-r)$. Let ω_X be a non-zero holomorphic 2-form on X. Then $E(\omega_X) = \langle \mathrm{Re}(\omega_X), \mathrm{Im}(\omega_X)\rangle$ is a 2-dimensional positive definite subspace of $T_X \otimes \mathbb{R}$ (see (4.2)). Since the signature of $T_X \otimes \mathbb{R}$ is $(2, 20-r)$, $E(\omega_X)$ and the orthogonal complement $E(\omega_X)^\perp$ in $T_X \otimes \mathbb{R}$ are negative definite. The definiteness of $E(\omega_X)$ and $E(\omega_X)^\perp$ implies that their orthogonal groups $\mathrm{O}(E(\omega_X))$, $\mathrm{O}(E(\omega_X^\perp))$ are compact. Denote by $\mathrm{Aut}(X)|T_X$ the restriction of $\mathrm{Aut}(X)$ to T_X. Since $\mathrm{Aut}(X)|T_X$ preserves $E(\omega_X)$ and $E(\omega_X)^\perp$, $\mathrm{Aut}(X)|T_X \subset \mathrm{O}(E(\omega_X)) \times \mathrm{O}(E(\omega_X^\perp))$. On the other hand, $\mathrm{O}(T_X)$ is a discrete subgroup of $\mathrm{O}(T_X \otimes \mathbb{R})$. Therefore $\mathrm{Aut}(X)|T_X$ is contained in the intersection of a discrete set and a compact set, and hence is a finite group. Thus the image of the homomorphism γ given in (8.3) is a finite group. In particular, $\gamma(g)$ is a root of unity for $g \in \mathrm{Aut}(X)$. Since the image of γ is a subgroup of a multiplicative group $\mathbb{C}^*$, it is a cyclic group.

Exercise 8.9. Show that a finite subgroup of $\mathbb{C}^*$ is cyclic.

Exercise 8.10. Let $x = (x_1, \ldots, x_6)$ be homogeneous coordinates of $\mathbb{P}^5$ and let

$$X = \left\{ x \in \mathbb{P}^5 : \textstyle\sum_{i=1}^6 x_i = \sum_{i=1}^6 x_i^2 = \sum_{i=1}^6 x_i^3 = 0 \right\}.$$

Show that

(1) X is a $K3$ surface;

(2) the symmetric group $\mathfrak{S}_6$ of permutations of coordinates acts on X, show that its subgroup $\mathfrak{A}_6$, the alternating group, acts on X symplectically.

Lemma 8.11. *Any $g \in \mathrm{Ker}(\gamma)$ acts on T_X trivially.*

Proof. For $x \in T_X$, the equations

$$\langle x, \omega_X \rangle = \langle g^*(x), g^*(\omega_X)\rangle = \langle g^*(x), \omega_X\rangle$$

imply that $x - g^*(x) \in (\omega_X)^\perp \cap H^2(X, \mathbb{Z}) = S_X$. Since $x \in T_X$, we have $x - g^*(x) \in T_X \cap S_X$. Since S_X is non-degenerate (Proposition 4.11), we obtain $S_X \cap T_X = 0$ and hence $g^*(x) = x$. □

Lemma 8.12. *Suppose that $g^*|T_X \neq 1$ for $g \in \mathrm{Aut}(X)$. Then g^* does not fix any non-zero elements of $T_X \otimes \mathbb{Q}$.*

Proof. Let $x \in T_X \otimes \mathbb{Q}$ with $g^*(x) = x$. Then

$$\langle x, \omega_X \rangle = \langle g^*(x), g^*(\omega_X) \rangle = \langle x, \gamma(g) \cdot \omega_X \rangle,$$

and the assumption $g|T_X \neq 1$ implies that $\gamma(g) \neq 1$ (Lemma 8.11), and hence $\langle x, \omega_X \rangle = 0$, that is, $x \in S_X \otimes \mathbb{Q} \cap T_X \otimes \mathbb{Q} = \{0\}$. Thus we have $x = 0$. □

Corollary 8.13. *The group* $\mathrm{Aut}(X)|T_X$ *is a finite cyclic group. Let* m *be its order. Then the representation*

$$\mathrm{Aut}(X) \to \mathrm{O}(T_X \otimes \mathbb{Q})$$

defined over $\mathbb{Q}$ *is a direct sum of irreducible representations of degree* $\varphi(m)$*. Here* φ *is the Euler function. In particular,* $\varphi(m)$ *is a divisor of the rank of* T_X*.*

Proof. It follows from Lemma 8.11 that $\mathrm{Aut}(X)|T_X$ is isomorphic to the image of γ, and hence is a finite cyclic group. Let g^* be its generator. Then $(g^*)^k$ $(k < m)$ does not fix any non-zero elements of $T_X \otimes \mathbb{Q}$ (Lemma 8.12). Therefore all eigenvalues of $g^*|T_X$ are primitive mth roots of unity. Moreover, g^* is defined over $\mathbb{Q}$, and hence any conjugates of primitive mth roots appear as eigenvalues. Now the assertion follows. □

Corollary 8.14. *If* $\mathrm{Aut}(X)|T_X$ *has order* m*, then* $m \leq 66$.

Proof. Since the rank of T_X is at most 21, the assertion follows from Corollary 8.13. □

Example 8.15. The elliptic surface given by

$$y^2 = x^3 + t^{12} - t, \quad g(x, y, t) = (\zeta_3 \zeta_{11}^4 x, -\zeta_{11}^6 y, \zeta_{11} t)$$

is a non-singular $K3$ surface and the automorphism g is an example of the equality $m = 66$ in Corollary 8.14. Here ζ_k is a primitive kth root of unity.

Example 8.16. Let $X \subset \mathbb{P}^3$ be a non-singular quartic surface and S_X its Néron–Severi lattice. Then we have

$$1 \leq \mathrm{rank}(S_X) \leq 20.$$

Now we assume that $\mathrm{rank}(S_X) = 1$ and show that the automorphism group $\mathrm{Aut}(X)$ is trivial. Let $h \in S_X$ be the cohomology class of a hyperplane section. Then $\langle h, h \rangle = 4$ and hence h is primitive and S_X is a positive definite lattice generated by h. Any $g \in \mathrm{Aut}(X)$ preserves S_X, and hence $g^*|S_X = 1$ or -1. If $g^*|S_X = -1$, then $g^*(h) = -h$, which is a contradiction. Hence $g^*|S_X = 1$. It now follows from Lemma 8.11 that $\mathrm{Ker}(\gamma)$ acts on $S_X \oplus T_X$ trivially. Since $S_X \oplus T_X$ is of finite index in $H^2(X, \mathbb{Z})$, $\mathrm{Ker}(\gamma)$ acts trivially on $H^2(X, \mathbb{Z})$ and hence we have $\mathrm{Ker}(\gamma) = 1$ by

Theorem 6.56. Thus we conclude that $\mathrm{Aut}(X)$ is a finite cyclic group. Let m be the order of $\mathrm{Aut}(X)$. Then it follows from Corollary 8.13 that $\varphi(m)$ is a divisor of the rank of T_X. The assumption $\mathrm{rank}(S_X) = 1$ implies $\mathrm{rank}(T_X) = 21$. If $m > 2$, then $\varphi(m)$ is an even integer, which is a contradiction. Hence we have $m \leq 2$. If $m = 2$, then a generator g of $\mathrm{Aut}(X)$ satisfies $g^*|S_X = 1$, $g^*|T_X = -1$. On the other hand, it follows from Corollary 1.33 and $g^*|S_X = 1$ that g^* acts on $T_X^*/T_X \cong S_X^*/S_X$ trivially. Lemma 1.45 implies that any element in $H^2(X,\mathbb{Z})$ of norm 4 is unique up to the action of the orthogonal group $\mathrm{O}(H^2(X,\mathbb{Z}))$. Hence we have

$$T_X \cong U \oplus U \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle. \tag{8.4}$$

Let u be a generator of the component $\langle -4 \rangle$ of the decomposition (8.4). Then $\frac{1}{4}u \bmod T_X$ is a generator of T_X^*/T_X. This implies that -1_{T_X} cannot act trivially on T_X^*/T_X. Hence $m = 1$. Thus we have the following.

Proposition 8.17. *Let X be a non-singular quartic surface whose Néron–Severi lattice has rank* 1. *Then* $\mathrm{Aut}(X) = \{1\}$.

Remark 8.18. By using the same argument as above we can determine the automorphism group of a projective $K3$ surface with Picard number 1 as follows:

(1) If $S_X \cong \langle 2m \rangle$ $(m \neq 1)$, then $\mathrm{Aut}(X) = \{1\}$.

(2) If $S_X \cong \langle 2 \rangle$, then $\mathrm{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z}$.

Exercise 8.19. In the case $S_X \cong \langle 2 \rangle$, prove $\mathrm{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 8.20. If a non-singular quartic surface is special, then its automorphism group might be large. For example, the group $(\mathbb{Z}/4\mathbb{Z})^3 \cdot \mathfrak{S}_4$ acts on the Fermat quartic surface

$$F_4 = \left\{x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0\right\}$$

as projective transformations. Here $\mathfrak{S}_4$ is the projective transformation induced from the linear transformation of $\mathbb{C}^4$,

$$(x_1, x_2, x_3, x_4) \longrightarrow (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}), \quad \sigma \in \mathfrak{S}_4, \tag{8.5}$$

and $(\mathbb{Z}/4\mathbb{Z})^3$ is the one induced from

$$(x_1, x_2, x_3, x_4) \longrightarrow \left(x_1, \sqrt{-1}^{\,a} \cdot x_2, \sqrt{-1}^{\,b} \cdot x_3, \sqrt{-1}^{\,c} \cdot x_4\right), \quad (a, b, c \in \mathbb{Z}). \tag{8.6}$$

The surface F_4 contains 48 lines. For example, if we denote by ζ_8 a primitive 8th root of 1, then

$$\ell_1 : x_1 = \zeta_8 x_2, \ \ x_3 = \zeta_8 x_4, \quad \ell_2 : x_1 = \zeta_8^3 x_2, \ \ x_3 = \zeta_8^3 x_4$$

define skew lines ℓ_1, ℓ_2 on F_4.

Exercise 8.21. Show that the Fermat quartic surface F_4 contains 48 lines.

Exercise 8.22. Let $X \subset \mathbb{P}^3$ be a non-singular quartic surface containing two skew lines ℓ_1, ℓ_2. Let $p \in X$ be a point not lying on ℓ_1, ℓ_2. Then show that there exists a unique line passing p and meeting both lines ℓ_1, ℓ_2.

Let ℓ be the line in Exercise 8.22 and let $\ell \cap X = \{\ell_1 \cap \ell, \ell_2 \cap \ell, p, q\}$. Associating q with p we have an automorphism of X of order 2. This automorphism is not induced from any projective transformation. Thus the Fermat quartic surface F_4 has an automorphism not induced from any projective transformation of $\mathbb{P}^3$. Moreover, it is known that the automorphism group of F_4 is infinite.

Remark 8.23. The results in this section are mainly due to Nikulin [Ni3]. Nikulin classified finite abelian groups of symplectic automorphisms of $K3$ surfaces and then Mukai [Muk1] gave the classification in the general case. Moreover, Mukai discovered that the character of the action of a finite symplectic automorphism on the cohomology ring $H^*(X, \mathbb{Q})$ coincides with that of the action of the Mathieu group M_{23}, one of the sporadic finite simple groups, on the set of 24 points, and proved that finite groups of symplectic automorphisms of $K3$ surfaces are subgroups of M_{23} satisfying some condition.[1]

8.3 A finite group that can be realized as an automorphism group of a $K3$ surface

In this section we give a necessary and sufficient condition for which a finite group G can act on a projective $K3$ surface as automorphisms, which will be used later. First, assume that G acts on the lattice $H^2(X, \mathbb{Z})$ effectively. For simplicity we denote $H^2(X, \mathbb{Z})$ by L. We define sublattices L^G, L_G of L as

$$L^G = \{x \in L : g(x) = x \; \forall g \in G\}, \quad L_G = (L^G)^\perp. \tag{8.7}$$

Now suppose that G acts on X as automorphisms. For an ample class h, $\sum_{g \in G} g^*(h)$ is a G-invariant ample class. Therefore $L^G \cap P^+(X) \neq \emptyset$. If G is symplectic, then it acts on the transcendental lattice T_X trivially (Lemma 8.11) and hence L^G is a lattice of signature $(3, n)$ and L_G is a negative definite lattice. On the other hand, if G is non-symplectic, then G does not fix non-zero elements in T_X, and hence L^G is a lattice of signature $(1, n)$ and L_G is one of signature $(2, 19 - n)$.

[1] Added in English translation: See Theorem 11.14.

Lemma 8.24. *Suppose that X is projective and let $G \subset \mathrm{O}(L)$ be a finite subgroup. Then there exists a $w \in W(X)$ such that the conjugate $w^{-1} \circ G \circ w$ acts on X as automorphisms if and only if*

(1) *G preserves holomorphic 2-forms on X,*

(2) *$L^G \cap P^+(X) \neq \{0\}$,*

(3) *$L_G \cap S_X$ contains no elements of norm -2.*

Proof. Assume that G acts on X as automorphisms. Then condition (1) holds obviously. And as mentioned above, condition (2) also holds. Assume that $L_G \cap S_X$ contains an element δ of norm -2. Then

$$\sum_{g \in G} g^*(\delta) \in L_G \cap L^G.$$

By the Riemann–Roch theorem we may assume that δ is effective (Lemma 4.16). Since any automorphism preserves effective divisors, $\sum_{g\in G} g^*(\delta)$ is a G-invariant effective divisor. On the other hand, as mentioned above, if G acts on X as automorphisms, then both L^G, L_G are non-degenerate; in particular $L^G \cap L_G = \{0\}$. This contradicts the fact that $\sum_{g\in G} g^*(\delta) \neq 0$.

Conversely, suppose that conditions (1), (2), (3) hold. Since $W(X)$ acts trivially on holomorphic 2-forms on X (Remark 4.18), the conjugate of G by any element of $W(X)$ satisfies condition (1). Therefore it suffices to prove that there exists $w \in W(X)$ such that $w^{-1} \circ G \circ w$ preserves the Kähler cone, and then the assertion follows from the Torelli-type theorem. Condition (2) implies that $L^G \cap P^+(X) \neq \{0\}$, and condition (3) implies that $L^G \cap P^+(X)$ is not contained in any face of $W(X)$. In fact, if it is contained in a face, then there exists a $\delta \in \Delta(X)$ satisfying $L^G \subset \delta^\perp$. This implies $\delta \in L_G$, which is a contradiction. Now it follows from Theorem 2.9 that there exists a $w \in W(X)$ satisfying $w^{-1}(L^G) \cap D(X) \neq \{0\}$. Then, for any $x \in L^G$, $(w^{-1} \circ G \circ w)(w^{-1}(x)) = w^{-1}(x)$ and hence $w^{-1} \circ G \circ w$ preserves the Kähler cone. □

The reference for this section is Namikawa [Na2].

8.4 Automorphisms of $K3$ surfaces of order 2

Finally, we mention automorphisms of $K3$ surfaces of order 2. In the following we do not assume that a $K3$ surface X is projective. Let g be an automorphism of X of order 2. Then

$$g^*(\omega_X) = \pm\omega_X.$$

If $g^*(\omega_X) = \omega_X$, then g is symplectic. Assume that g has a fixed point $p \in X$. Since g is of finite order, the action of g on the tangent space $T_p(X)$ of p can be diagonalized as

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ if } g^*(\omega_X) = \omega_X, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ if } g^*(\omega_X) = -\omega_X$$

(see the proof of Theorem 6.56). Thus we have the following.

Lemma 8.25. *Assume that $g \in \mathrm{Aut}(X)$ is of order 2 and has a fixed point. If g is symplectic, then any fixed point is isolated, and if non-symplectic, then the fixed point set is a non-singular curve.*

Exercise 8.26. Let g be a finite symplectic automorphism of a $K3$ surface X. Prove that g has a fixed point on X.

Automorphisms of order 2 are completely classified as follows. We state only the result, without its proof. Let X be a $K3$ surface and let $g \in \mathrm{Aut}(X)$ be an automorphism of order 2. Define

$$L_{\pm} = \{x \in H^2(X, \mathbb{Z}) : g^*(x) = \pm x\}.$$

Then $L_{\pm}$ is a 2-elementary lattice.

Exercise 8.27. Show that $L_{\pm}$ is a 2-elementary lattice.

Proposition 8.28. *Let g be a symplectic automorphism of order 2:*

(1) *g has exactly 8 isolated fixed points on X.*

(2) *L_- is isomorphic to $E_8(2)$.*

Proposition 8.29. *Let g be a non-symplectic automorphism of order 2 and let F be the set of fixed points of g. Let $(r = t_+ + t_-, \ell, \delta)$ be the invariants of the 2-elementary lattice L_+ (Proposition 1.39). Then the following hold:*

(1) *$F = \emptyset$. In this case, $(r, \ell, \delta) = (10, 10, 0)$ and L_+ is isomorphic to $U(2) \oplus E_8(2)$.*

(2) *F is a disjoint union of two non-singular elliptic curves. In this case, $(r, \ell, \delta) = (10, 8, 0)$ and L_+ is isomorphic to $U \oplus E_8(2)$.*

(3) *$F = C + E_1 + \cdots + E_k$. Here C is a non-singular curve of genus g, $E_1, \ldots, E_k$ are non-singular rational curves, and (r, ℓ) satisfies*

$$g = \frac{22 - r - \ell}{2}, \quad k = \frac{r - \ell}{2}.$$

Moreover, the cohomology class of F is divisible by 2 in L_+ if and only if $\delta = 0$.

Remark 8.30. We refer the reader to Nikulin [Ni3] for the proof of Proposition 8.28[2] and Nikulin [Ni5] for Proposition 8.29.[3]

[2]Added in English translation: See Theorem 11.12.

[3]Added in English translation: There are many papers concerning a generalization of Proposition 8.29 to higher order. Here we add only Artebani, Sarti, Taki [AST] as a reference.

9

Enriques surfaces

As an application of the Torelli-type theorem for $K3$ surfaces and the surjectivity of the period map, we give the Torelli-type theorem for Enriques surfaces and the surjectivity of the period map. Moreover, we discuss non-singular rational curves and elliptic curves on Enriques surfaces, and give a description of the automorphism group of a generic Enriques surface. Finally, we introduce concrete examples of Enriques surfaces.

9.1 Periods of Enriques surfaces

9.1.1 Enriques surfaces and their covering $K3$ surfaces. Let Y be an Enriques surface. As mentioned in Section 3.2, Y is a surface satisfying $p_g(Y) = q(Y) = 0$ and $K_Y^{\otimes 2}$ is trivial. Thus $c_1(Y)^2 = 0$. By Noether's formula

$$c_1(Y)^2 + c_2(Y) = 12(p_g(Y) - q(Y) + 1),$$

we have $c_2(Y) = 12$. It follows from Theorem 3.5 that

$$h^{1,0}(Y) = h^{0,1}(Y) = h^{2,0}(Y) = h^{0,2}(Y) = 0, \quad h^{1,1}(Y) = 10.$$

Moreover, by Theorem 3.5, $b^+(Y) = 1$ and hence Enriques surfaces are algebraic. Let C be an irreducible curve on Y. Then it follows from the adjunction formula (Theorem 3.3) and the Riemann–Roch theorem (Theorem 3.1) that

$$C^2 = 2p_a(C) - 2, \quad \dim H^0(Y, \mathcal{O}(C)) \geq \tfrac{1}{2}C^2 + 1 = p_a(C).$$

In particular, $C^2 \geq -2$, and the equality $C^2 = -2$ holds if and only if C is a non-singular rational curve. Note that C^2 is even and hence Y is minimal. Now we conclude the following.

Proposition 9.1. *An Enriques surface is a minimal algebraic surface.*

Remark 9.2. Contrary to the case of $K3$ surfaces, it might happen that both divisors D and $-D$ of norm -2 are non-effective.

Proposition 9.3. *Let Y be an Enriques surface. Then the fundamental group of Y is $\mathbb{Z}/2\mathbb{Z}$ and the universal covering of Y is a $K3$ surface. Conversely, let X be a $K3$ surface with a fixed-point-free automorphism σ of order 2. Then the quotient surface $X/\langle\sigma\rangle$ is an Enriques surface.*

Proof. There exists an unramified double covering of Y,

$$\pi\colon X \to Y,$$

corresponding to the torsion element K_Y of order 2. Then $e(X) = 2e(Y) = 24$ and $K_X = \pi^*(K_Y^{\otimes 2})$ is trivial. It follows from Noether's formula that $q(X) = 0$, and hence X is a $K3$ surface. A $K3$ surface is simply connected (Corollary 6.40) and hence the first half of the lemma is proved.

Conversely, let X be a $K3$ surface, and σ an automorphism of order 2 with no fixed point. Then $Y = X/\langle\sigma\rangle$ is a non-singular minimal surface. Since $\pi^*(K_Y) = K_X = 0$, we have $K_Y^{\otimes 2} = 0$. Since $K3$ surfaces are Kähler, Y is a Kähler minimal surface with Kodaira dimension 0. Such surfaces are abelian surfaces, bielliptic surfaces, $K3$ surfaces, Enriques surfaces, and their invariants are given by $(p_g, q) = (1,2)$, $(0,1)$, $(1,0)$, $(0,0)$ respectively (Section 3.2). In the above case, it follows from Noether's formula that $2c_1(Y)^2 = c_1(X)^2 = 0$ and $2c_2(Y) = c_2(X) = 24$, and hence $p_g(Y) - q(Y) + 1 = 1$. Thus Y is an Enriques surface. □

Example 9.4. Let X be the intersection of three quadrics in $\mathbb{P}^5$,

$$X\colon Q_{i,1}(x_0, x_1, x_2) + Q_{i,2}(x_3, x_4, x_5) = 0 \quad (i = 1, 2, 3).$$

Here $(x_0, x_1, x_2, x_3, x_4, x_5)$ are homogeneous coordinates of $\mathbb{P}^5$ and $Q_{i,j}$ is a homogeneous polynomial of degree 2 in 3 variables. We assume that X is non-singular. Then X is a $K3$ surface (Example 4.1). Let σ be the projective transformation of $\mathbb{P}^5$ of order 2 given by

$$\sigma\colon (x_0, x_1, x_2, x_3, x_4, x_5) \to (x_0, x_1, x_2, -x_3, -x_4, -x_5).$$

Then the fixed point set of σ is the union of two planes $\{x_0 = x_1 = x_2 = 0\}$, $\{x_3 = x_4 = x_5 = 0\}$. Now we assume that

$$\{Q_{1,i} = 0\} \cap \{Q_{2,i} = 0\} \cap \{Q_{3,i} = 0\} = \emptyset \quad (i = 1, 2).$$

Then σ induces an automorphism, denoted by the same σ, of X of order 2 with no fixed points, and the quotient surface $Y = X/\langle\sigma\rangle$ is an Enriques surface. The three quadrics given in (4.6) satisfy the above assumption. Thus the Kummer surface $\mathrm{Km}(C)$ associated with a curve C of genus 2 is the covering $K3$ surface of an Enriques surface.

Exercise 9.5. What is the dimension of the family of Enriques surfaces given in Example 9.4?

Lemma 9.6. *Let Y be an Enriques surface. Then the following holds*:

$$\mathrm{Pic}(Y) \cong H^2(Y,\mathbb{Z}) \cong \mathbb{Z}^{\oplus 10} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Proof. The fact $c_2(Y) = 12$, $q(Y) = 0$ implies that $b_2(Y) = 10$. It follows from Proposition 9.3 that $H_1(Y,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Therefore $H^2(Y,\mathbb{Z}) \cong \mathbb{Z}^{10} \oplus \mathbb{Z}/2\mathbb{Z}$. By the exact sequence of the cohomology (3.2) and $p_g(Y) = q(Y) = 0$, we obtain $\mathrm{Pic}(Y) \cong H^2(Y,\mathbb{Z})$. □

Proposition 9.7. *Let Y be an Enriques surface. Then the following hold*:

$$\dim H^0(Y,T_Y) = \dim H^2(Y,T_Y) = 0, \quad \dim H^1(Y,T_Y) = 10.$$

Proof. First, note that a holomorphic vector field on Y induces that of the universal covering $\pi\colon X \to Y$ of Y. Hence Lemma 5.17 implies $H^0(Y,T_Y) = 0$. On the other hand, it follows from Serre duality that $H^2(Y,T_Y) \cong H^0(Y,K_Y \otimes \Omega^1_Y)$. If this is not zero, then we have $H^0(X,\Omega^1_X) = H^0(X,\pi^*(K_Y \otimes \Omega^1_Y)) \neq 0$, which contradicts $q(X) = 0$. Finally, by applying the Riemann–Roch theorem to the vector bundle T_Y of rank 2 (e.g., Barth, Hulek, Peters, Van de Ven [BHPV, Chap. I, Sect. 5]), we have

$$\sum_i (-1)^i \dim H^i(Y,T_Y) = 2(p_g(Y) - q(Y) + 1) + c_1(Y)^2 - c_2(Y)$$

and hence $H^1(Y,T_Y) = 10$. □

Combining Theorems 5.14 and 5.15, we have the following.

Corollary 9.8. *An Enriques surface has a* 10*-dimensional complete deformation family.*

Definition 9.9. Let Y be an Enriques surface. We denote by $H^2(Y,\mathbb{Z})_f$ the quotient of $H^2(Y,\mathbb{Z})$ by the torsion subgroup.

Lemma 9.10. *Let Y be an Enriques surface. Then $H^2(Y,\mathbb{Z})_f$ with intersection form has the structure of a lattice isomorphic to $U \oplus E_8$.*

Proof. By Proposition 9.1 and Lemma 9.6, $H^2(Y,\mathbb{Z})$ is generated by classes of irreducible curves. For any irreducible curve C, we have $C^2 = C^2 + C\cdot K_Y = 2g(C) - 2$, and hence the lattice is even. By Poincaré duality, it is unimodular, and by $b^+(Y) = 1$ it has signature $(1,9)$. Therefore it follows from Theorem 1.27 that it is isomorphic to $U \oplus E_8$. □

Let Y be an Enriques surface and let $\pi\colon X \to Y$ be the unramified double covering. Then X is a $K3$ surface (Proposition 9.3). Let $L_X = H^2(X,\mathbb{Z})$. Then $L_X \cong U^{\oplus 3} \oplus E_8^{\oplus 2}$ (Theorem 4.5). Denote by σ the covering transformation of π and define

$$L_X^+ = \{x \in L_X : \sigma^*(x) = x\}, \quad L_X^- = \{x \in L_X : \sigma^*(x) = -x\}. \tag{9.1}$$

Then L_X^+ and L_X^- are orthogonal complements to each other in L_X.

Lemma 9.11. $L_X^+ \cong U(2) \oplus E_8(2)$, $L_X^- \cong U \oplus U(2) \oplus E_8(2)$.

Proof. For $y, y' \in H^2(Y,\mathbb{Z})$, we have $\langle \pi^*(y), \pi^*(y')\rangle = 2\langle y, y'\rangle$, and by Lemma 9.10, we obtain

$$L_X^+ \cong U(2) \oplus E_8(2).$$

Obviously,

$$A_{L_X^+} \cong (\mathbb{Z}/2\mathbb{Z})^{10}.$$

Lemma 1.31 implies $A_{L_X^-} \cong (\mathbb{Z}/2\mathbb{Z})^{10}$. On the other hand, $\mathrm{rank}(L_X^-) = 12$ and hence, by Proposition 1.37 the isomorphism class of L_X^- is determined by $q_{L_X^-}$. Now it follows from the isomorphisms

$$q_{U\oplus U(2)\oplus E_8(2)} \cong -q_{U(2)\oplus E_8(2)} = -q_{L_X^+} \cong q_{L_X^-}$$

that $L_X^- \cong U \oplus U(2) \oplus E_8(2)$. □

Remark 9.12. The involution σ is case (1) in Proposition 8.29.

Definition 9.13. Let L be an even unimodular lattice of signature $(3,19)$. Fix an orthogonal decomposition

$$L = U \oplus U \oplus U \oplus E_8 \oplus E_8$$

of L, and let x_i be coordinates of the ith U, y_j coordinates of the jth E_8. Define an isomorphism ι of the lattice and its invariant sublattice by

$$\iota(x_1, x_2, x_3, y_1, y_2) = (-x_1, x_3, x_2, y_2, y_1), \tag{9.2}$$

$$L^+ = \{x \in L : \iota^*(x) = x\}, \quad L^- = \{x \in L : \iota^*(x) = -x\}. \tag{9.3}$$

Then we obtain

$$L^+ \cong U(2) \oplus E_8(2), \quad L^- \cong U \oplus U(2) \oplus E_8(2).$$

Lemma 9.14. *The action of σ^* on L_X is conjugate with that of ι on L.*

Proof. We identify L_X and L by fixing an isomorphism between them. Let $\varphi\colon L_X^+ \to L^+$ be an isomorphism. Then the surjectivity of the natural map

$$\mathrm{O}(L^-) \to \mathrm{O}(q_{L^-})$$

(Proposition 1.37) implies that there exists an isomorphism $\psi\colon L_X^- \to L^-$ satisfying condition (2) in Corollary 1.33. Therefore it follows from Corollary 1.33 that (φ, ψ) can be extended to an isomorphism $\tilde{\varphi}\colon L \to L$. Then we obtain $\tilde{\varphi}^{-1} \circ \iota \circ \tilde{\varphi} = \sigma^*$. □

It follows from Exercise 8.27 that $L^\pm$ are even 2-elementary lattices. By Proposition 1.39, the following also holds.

Lemma 9.15. *Any automorphism of L^+ or L^- can be extended to an automorphism of L. That is, the restriction maps*

$$\mathrm{O}(L) \to \mathrm{O}(L^\pm)$$

are surjective.

Definition 9.16. Let $L, L^\pm, \iota$ be the same as in Definition 9.13 and put $M = L^+(1/2) \cong U \oplus E_8$. Let Y be an Enriques surface, $\pi\colon X \to Y$ the unramified double covering, and σ the covering transformation. Take an isomorphism

$$\alpha_Y\colon H^2(Y, \mathbb{Z})_f \to M$$

of lattices. Then α_Y induces an isomorphism

$$\tilde{\alpha}_Y\colon L_X^+ \to L^+$$

of lattices, and then by Lemma 9.15, $\tilde{\alpha}_Y$ can be extended to an isomorphism

$$\alpha_X\colon L_X = H^2(X, \mathbb{Z}) \to L$$

of lattices satisfying $\alpha_X \circ \sigma^* = \iota \circ \alpha_X$. The pair (Y, α_X) of an Enriques surface Y and an isomorphism of lattices obtained as above is called a *marked Enriques surface*.

9.1.2 Periods and the period domain of Enriques surfaces. We define the periods and the period domain of Enriques surfaces. Let (Y, α_X) be a marked Enriques surface and let ω_X be a non-zero holomorphic 2-form on X. The property $p_g(Y) = 0$ implies that there are no non-zero σ-invariant holomorphic 2 forms on X, and hence

$$\sigma^*(\omega_X) = -\omega_X, \tag{9.4}$$

that is, $\omega_X \in L_X^- \otimes \mathbb{C}$. Now defining

$$\Omega(L^-) = \{\omega \in \mathbb{P}(L^- \otimes \mathbb{C}) : \langle \omega, \omega \rangle = 0,\ \langle \omega, \bar{\omega} \rangle > 0\}, \tag{9.5}$$

we have $\alpha_X(\omega_X) \in \Omega(L^-)$. Since L^- has signature $(2, 10)$, $\Omega(L^-)$ is the disjoint union of two bounded symmetric domains of type IV (see Section 5.1). We define

$$\Gamma = \mathrm{O}(L^-). \tag{9.6}$$

Two connected components of $\Omega(L_-)$ are interchanged by an element of Γ. The group Γ acts on $\Omega(L^-)$ properly discontinuously, and hence its quotient $\Omega(L^-)/\Gamma$ has the structure of a complex analytic space and is even quasi-projective (see Section 5.1.2).

Definition 9.17. We call $\alpha_X(\omega_X) \in \Omega(L^-)$ the *period* of a marked Enriques surface and $\Omega(L^-)$ the *period domain*. For an Enriques surface Y,

$$\alpha_X(\omega_X) \bmod \Gamma \in \Omega(L^-)/\Gamma$$

is independent of the choice of a marking. Let $\mathcal{M}_E$ be the set of isomorphism classes of Enriques surfaces. Then we have a map

$$p \colon \mathcal{M}_E \to \Omega(L^-)/\Gamma, \tag{9.7}$$

which is called the *period map*.

Contrary to the case of $K3$ surfaces, some points of $\Omega(L^-)$ do not correspond to the periods of Enriques surfaces. In the following we explain this fact. Note that L^- has an orthogonal direct summand U and hence contains an element δ of norm -2. Consider a point $\omega \in \Omega(L^-)$ perpendicular to δ. It follows from the surjectivity of the period map of $K3$ surfaces that there exists a marked $K3$ surface (X, α_X) satisfying $\alpha_X(\omega_X) = \omega$. The isomorphism

$$\iota_X = \alpha_X^{-1} \circ \iota \circ \alpha_X \colon H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

of lattices preserves holomorphic 2-forms because $\iota_X(\omega_X) = -\omega_X$. On the other hand, if we put $\delta_X = \alpha_X^{-1}(\delta)$, then δ_X is the class of a divisor because $\langle \delta_X, \omega_X \rangle = 0$. We may assume that δ_X is effective because $\delta_X^2 = -2$. Moreover, $\delta \in L_-$ implies that $\iota_X(\delta_X) = -\delta_X$. However, if ι_X is induced from an automorphism of X, then it should preserve effective classes, which is a contradiction. Thus we have the following.

Lemma 9.18. *The period $\alpha_X(\omega_X)$ of a marked Enriques surface is not perpendicular to any element in L^- of norm -2.*

Definition 9.19. For $\delta \in L^-$, $\delta^2 = -2$, we define $\mathcal{H}_\delta = \{\omega \in \Omega(L^-) : \langle \omega, \delta \rangle = 0\}$ and

$$\mathcal{H} = \bigcup_{\delta \in L^-, \, \delta^2 = -2} \mathcal{H}_\delta. \tag{9.8}$$

Then it follows from Lemma 9.18 that the period $\alpha_X(\omega_X)$ of a marked Enriques surface is contained in $\Omega(L^-) \setminus \mathcal{H}$. In particular, we can refine the period map (9.7) as

$$p \colon \mathcal{M}_E \to (\Omega(L^-) \setminus \mathcal{H})/\Gamma. \tag{9.9}$$

The Torelli-type theorem for Enriques surfaces is nothing but the injectivity of p, and the surjectivity of the period map means the surjectivity of (9.9).

9.1.3 The Torelli-type theorem for Enriques surfaces. We prepare to prove the Torelli-type theorem for Enriques surfaces.

Definition 9.20. Let Y be an Enriques surface. Recall that the signature of $H^2(Y, \mathbb{R})$ is $(1, 9)$, and hence

$$P(Y) = \{x \in H^2(Y, \mathbb{R}) : \langle x, x \rangle > 0\}$$

has two connected components. We denote by $P^+(Y)$ the one containing an ample class. Define

$$\Delta(Y)^+ = \left\{\delta \in H^2(Y, \mathbb{Z}) : \delta \text{ is the class of an effective divisor with } \delta^2 = -2\right\}.$$

To each δ in $\Delta(Y)^+$, we associate the reflection

$$s_\delta(x) = x + \langle x, \delta \rangle \delta, \quad x \in H^2(Y, \mathbb{Z})$$

of $H^2(Y, \mathbb{Z})$. Denote by $W(Y)$ the group generated by reflections $\{s_\delta : \delta \in \Delta(Y)^+\}$ and define

$$D(Y) = \{x \in P(Y)^+ : \langle x, \delta \rangle > 0 \, \forall \, \delta \in \Delta(Y)^+\}.$$

Then $D(Y)$ is a fundamental domain of $W(Y)$ with respect to the action on $P^+(Y)$ (Theorem 2.9). By Nakai's criterion for ampleness and the Schwarz inequality, we can see that $D(Y) \cap H^2(Y, \mathbb{Z})$ is nothing but the set of ample classes on Y.

Remark 9.21. Recall that in the case of $K3$ surfaces we denote by $\Delta(X)$ the set of elements of norm -2 in S_X and then have the decomposition $\Delta(X) = \Delta(X)^+ \cup \Delta(X)^-$ (Definition 4.15). As mentioned in Remark 9.2, in the case of Enriques surfaces both $\pm\delta$ of norm -2 might be not effective, and hence we need to add the effectivity in the definition of $\Delta(Y)^+$. As proved later in Section 9.2.1, it happens that $\Delta(Y)^+ = \emptyset$. In the case of $K3$ surfaces, the Néron–Severi lattice depends on a $K3$ surface X, but effective divisors of norm -2 (or non-singular rational curves) are determined by the Néron–Severi lattice. On the other hand, the Néron–Severi lattices of Enriques surfaces are isomorphic, but effective divisors of norm -2 (or non-singular rational curves) depend on an Enriques surface Y.

Let $\pi\colon X \to Y$ be the universal covering of an Enriques surface Y and let σ be the covering transformation. Let $D(X)$ be the Kähler cone of the $K3$ surface X. Then the following holds:

Lemma 9.22. $\pi^*(D(Y)) = L_X^+ \otimes \mathbb{R} \cap D(X)$.

Proof. Obviously, the right-hand side is contained in the left-hand side. Let $x = \pi^*(y) \in \pi^*(D(Y))$. It suffices to prove that $\langle x, \tilde{\delta} \rangle > 0$ for the cohomology class $\tilde{\delta}$ of any non-singular rational curve on X. Note that any non-trivial automorphism of a non-singular rational curve has a fixed point. On the other hand, σ has no fixed points and hence $\sigma^*(\tilde{\delta})$ and $\tilde{\delta}$ are classes of different irreducible curves. Therefore we have $\langle \sigma^*(\tilde{\delta}), \tilde{\delta} \rangle \geq 0$. If $\langle \sigma^*(\tilde{\delta}), \tilde{\delta} \rangle = 0$, then there exists a $\delta \in \Delta(Y)^+$ with $\pi^*(\delta) = \sigma^*(\tilde{\delta}) + \tilde{\delta}$. In this case, $\langle x, \tilde{\delta} \rangle = \langle \sigma^*(x), \sigma^*(\tilde{\delta}) \rangle = \langle x, \sigma^*(\tilde{\delta}) \rangle$ and hence

$$2\langle x, \tilde{\delta} \rangle = \langle x, \tilde{\delta} + \sigma^*(\tilde{\delta}) \rangle = \langle \pi^*(y), \pi^*(\delta) \rangle = 2\langle y, \delta \rangle > 0.$$

In the case that $\langle \sigma^*(\tilde{\delta}), \tilde{\delta} \rangle > 0$, $\langle \sigma^*(\tilde{\delta}), \tilde{\delta} \rangle$ is an even number because σ has no fixed points. This implies that $(\tilde{\delta} + \sigma^*(\tilde{\delta}))^2 \geq 0$, and it now follows from Lemma 2.3 that $\langle x, \tilde{\delta} + \sigma^*(\tilde{\delta}) \rangle > 0$. □

Theorem 9.23 (Torelli-type theorem for Enriques surfaces). *Let Y, Y' be Enriques surfaces and X, X' the covering $K3$ surfaces, respectively. Let*

$$\phi\colon H^2(Y, \mathbb{Z}) \to H^2(Y', \mathbb{Z})$$

be an isomorphism of lattices satisfying the following:

(a) *ϕ can be extended to an isomorphism $\tilde{\phi}\colon H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ of lattices satisfying $\tilde{\phi}(\omega_X) \in \mathbb{C}\omega_{X'}$.*

(b) *$\phi(D(Y)) = D(Y')$.*

Then there exists an isomorphism $\varphi\colon Y' \to Y$ of complex manifolds with $\varphi^* = \phi$.

Proof. It follows from Lemma 9.22 and Theorem 6.1 that there exists an isomorphism $\tilde{\varphi}\colon X' \to X$ with $\tilde{\varphi}^* = \tilde{\phi}$. Let σ be the covering transformation of the covering $X \to Y$ and let σ' be that of $X' \to Y'$. Then $\tilde{\phi} \circ \sigma^* = (\sigma')^* \circ \tilde{\phi}$ and hence $\tilde{\varphi} \circ \sigma' = \sigma \circ \tilde{\varphi}$ by the Torelli-type theorem for $K3$ surfaces. Therefore $\tilde{\varphi}$ induces an isomorphism $\varphi\colon Y' \to Y$ with $\varphi^* = \phi$. □

Remark 9.24. There exist Enriques surfaces with an automorphism acting trivially on $H^2(Y,\mathbb{Z})$ (see Remark 9.39). Thus it may happen that the uniqueness of φ does not hold.

Corollary 9.25. *Let Y, Y' be Enriques surfaces and let X, X' be the covering $K3$ surfaces, respectively. Suppose that there is an isomorphism $\phi\colon H^2(Y,\mathbb{Z}) \to H^2(Y',\mathbb{Z})$ of lattices which can be extended to an isomorphism $\tilde{\phi}\colon H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ of lattices with $\tilde{\phi}(\omega_X) \in \mathbb{C}\omega_{X'}$. Then Y and Y' are isomorphic. In particular, the period map given in* (9.9) *is injective.*

9.1.4 Surjectivity of the period map for Enriques surfaces.

Next we show the surjectivity of the period map.

Theorem 9.26 (Surjectivity of the period map of Enriques surfaces). *Let $\omega \in \Omega(L^-)\setminus \mathcal{H}$. Then there exists a marked Enriques surface (Y,α_X) with $\alpha_X(\omega_X) = \omega$.*

Proof. It follows from the surjectivity of the period map of $K3$ surfaces (Theorem 7.5) that there exists a marked $K3$ surface (X,α_X) with $\alpha_X(\omega_X) = \omega$. Since $\omega \notin \mathcal{H}$, an isomorphism

$$\iota_X = \alpha_X^{-1} \circ \iota \circ \alpha_X\colon H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$$

of lattices satisfies the assumption in Lemma 8.24. Therefore there exists a $w \in W(X)$ such that $w^{-1} \circ \iota_X \circ w$ can be represented by an automorphism σ of X. If we prove that σ has no fixed points, then $Y = X/\langle\sigma\rangle$ is the desired Enriques surface.

Now assume that σ has a fixed point p. By definition, σ is a non-symplectic automorphism of order 2, and hence the set of its fixed points is a non-singular curve. Assume that it is a disjoint union $C_1 + \cdots + C_n$ of non-singular irreducible curves $C_1,\dots,C_n$. Let $Y = X/\langle\sigma\rangle$. Since the set of fixed points is non-singular, Y is a non-singular surface. The natural map $\pi\colon X \to Y$ is a double covering branched along the image $\bar{C}_i$ of C_i and Y is a rational surface because $\bar{C}_1 + \cdots + \bar{C}_n \in |-2K_Y|$. By $2\bar{C}_i^2 = \pi^*(\bar{C}_i)^2 = (2C_i)^2 = 4C_i^2$, $\bar{C}_i^2$ is an even integer. Moreover, Y is a minimal surface. In fact, if there is a non-singular rational curve $C \subset Y$ with $C^2 = -1$, then $C \neq \bar{C}_i$ and $\pi^*(C)^2 = 2C^2 = -2$. This contradicts the fact that $L_X^+ \cong U(2) \oplus E_8(2)$ contains no (-2)-elements. Thus Y is minimal and hence its Euler number $e(Y)$ is 3 or 4. On the other hand, by the Lefschetz fixed point formula (e.g., see Ueno [U]) we have

$$\sum_{i=1}^{n} e(C_i) = \sum_{k=0}^{4} (-1)^k \operatorname{trace}(\sigma^*|H^k(X,\mathbb{Z})) = 2 + 10 - 12 = 0.$$

Since $\pi\colon X \to Y$ is a double covering branched along $\bar{C}_1 + \cdots + \bar{C}_n$, we have

$$24 = e(X) = 2 \cdot e(Y) - \sum_{i=1}^{n} e(C_i).$$

Therefore we have $e(Y) = 12$, which is a contradiction. Thus σ has no fixed points and Y is an Enriques surface. □

Remark 9.27. (i) The Torelli-type theorem and the surjectivity of the period map for Enriques surfaces are due to Horikawa [Ho2]. The reference for the proofs given in this section is Namikawa [Na2].

(ii) One can prove that $\Gamma = \mathrm{O}(L^-)$ acts on the set of elements in L^- of norm -2 transitively. In particular, $\mathcal{H}/\Gamma$ is an irreducible hypersurface in $\Omega(L^-)/\Gamma$ (Namikawa [Na2]).

(iii) The quotient space $(\Omega(L^-) \setminus \mathcal{H})/\Gamma$ is called the moduli space of Enriques surfaces. Here we do not consider a polarization. No algebraic construction of this space is known. However, it is known that $\Omega(L^-)/\Gamma$ is rational, that is, it is birational to $\mathbb{P}^{10}$ (Kondo [Kon2]).

(iv) It is known that there exists an automorphic form Ψ on $\Omega(L^-)$ whose zero divisor (Ψ) coincides with $\mathcal{H}$ (Borcherds [Bor3]). This implies that $(\Omega(L^-) \setminus \mathcal{H})/\Gamma$ is a quasi-affine variety. Here we recall the definition of automorphic forms. Let

$$\Omega(L^-)^* = \{\omega \in L^- \otimes \mathbb{C} : \langle \omega, \omega \rangle = 0,\ \langle \omega, \bar{\omega} \rangle > 0\}.$$

The natural map $\Omega(L^-)^* \to \Omega(L^-)$ is a $\mathbb{C}^*$-bundle. Let $\Gamma \subset \mathrm{O}(L^-)$ be a subgroup of finite index. Then a holomorphic function

$$F \colon \Omega(L^-)^* \to \mathbb{C}$$

is called a (holomorphic) *automorphic form on* $\Omega(L^-)$ *of weight* k *with respect to* Γ if it satisfies the following two conditions:

(1) $F(g(\omega)) = F(\omega)\ \forall\, g \in \Gamma$.

(2) $F(\alpha\omega) = \alpha^{-k} F(\omega)\ \forall\, \alpha \in \mathbb{C}^*$.

In this case, contrary to the case of modular forms of one variable, the holomorphicity at infinity (the cusp) is automatically satisfied. The above example by Borcherds is the first application of automorphic forms on bounded symmetric domains of type IV to algebraic geometry (in the sense of concrete examples). This example is derived from the denominator formula of a generalized Kac–Moody algebra.

9.2 Non-singular rational curves and elliptic curves on Enriques surfaces

9.2.1 Non-singular rational curves on Enriques surfaces. First of all, we will show that a generic Enriques surface contains no non-singular rational curves. Let Y be an Enriques surface, $\pi\colon X \longrightarrow Y$ the covering $K3$ surface, and σ the covering transformation. Let $C \subset Y$ be a non-singular rational curve. Then C is simply connected and hence $\pi^*(C)$ is a disjoint union of two non-singular rational curves C^+, C^- on X. Since $\sigma(C^{\pm}) = C^{\mp}$, we have $C^+ + C^- \in L_X^+$, $C^+ - C^- \in L_X^-$. Since any holomorphic 2-form ω_X is perpendicular to classes of curves, we have $\langle \omega_X, C^+ - C^- \rangle = 0$. Obviously,

$$(C^+ \pm C^-)^2 = -4, \quad \frac{C^+ + C^-}{2} + \frac{C^+ - C^-}{2} = C^+.$$

Under this observation, we consider $\delta^{\pm}$ satisfying

$$\delta^{\pm} \in L^{\pm}, \quad (\delta^{\pm})^2 = -4, \quad \frac{\delta^+ + \delta^-}{2} \in L. \tag{9.10}$$

Let $\mathcal{R}$ be the set of elements δ_- of L_- of norm -4 such that there exists a δ_+ satisfying condition (9.10), and define

$$\mathcal{N}_{\delta_-} = \{\omega \in \Omega(L^-) : \langle \omega, \delta_- \rangle = 0\}, \quad \mathcal{N} = \sum_{\delta_- \in \mathcal{R}} \mathcal{N}_{\delta_-}.$$

If $\langle \delta^-, \omega_X \rangle = 0$, then δ^- is represented by a divisor. We may assume that $\frac{\delta^+ + \delta^-}{2} \in L_X$ is effective because it is of norm -2. Since σ is an automorphism of X, $\sigma(\frac{\delta^+ + \delta^-}{2}) = \frac{\delta^+ - \delta^-}{2}$ is also an effective divisor of norm -2. Therefore their sum δ^+ is effective and thus Y contains an effective divisor of norm -2. If Y contains no non-singular rational curves, then any effective divisor has a non-negative norm. Hence Y contains a non-singular rational curve. Thus we have the following.

Proposition 9.28. *An Enriques surface contains a non-singular rational curve if and only if its period is contained in $\mathcal{N}$.*

Next we introduce the notion of the root invariant which measures how many non-singular rational curves sit on an Enriques surface. Let Y be an Enriques surface, $\pi\colon X \longrightarrow Y$ the covering $K3$ surface and σ the covering transformation. Recall that L_X^- has the signature $(2, 10)$, and $\mathrm{Re}(\omega_X)$, $\mathrm{Im}(\omega_X)$ generate a positive definite 2-dimensional subspace in $L_X^- \otimes \mathbb{R}$. This implies that the sublattice in L_X^- generated by δ^- satisfying (9.10) and perpendicular to ω_X is negative definite. If $\delta_1^{\pm}$, $\delta_2^{\pm}$ satisfy (9.10), then $\langle \delta_1^+ + \delta_1^-, \delta_2^+ + \delta_2^- \rangle \in 4\mathbb{Z}$ and $\langle \delta_1^+, \delta_2^+ \rangle \in 2\mathbb{Z}$, and hence

$$\langle \delta_1^-, \delta_2^- \rangle \in 2\mathbb{Z}.$$

Therefore the sublattice in L_X^- generated by δ^- satisfying (9.10) and perpendicular to ω_X is isomorphic to the lattice $R(2)$ obtained from a root lattice R by multiplying the bilinear form by 2. We define a map

$$d\colon \tfrac{1}{2}R(2)/R(2) \longrightarrow (L_X^+)^*/L_X^+ \tag{9.11}$$

by

$$d\left(\frac{\delta^-}{2} \bmod R(2)\right) = \frac{\delta^+}{2} \bmod L_X^+,$$

and denote $\mathrm{Ker}(d)$ by K. Note that K is a 2-elementary finite abelian group. The pair (R, K) is called the *root invariant* of an Enriques surface. We now consider the meaning of K. Assume that $K \neq 0$ and let $\alpha \in K$, $\alpha \neq 0$. By definition of $R(2)$, there is a $\beta \in (L_X^+)^*$ with $\alpha + \beta \in L_X$. Since $\beta = d(\alpha) = 0$, we have $\beta \in L_X^+$, and hence $\alpha \in L_X^-$. This means the existence of a non-trivial overlattice of $R(2)$ in L_X^- (Theorem 1.19).

Remark 9.29. The root invariant was introduced by Nikulin [Ni6].[1]

9.2.2 Elliptic curves on Enriques surfaces. We will study elliptic curves on Enriques surfaces. First, we recall the case of algebraic $K3$ surfaces.

Theorem 9.30. *Let X be an algebraic $K3$ surface and let F be a primitive effective divisor on X satisfying*

(1) $F^2 = 0$,

(2) $C \cdot F \geq 0$ *for any irreducible curve C.*

Then the complete linear system $|F|$ contains an elliptic curve.

Proof. First of all, we show that the complete linear system $|F|$ contains an irreducible curve. To do this, we consider an element D of $|F|$ such that

$$D = \sum_{i=1}^{r} m_i C_i \quad (m_i > 0,\ r \geq 2,\ C_i \neq C_j,\ i \neq j).$$

Here C_i is an irreducible curve. Since $D^2 = 0$ and $D \cdot C_i \geq 0$, we have $D \cdot C_i = 0$. Moreover, $C_i \cdot C_j \geq 0$ $(i \neq j)$ implies that $C_i^2 \leq 0$. We will show that there exists an element D which has an irreducible component C_i with $C_i^2 = 0$. Assume that X is embedded in a projective space $\mathbb{P}^N$ and let H be the hyperplane section. Since $H \cdot D = \sum m_i H \cdot C_i$, the degree $H \cdot C_i$ of C_i is bounded. Note that the number

[1] Added in English translation: S. Mukai [Muk4] gave a refinement of the notion of root invariants based on a remark by Allcock [All].

of non-singular rational curves with bounded degree is finite. On the other hand, it follows from the Riemann–Roch theorem (Theorem 3.1) that $\dim |F| \geq 1$ and in particular $|F|$ contains infinitely many members. Therefore there exists a member $D \in |F|$ that has an irreducible component with $C_i^2 = 0$. We denote it by

$$D = mE + D' \quad (m \geq 1,\ E^2 = 0,\ E \neq D').$$

Here we assume that E does not appear as an irreducible component of D'. Then by the assumption on F, we have $D^2 = 0$, $D' \cdot E = D \cdot E \geq 0$, $D \cdot D' \geq 0$, and hence

$$2mE \cdot D' + (D')^2 = 0, \quad mE \cdot D' + (D')^2 \geq 0.$$

Thus we have

$$E \cdot D' = (D')^2 = 0.$$

This means that if $D' \neq 0$, then $D' = kE$, which contradicts the assumption on D'. Hence $D' = 0$, that is, $mE \in |F|$. Since F is primitive, we have $m = 1$. The linear system $|E|$ has no base points by $E^2 = 0$, and by Bertini's theorem (e.g., Griffiths, Harris [GH, p. 137]), $|E|$ contains a non-singular curve. It is an elliptic curve by the adjunction formula. □

Exercise 9.31. Let X be a $K3$ surface and let H be an ample divisor with $H^2 = 2d$. Show that for a natural number N, the number of non-singular rational curves C on X with $H \cdot C \leq N$ is finite.

Corollary 9.32. *A $K3$ surface X has the structure of an elliptic fibration if and only if S_X contains the class of a non-zero divisor F with $F^2 = 0$.*

Proof. If $F = mF'$, then we consider F' instead of F. Thus we may assume that F is primitive. It follows from Theorem 2.9 that there exists an element w in $W(X)$ such that $w(F) \in \overline{D}(X)$. Here $\overline{D}(X)$ is the closure of the Kähler cone $D(X)$. Then $w(F)$ satisfies the condition of Theorem 9.30, and hence $|w(F)|$ contains an elliptic curve E. By the Riemann–Roch theorem (Theorem 3.1) we have

$$\dim H^0(X, \mathcal{O}(E)) \geq \tfrac{1}{2}E^2 + p_g(X) - q(X) + 1 = 2.$$

Since $E^2 = 0$ we also have $\mathcal{O}_X(E)|_E = \mathcal{O}_E$. Therefore the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(E) \to \mathcal{O}_X(E)|_E \to 0$$

induces the exact sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(E)) \to H^0(E, \mathcal{O}_E) \to 0,$$

which implies $\dim H^0(X, \mathcal{O}_X(E)) = 2$. Again by $E^2 = 0$, the complete linear system $|E|$ has no base points and induces the structure of an elliptic fibration $\varphi_{|E|} \colon X \to \mathbb{P}^1$. □

By combining with Proposition 1.24 we have the following.

Corollary 9.33. *Any algebraic $K3$ surface with Picard number greater than or equal to 5 has the structure of an elliptic fibration.*

A $K3$ surface with the structure of an elliptic fibration is special. For example, any algebraic $K3$ surface with Picard number 1 has no elliptic fibration.

Now we return to Enriques surfaces. The Néron–Severi group of an Enriques surface has rank 10 and hence it always contains an isotropic element. Thus the following hold.

Proposition 9.34. *Let Y be an Enriques surface*:

(1) *Y has an elliptic fibration.*
(2) *Let $\pi\colon Y \to \mathbb{P}^1$ be an elliptic fibration. Then π has exactly two multiple fibers $2F_1$, $2F_2$ and $K_Y = F_1 - F_2$.*

Proof. Since $H^2(Y,\mathbb{Z})_f \cong U \oplus E_8$ (Lemmas 9.6, 9.10), there exists a primitive divisor F on Y with $F^2 = 0$. If necessary by applying the reflections associated with non-singular rational curves mentioned in Definition 9.20, we may assume that $\langle F, C\rangle \geq 0$ for any irreducible curve C. Let $\pi\colon X \to Y$ be the covering $K3$ surface and σ the covering transformation. Then by Lemma 9.22, $\pi^*(F)$ satisfies the condition in Theorem 9.30. The complete linear system $|\pi^*(F)|$ defines an elliptic fibration $\varphi\colon X \to \mathbb{P}^1$ which induces an elliptic fibration

$$\psi\colon Y \to \mathbb{P}^1$$

on Y because $\pi^*(F)$ is σ-invariant. Thus we have assertion (1). Next we show assertion (2). First, we prove that σ acts on the base $\mathbb{P}^1$ of φ as an automorphism of order 2. Assume that it acts on the base trivially. Then σ preserves each fiber. Since σ has no fixed points, it acts on each fiber as a translation. Since it acts on the base trivially and acts on each fiber as a translation, σ acts on holomorphic 2-forms trivially, which contradicts equation (9.4). Therefore σ acts on the base as an automorphism of order 2. Any automorphism of $\mathbb{P}^1$ of order 2 has two fixed points. Let D_1, D_2 be the fibers over the fixed points. Since $\sigma(D_i) = D_i$ $(i = 1, 2)$, $F_1 = \pi(D_1)$, $F_2 = \pi(D_2)$ are multiple fibers and $|2F_1| = |2F_2|$ defines the elliptic fibration. Since F_1 is not linearly equivalent to F_2, $F_1 - F_2$ is a torsion in $\mathrm{Pic}(Y)$ of order 2 and hence $K_Y = F_1 - F_2$. □

Now let e, f be a basis of the lattice U with $e^2 = f^2 = 0$, $\langle e, f\rangle = 1$, and let $\{r_1, \dots, r_8\}$ be a basis of the root lattice E_8 given in Figure 1.2. Then $\{e, f, r_1, \dots, r_8\}$ is a basis of the lattice $U \oplus E_8$. We now define

$$w_1 = e, \quad w_2 = f, \quad w_i = e + f + r_1 + \cdots + r_{i-2} \quad (3 \leq i \leq 10).$$

We can easily see that $\langle w_i, w_j\rangle = 1 - \delta_{ij}$. Here δ_{ij} is the Kronecker delta.

Corollary 9.35. *Suppose that an Enriques surface Y contains no non-singular rational curve. Then there exist 10 irreducible curves $E_1, \ldots, E_{10}$ satisfying*

$$p_a(E_i) = 1, \quad \langle E_i, E_j \rangle = 1 \quad (i \neq j).$$

Each $|2E_i|$ defines an elliptic fibration on Y.

Proof. The classes $w_1, \ldots, w_{10}$ given above satisfy $\langle w_i, w_j \rangle = 1$ $(i \neq j)$, and hence are primitive. Since Y contains no non-singular rational curves, we have $D(Y) = P^+(Y)$. If necessary by multiplying them by -1, $w_1, \ldots, w_{10}$ are nef divisors. Then the pullbacks of these divisors to the $K3$ surface satisfy the assumption of Theorem 9.30 and hence define elliptic fibrations on the $K3$ surface. Thus we have finished the proof by using the proof of Proposition 9.34. □

Example 9.36. We show the existence of elliptic fibrations on the Enriques surface given in Example 9.4. Consider two families

$$\{t_1 Q_{1,j} + t_2 Q_{2,j} + t_3 Q_{3,j} = 0\}_{(t_1,t_2,t_3)\in\mathbb{P}^2}, \quad j = 1, 2$$

of conics and define

$$\det(t_1 Q_{1,j} + t_2 Q_{2,j} + t_3 Q_{3,j}) = 0. \tag{9.12}$$

Equation (9.12) is the determinant of a matrix of degree 3 with entries that are polynomials of degree 1 in t_1, t_2, t_3, and hence it defines a cubic curve C_j in $\mathbb{P}^2$. The conic corresponding to a point (t_1, t_2, t_3) on the cubic curve is the union of two lines. A point in $C_1 \cap C_2$ corresponds to two conics, both of which decompose into two lines. For simplicity, we assume that $Q_{1,1}$, $Q_{1,2}$ define such conics. Then the quadric hypersurface Q in $\mathbb{P}^5$ given by $Q_{1,1} + Q_{1,2} = 0$ is defined by a symmetric matrix of rank 4, and hence it contains two 1-dimensional families of 3-dimensional subspaces in $\mathbb{P}^5$. For example, if we assume

$$Q_{1,1} + Q_{1,2} = x_0 x_1 + x_3 x_4,$$

then Q contains the family of 3-dimensional subspaces

$$P_{a,b} = \{a x_0 + b x_3 = b x_1 - a x_4 = 0\} \quad ((a, b) \in \mathbb{P}^1).$$

In this case,

$$E_{a,b} = X \cap P_{a,b} = P_{a,b} \cap \{Q_{2,1} + Q_{2,2} = 0\} \cap \{Q_{3,1} + Q_{3,2} = 0\}$$

is the intersection of two quadrics in $\mathbb{P}^3 = P_{a,b}$ and is non-singular if $(a, b) \in \mathbb{P}^1$ is general, and hence it is an elliptic curve by the adjunction formula. Thus the family

$$\{E_{a,b}\}_{(a,b)\in\mathbb{P}^1}$$

gives an elliptic fibration on X. Since σ preserves the family $\{P_{a,b}\}_{(a,b)\in\mathbb{P}^1}$, this fibration induces the structure of an elliptic fibration on the Enriques surface $Y = X/\langle\sigma\rangle$.

Exercise 9.37. Show that the elliptic fibration on Y given above has exactly two multiple fibers.

Remark 9.38. Theorem 9.30 is due to Piatetskii-Shapiro, Shafarevich [PS].

9.3 Automorphism groups of Enriques surfaces

As an application of the Torelli-type theorem for Enriques surfaces, we mention the automorphism groups of Enriques surfaces. In particular we show that a generic Enriques surface has an infinite group of automorphisms. This is a phenomenon peculiar to Enriques surfaces and contrary to the case of $K3$ surfaces. A general algebraic $K3$ surface has Picard number 1 and hence its automorphism group is $\{1\}$ or $\mathbb{Z}/2\mathbb{Z}$, as mentioned in Remark 8.18.

Let Y be an Enriques surface, $\pi\colon X \to Y$ the covering $K3$ surface, and σ the covering transformation. We denote by $\mathrm{Aut}(X)$, $\mathrm{Aut}(Y)$ the automorphism group of X, Y, respectively. Since X is the universal covering of Y, we obtain

$$\mathrm{Aut}(Y) \cong \{g \in \mathrm{Aut}(X) : g\circ\sigma = \sigma\circ g\}/\{1,\sigma\}. \tag{9.13}$$

Consider the action of the automorphism group on the cohomology group

$$\rho\colon\ \mathrm{Aut}(Y) \to \mathrm{O}(H^2(Y,\mathbb{Z})). \tag{9.14}$$

By the same proof as in $K3$ surfaces (Theorem 6.56, Proposition 9.7), $\mathrm{Ker}(\rho)$ is a finite group.

Remark 9.39. In the case of Enriques surfaces, $\mathrm{Ker}(\rho)$ is not necessarily trivial. Moreover, $\mathrm{Ker}(\rho')$ of the map

$$\rho'\colon\ \mathrm{Aut}(Y) \to \mathrm{O}(H^2(Y,\mathbb{Q}))$$

happens to be non-trivial. Such Enriques surfaces and $\mathrm{Ker}(\rho)$, $\mathrm{Ker}(\rho')$ are completely classified (Mukai, Namikawa [MuN], [Muk3]). See Exercise 9.46.

As in the case of $K3$ surfaces (Theorem 8.1), we have the following corollary by the Torelli-type theorem for Enriques surfaces (Theorem 9.23). In this case, we need a small modification because $W(Y)$ is not necessarily a normal subgroup in $\mathrm{O}(H^2(Y,\mathbb{Z})_f)$ (see Dolgachev [Do1]).

Corollary 9.40. *Let* $G(Y)$ *be the subgroup of* $\mathrm{O}(H^2(Y,\mathbb{Z})_f)$ *generated by* $\mathrm{Im}(\rho)$ *and* $W(Y)$*. Then* $W(Y)$ *is a normal subgroup of* $G(Y)$ *and* $G(Y)$ *is a subgroup of finite index in* $\mathrm{O}(H^2(Y,\mathbb{Z})_f)$*.*

In the case of $K3$ surfaces X, the Néron–Severi lattice S_X depends on X, but in the case of Enriques surfaces Y, all $H^2(Y,\mathbb{Z})$ are isomorphic and independent on Y. On the other hand, $W(X) = W(S_X)$ is uniquely determined by S_X, but a divisor on an Enriques surface of self-intersection -2 is not necessarily effective and $W(Y)$ depends on Y.

Corollary 9.41. *The group* $\mathrm{Aut}(Y)$ *is finite if and only if* $[\mathrm{O}(H^2(Y,\mathbb{Z})_f) : W(Y)] < \infty$.

Next we consider Enriques surfaces Y whose covering $K3$ surface has the smallest Néron–Severi lattice S_X, that is, the case of $\mathrm{rank}(S_X) = 10$. It follows from the Torelli-type theorem and the surjectivity of the period map that these Enriques surfaces form a 10-dimensional family. Moreover, $L_X^+ \subset S_X$ and both L_X^+ and S_X are primitive in L_X, and hence $S_X = L_X^+$. Therefore, any element in $\mathrm{Aut}(X)$ commutes with σ. Hence by formula (9.13) we obtain

$$\mathrm{Aut}(Y) \cong \mathrm{Aut}(X)/\{1,\sigma\}.$$

On the other hand, Proposition 9.28 implies that these Enriques surfaces contain no non-singular rational curves, and hence $D(Y) = P^+(Y)$. The restriction $\mathrm{Aut}(X)|T_X$ of $\mathrm{Aut}(X)$ to T_X is a cyclic group by Corollary 8.13. If m is its order, then $\varphi(m)$ is a divisor of $\mathrm{rank}(T_X) = \mathrm{rank}(L_X^-) = 12$. Now we define

$$M = \{m \in \mathbb{Z} : m > 2,\ \varphi(m)|12\}.$$

Let ζ_m be a primitive mth root of unity and let $\mathcal{S}$ be the subset of the period domain $\Omega(L^-)$ of Enriques surfaces defined by

$$\mathcal{S} = \bigcup_{m\in M} \{\omega \in \Omega(L^-) : \omega \text{ is defined over } \mathbb{Q}(\zeta_m)\}.$$

In the following we assume that the period of Y is contained in $\Omega(L^-) \setminus \mathcal{S}$. Then we have $\mathrm{Aut}(X)|T_X = \{1,\sigma\}$ (Corollary 8.13). Thus, to calculate $\mathrm{Aut}(Y)$ it suffices to determine the subgroup of $\mathrm{O}(S_X) = \mathrm{O}(L^+)$ realized as automorphisms of X. Let q_{L^+} be the discriminant quadratic form of the lattice L^+. The natural map $\mathrm{O}(L^+) \to \mathrm{O}(q_{L^+})$ is surjective (Proposition 1.39). If we denote by $\widetilde{\mathrm{O}}(L^+)$ the kernel $\mathrm{Ker}\{\mathrm{O}(L^+) \to \mathrm{O}(q_{L^+})\}$ of this map, then we have $\mathrm{O}(L^+)/\widetilde{\mathrm{O}}(L^+) \cong \mathrm{O}(q_{L^+})$.

Theorem 9.42. *Let* Y *be an Enriques surface,* X *the covering* $K3$ *surface, and* σ *the covering transformation. Suppose that* $S_X = L_X^+$ *and* $\alpha_X(\omega_X) \in \Omega(L^-) \setminus \mathcal{S}$*. Then*

$$\mathrm{Aut}(Y) \cong \widetilde{\mathrm{O}}(L^+)/\{\pm 1\}.$$

Proof. Recall that $\mathrm{Aut}(X)|L_X^- = \{1, \sigma\}$. Since $\sigma^*|L_X^- = -1$ and $(L_X^-)^*/L_X^-$ is a 2-elementary abelian group, σ^* acts on $(L_X^-)^*/L_X^-$ trivially. Since $(L_X^+)^*/L_X^+ \cong (L_X^-)^*/L_X^-$, any automorphism of X acts trivially on $(L_X^+)^*/L_X^+$. Obviously, $-1_{L_X^+}$ sends $P^+(X)$ to $-P^+(X)$, and hence it cannot be realized as automorphisms of X. Since $\mathrm{Aut}(X)/\{1, \sigma\}$ preserves $P^+(X)$, it is a subgroup of $\widetilde{\mathrm{O}}(L_X^+)/\{\pm 1\}$. Hence the map

$$\mathrm{Aut}(X)/\{1, \sigma\} \to \widetilde{\mathrm{O}}(L^+)/\{\pm 1\}$$

is injective.

Conversely, let ϕ be any element in $\widetilde{\mathrm{O}}(L_X^+)$. If necessary by considering $-\phi$, we may assume that ϕ preserves $P^+(X)$. It follows from Corollary 1.33 that $(\phi, 1_{L_X^-}) \in \mathrm{O}(L_X^+) \times \mathrm{O}(L_X^-)$ can be extended to an isomorphism $\tilde{\phi}$ of $H^2(X, \mathbb{Z})$. The condition $\tilde{\phi}|L_X^- = 1$ implies that $\tilde{\phi}$ preserves holomorphic 2-forms. Since $P^+(Y) = D(Y)$, $\tilde{\phi}$ also preserves the Kähler cone. Therefore it now follows from the Torelli-type theorem for $K3$ surfaces that there exists a unique $g \in \mathrm{Aut}(X)$ satisfying $g^* = \tilde{\phi}$. □

Remark 9.43. Theorem 9.42 was given by Barth, Peters [BP] and independently by Nikulin [Ni5]. Thus we know that a generic Enriques surface has an infinite group of automorphisms. On the other hand, by Corollary 9.41 the automorphism group of Y is finite if and only if $W(Y)$ is of finite index in $\mathrm{O}(H^2(Y, \mathbb{Z})_f)$. In fact, Enriques surfaces with finite automorphism group exist and are classified into seven types (Kondo [Kon1], Nikulin [Ni6]). The classification is obtained by determining all non-singular rational curves on Y by using a criterion for which $W(Y)$ is of finite index in $\mathrm{O}(H^2(Y, \mathbb{Z})_f)$ (Vinberg [V1]). In the 10-dimensional family of Enriques surfaces, two types among the seven form 1-dimensional irreducible families and each of the remaining five types is a unique Enriques surface. The number of non-singular rational curves is 12 in the case of two 1-dimensional families and 20 in the remaining five types. It is not known why the numbers of non-singular rational curves on such Enriques surfaces are 12 and 20.[2] Contrary to the case of $K3$ surfaces, Enriques surfaces with finite automorphism group are very rare. We give examples of such Enriques surfaces in Remarks 9.53, 9.54.

Remark 9.44. The classification of Enriques surfaces whose periods are contained in $\mathcal{S}$ is not known.

[2] Added in English translation: Recently, Brandhorst, Shimada [BS] discovered 17 polyhedrons in the positive cone of Enriques surfaces, in which these seven types of polyhedrons appear.

9.4 Examples of Enriques surfaces

9.4.1 Enriques surfaces associated with Kummer surfaces of product type. We consider the Kummer surface $\mathrm{Km}(E \times F)$ given in Example 4.24. We assume that p_1, q_1 are the origin of E, F, respectively. We take a point $a = (p_i, q_j)$ ($p_i \neq p_1$, $q_j \neq q_1$) of $E \times F$ of order 2. Let

$$t_a \colon E \times F \to E \times F$$

be the translation by a. Then t_a induces an automorphism $\bar{t}_a$ of $\mathrm{Km}(E \times F)$ of order 2 because $t_a \circ (-1_{E\times F}) = -1_{E\times F} \circ t_a$. On the other hand, we denote by τ the automorphism of $\mathrm{Km}(E \times F)$ of order 2 induced by $(1_E, -1_F)$. Since τ and $\bar{t}_a$ commute, $\sigma = \tau \circ \bar{t}_a$ is of order 2. We now show that σ has no fixed points. To do this, we consider an elliptic fibration

$$\pi \colon \mathrm{Km}(E \times F) \to F/(-1_F) = \mathbb{P}^1$$

given in Example 4.24. Note that π has 4 singular fibers of type I_0^* over the points on the base $\mathbb{P}^1$ corresponding to the points q_k of order 2 and other fibers are non-singular. The involution τ acts trivially on the base $\mathbb{P}^1$ of π and hence preserves every fiber, and the set of fixed points is the union of 8 non-singular rational curves E_i, F_j. On the other hand, $\bar{t}_a$ acts on the base $\mathbb{P}^1$ non-trivially and hence it preserves 2 fibers G_1, G_2. We can see that G_1, G_2 are non-singular elliptic curves on which $\bar{t}_a$ acts as a translation. Thus σ has no fixed points and $Y = \mathrm{Km}(E \times F)/\langle\sigma\rangle$ is an Enriques surface. The images of 24 non-singular rational curves on $\mathrm{Km}(E \times F)$ given in Figure 4.1 are described as in Figure 9.1.

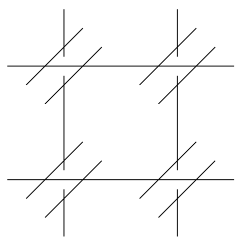


Figure 9.1.

Exercise 9.45. Prove that 12 non-singular rational curves in Figure 9.1 generate $H^2(Y, \mathbb{Q})$.

Exercise 9.46. Note that t_a (or τ) induces an automorphism of Y of order 2. Show that this automorphism acts on $H^2(Y, \mathbb{Q})$ trivially (see Remark 9.39).

Note that abelian surfaces of product type of two elliptic curves form a 2-dimensional family. Thus we get a 2-dimensional family of Enriques surfaces by the above construction. The fibration π induces an elliptic fibration $\bar{\pi}\colon Y \to \mathbb{P}^1$ on Y. The fibration $\bar{\pi}$ has 2 singular fibers of type I_0^* and other fibers are non-singular. The images of G_1, G_2 are multiple fibers of $\bar{\pi}$ with multiplicity 2.

We give another construction of this Enriques surface. Recall that the set of fixed points of τ is the union of 8 non-singular rational curves E_i, F_j, and each of 16 non-singular rational curves N_{ij} is preserved under the action of τ. Let S be the quotient of $\mathrm{Km}(E \times F)$ by τ and let

$$p\colon \ \mathrm{Km}(E \times F) \to S$$

be the projection. Then S is non-singular because the fixed points of τ are non-singular curves. Let $\bar{E}_i$, $\bar{F}_j$, $\bar{N}_{ij}$ be the images of E_i, F_j, N_{ij} by p. Then we have

$$(2E_i)^2 = (p^*(\bar{E}_i))^2 = 2(\bar{E}_i)^2, \quad (N_{ij})^2 = (p^*(\bar{N}_{ij}))^2 = 2(\bar{N}_{ij})^2,$$

and hence $(\bar{E}_i)^2 = (\bar{F}_j)^2 = -4$, $(\bar{N}_{ij})^2 = -1$. By contracting 16 exceptional curves $\bar{N}_{ij}$, we obtain $\mathbb{P}^1 \times \mathbb{P}^1$. A divisor D on $\mathbb{P}^1 \times \mathbb{P}^1$ is said to be of type (m, n) if the defining equation of D is given by a homogeneous polynomial of degree m in the coordinates of the first component $\mathbb{P}^1$ and of degree n in the coordinates of the second component $\mathbb{P}^1$. Then, respectively, the image of the branch divisor $\bar{E}_i$, $\bar{F}_j$ is of type $(1, 0)$, type $(0, 1)$. By changing the coordinates we may assume that $\bar{t}_a$ induces the following automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ of order 2:

$$t\colon ((t_0 : t_1), (s_0 : s_1)) \to ((t_0 : -t_1), (s_0 : -s_1)). \tag{9.15}$$

Note that the images of the branch divisors $\bar{E}_i$, $\bar{F}_j$ do not contain the 4 fixed points

$$((1:0),(1:0)), \quad ((1:0),(0:1)), \quad ((0:1),(1:0)), \quad ((0:1),(0:1))$$

of t. Conversely, consider a divisor of type $(4, 4)$ consisting of 4 divisors of type $(1, 0)$ and 4 divisors of type $(0, 1)$ not passing the fixed points of t. Let X be the minimal non-singular model X of the double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along this divisor of type $(4, 4)$. Then X contains 16 disjoint non-singular rational curves N_{ij} satisfying

$$\frac{1}{2}\sum_{ij} N_{ij} \in S_X.$$

Therefore X is a Kummer surface by Corollary 6.20. The involution t induces an automorphism of the Kummer surface of order 2. Thus we obtain an Enriques surface by taking the quotient (see Lemma 9.48). This method is a special case of Horikawa's model mentioned in the next subsection.

Exercise 9.47. Show that $\frac{1}{2}\sum_{ij} N_{ij} \in S_X$.

Finally, we calculate the root invariant of this Enriques surface. Assume that E, F are general, that is, the Néron–Severi group of the abelian surface $E \times F$ is generated by E, F. Since $E^2 = F^2 = 0$, $E \cdot F = 1$, its Néron–Severi lattice is isomorphic to U. Recall that $H^2(E \times F, \mathbb{Z})$ is an even unimodular lattice of signature $(3,3)$ (formula (4.11)). It follows from Theorem 1.32 that the transcendental lattice $T_{E\times F}$ of $E \times F$ is an even unimodular lattice of signature $(2,2)$, and hence $T_{E\times F}$ is isomorphic to $U \oplus U$ by Theorem 1.27. It now follows from Corollary 6.26 that the transcendental lattice T_X of $X = \mathrm{Km}(E \times F)$ is isomorphic to $U(2) \oplus U(2)$. Again by Theorem 1.32, we can see that the Néron–Severi lattice S_X of X has signature $(1, 17)$ and discriminant $d(S_X) = 2^4$. In the following we show that S_X is generated by non-singular rational curves E_i, F_j, N_{ij} $(i, j = 1, \dots, 4)$. The linear system $|2E_1 + N_{11} + N_{21} + N_{31} + N_{41}|$ gives an elliptic fibration $p\colon X \to \mathbb{P}^1$ on X which has 4 singular fibers of type I_0^* and 4 sections $F_1, \dots, F_4$. A section and components of fibers generate a sublattice of S_X which is isomorphic to $U \oplus D_4^{\oplus 4}$ and is of index 4 in S_X (see Exercise 4.26). By adding the remaining sections to this sublattice, we can prove that E_i, F_j, N_{ij} $(i, j = 1, \dots, 4)$ generate S_X. Now, if necessary by changing the numbering, we may assume that

$$\sigma(E_1) = E_2, \quad \sigma(E_3) = E_4, \quad \sigma(F_1) = F_2, \quad \sigma(F_3) = F_4.$$

Then

$$\begin{aligned} &N_{11} - N_{22}, \quad N_{12} - N_{21}, \quad F_1 - F_2, \quad N_{14} - N_{23}, \\ &E_4 - E_3, \quad N_{44} - N_{33}, \quad F_4 - F_3, \quad N_{42} - N_{31} \end{aligned}$$

generate $D_8(2)$ and hence we have $D_8 \subset R$. Moreover, by adding $(\delta^+ + \delta^-)/2$ we obtain an overlattice S of $L_X^+ \oplus D_8(2)$ of index 2^8 where δ^- runs over the generator of $D_8(2)$ given as above. Since $S \subset S_X$ and

$$\det(S) \cdot [S : L_X^+ \oplus D_8(2)]^2 = \det(L_X^+ \oplus D_8(2)),$$

we get $\det(S) = 2^4 = \det(S_X)$. Therefore we have $S_X = S$ and hence $R = D_8$. On the other hand, t_a acts on L_X^+ and T_X trivially, and acts on $D_8(2)$ as -1. It is known that the anti-invariant sublattice in $H^2(X, \mathbb{Z})$ of a symplectic automorphism of a $K3$ surface of order 2 is $E_8(2)$ (Proposition 8.28). Since $E_8(2)/D_8(2) \subset D_8(2)^*/D_8(2)$ is contained in the kernel of d, we have $K = E_8(2)/D_8(2) \cong \mathbb{Z}/2\mathbb{Z}$, where d is the homomorphism given in (9.11). Thus the root invariant of this Enriques surface is $(R, K) = (D_8, \mathbb{Z}/2\mathbb{Z})$.

9.4.2 Horikawa model. In the previous subsection we considered a special divisor of type (4,4) in $\mathbb{P}^1 \times \mathbb{P}^1$. Now we consider a t-invariant general reduced divisor D of type (4,4) satisfying the following two conditions:

(1) D does not pass any fixed points of t. Here t is the automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ of order 2 given in (9.15).

(2) The double covering $\bar{X}$ of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along D has only rational double points as singularities.

Denote by X the minimal non-singular model of $\bar{X}$. Then by condition (2), X is a $K3$ surface (see Remark 4.22). Let τ be the covering transformation.

Lemma 9.48. *The automorphism t is lifted to two automorphisms $\tilde{t}$, $\tilde{t} \circ \tau$ of X of order* 2*. One of them acts identically on $\mathbb{C}\omega_X$ and the other acts on it by multiplication by* -1*. The one acting by* -1 *has no fixed points.*

Proof. If t induces an automorphism $\tilde{t}$ of order 4, then $\tilde{t}^2 = \tau$ and the set of fixed points of $\tilde{t}$ is contained in that of τ. This contradicts the fact that the set of fixed points of t is not contained in D. Hence t is lifted to two automorphisms $\tilde{t}$, $\tilde{t} \circ \tau$ of order 2. Recall that τ acts on ω_X by multiplication by -1. Therefore exactly one of $\tilde{t}$, $\tilde{t} \circ \tau$ acts trivially on ω_X and the other acts on it by multiplication by -1. We may assume that $\tilde{t}^*(\omega_X) = -\omega_X$. If $\tilde{t}$ has a fixed point, then by Lemma 8.25, the set of fixed points of $\tilde{t}$ is a curve. This contradicts the fact that the set of fixed points of t consists of 4 points. □

Let (t_0, t_1) be homogeneous coordinates of the first factor of $\mathbb{P}^1 \times \mathbb{P}^1$ and (s_0, s_1) of the second factor. The vector space of t-invariant homogeneous polynomials $f(t_0, t_1, s_0, s_1)$ in 4 variables with homogeneous degree 4 in each pair of coordinates (t_0, t_1), (s_0, s_1) is generated by 13 monomials

$$\begin{aligned} &(t_0^k t_1^{2-k})^2 \cdot (s_0^l s_1^{2-l})^2 && (k, l = 0, 1, 2),\\ &t_0 t_1 s_0 s_1 \cdot (t_0^k t_1^{1-k})^2 \cdot (s_0^l s_1^{1-l})^2 && (k, l = 0, 1). \end{aligned}$$

Since the subgroup of the projective transformation group PGL(2) commuting with t has dimension 2, we obtain a 10-dimensional family of Enriques surfaces by Lemma 9.48.

A general fiber F of the projection $p_i\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ $(i = 1, 2)$ and the branch divisor meet at 4 distinct points. Therefore the pullback F to X is the double covering of $\mathbb{P}^1$ branched at 4 points, and hence is an elliptic curve. Thus the projection p_i defines an elliptic surface

$$\tilde{p}_i\colon X \to \mathbb{P}^1$$

on X, and hence an elliptic surface

$$\bar{p}_i : Y \to \mathbb{P}^1$$

on the obtained Enriques surface Y. There are exactly 2 fibers of p_i passing to the fixed points of t. We denote by F_i, F_i' their pullbacks to X. Then both F_i and F_i' are τ-invariant, and hence their images $\bar{F}_i$, $\bar{F}_i'$ on Y are the multiple fibers of $\bar{p}_i$. Let $\mathcal{Q}$ be the quotient surface of $\mathbb{P}^1 \times \mathbb{P}^1$ by the automorphism t. Then $\mathcal{Q}$ has 4 rational double points of type A_1 corresponding to 4 fixed points of t. By construction, Y is the minimal non-singular model of $\mathcal{Q}$. The branch locus of this covering consists of 4 rational double points of type A_1 and the image of the divisor D of type $(4, 4)$ on $\mathcal{Q}$:

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \longleftarrow & X \\ \downarrow & & \downarrow \\ \mathcal{Q} & \longleftarrow & Y. \end{array}$$

It is known that $\mathcal{Q}$ is the complete intersection of two quadrics Q_1, Q_2 of rank 3 in $\mathbb{P}^4$ which is called a 4-nodal quartic del Pezzo surface. The covering map $Y \to \mathcal{Q}$ coincides with the map $\varphi_{|2(\bar{F}_1+\bar{F}_2)|}$ associated with the complete linear system $|2(\bar{F}_1 + \bar{F}_2)|$.

Remark 9.49. Horikawa [Ho2] studied Enriques surfaces as mentioned here and their degenerations, and proved the surjectivity of the period map of Enriques surfaces.

9.4.3 An example by Enriques. All examples of Enriques surfaces as above are realized as the quotient surfaces of $K3$ surfaces by their fixed-point-free involutions. In this section, we introduce the projective model of the Enriques surface discovered by Enriques himself.

We assume that an Enriques surface Y contains 3 irreducible curves E_1, E_2, E_3 satisfying

$$E_i^2 = 0, \quad \langle E_i, E_j \rangle = 1 \quad (i \neq j)$$

(e.g., see Corollary 9.35). Let $E_i' = E_i + K_Y$. It is known that the map

$$\varphi = \varphi_{|D|} : Y \to \mathbb{P}(H^0(Y, \mathcal{O}_Y(D))^*) = \mathbb{P}^3$$

associated with a complete linear system $|D| = |E_1 + E_2 + E_3|$ is a birational map onto its image (Shafarevich [Sh, Chap. 10]). We assume this fact here and will show that the image of φ is a sextic surface with double points along 6 lines consisting of the intersections of 4 planes forming a tetrahedron. First of all, by $D^2 = 6$, the image of φ, denoted by S, is a sextic surface in $\mathbb{P}^3$. Since $2K_Y = 0$, we can see that

$$E_1 + E_2 + E_3, \quad E_1 + E_2' + E_3', \quad E_1' + E_2 + E_3', \quad E_1' + E_2' + E_3$$

are members of $|D|$ and are the pullbacks of hyperplanes in $\mathbb{P}^3$. Since $\langle E_i, D\rangle > 0$, the images of E_i, E'_i by φ are curves. Moreover, for example, E_1 is the intersection of $E_1 + E_2 + E_3$ and $E_1 + E'_2 + E'_3$, and hence the image of each E_i, E'_i by φ is the intersection of two hyperplanes, that is, it is a line. Since $\langle D, E_i\rangle = 2$, the degree of the map $E_i \to \varphi(E_i)$ is at most 2. Each E_i, E'_i has arithmetic genus 1 and hence a non-singular elliptic curve or singular rational curve, but its image is a non-singular rational curve. Therefore the degree of the map is 2. Thus we have proved that S contains 6 lines with multiplicity 2.

We can see how 6 lines E_i, E'_i are identified by φ as in the following. For example, assume that $\varphi(p) = \varphi(q)$ for $p \in E_1$, $q \in E_2$. This means that a member of $|D|$ containing p also passes through q. Since $E_1 + E'_2 + E'_3$ passes through p, it passes through q, and hence $q = E_1 \cap E_2$ or $q = E_2 \cap E'_3$.

Finally, we show that 4 hyperplanes do not meet at 1 point. If $p \in E_1$ identifies with $q \in E_2$, then we have $q = E_1 \cap E_2$ or $q = E_2 \cap E'_3$. Therefore $E_1 \cap E_2$ and $E_2 \cap E_3$ are not identified. Thus we have shown that S is a sextic containing 6 lines of the intersection of 4 hyperplanes as double lines.

Now we give the defining equation of S. Consider a tetrahedron consisting of 4 hyperplanes in $\mathbb{P}^3$. Let p_0, p_1, p_2, p_3 be its vertices, ℓ_{ij} the line passing through p_i and p_j, and H_{ijk} the hyperplane containing p_i, p_j, p_k. Then S is a sextic in $\mathbb{P}^3$ with double points along the union of 6 lines $\sum \ell_{ij}$. For simplicity, we assume that

$$p_0 = (1,0,0,0), \quad p_1 = (0,1,0,0), \quad p_2 = (0,0,1,0), \quad p_3 = (0,0,0,1). \tag{9.16}$$

Let us determine the homogeneous polynomial $f(x_0, x_1, x_2, x_3)$ defining S. Since the intersection of the hyperplane H_{ijk} and S is the sextic curve $2\ell_{ij} + 2\ell_{jk} + 2\ell_{ki}$, we have

$$f(0, x_1, x_2, x_3) = a_0 x_1^2 x_2^2 x_3^2.$$

Hence we obtain

$$f = a_0 x_1^2 x_2^2 x_3^2 + a_1 x_0^2 x_2^2 x_3^2 + a_2 x_0^2 x_1^2 x_3^2 + a_3 x_0^2 x_1^2 x_2^2 + q(x) x_0 x_1 x_2 x_3.$$

Here a_i is a non-zero constant and $q(x)$ is a homogeneous polynomial of degree 2. Permutations of homogeneous coordinates and

$$(x_0 : x_1 : x_2 : x_3) \to (\lambda_0 x_0 : \lambda_1 a_1 : \lambda_2 a_2 : \lambda_3 a_3) \quad (\lambda_i \in \mathbb{C}^*)$$

generate the group of projective transformations preserving the tetrahedron. By applying a projective transformation we may assume that the defining equation of S is given by

$$x_1^2 x_2^2 x_3^2 + x_0^2 x_2^2 x_3^2 + x_0^2 x_1^2 x_3^2 + x_0^2 x_1^2 x_2^2 + q(x) x_0 x_1 x_2 x_3 = 0.$$

The space of polynomials $q(x)$ has dimension

$$\dim H^0(\mathbb{P}^3, \mathcal{O}(2H)) = 10,$$

and thus the S form a 10-dimensional family.

Remark 9.50. The detail of the sextic model discovered by Enriques is mentioned in Shafarevich [Sh, Chap. 10] and Griffiths, Harris [GH, Chap. 4, Sect. 6]. On the other hand, the following projective model of degree 10 is also known. Consider an Enriques surface Y with elliptic curves $E_1, \ldots, E_{10}$ satisfying $\langle E_i, E_j \rangle = 1$ $(i \neq j)$ mentioned in Corollary 9.35. By construction, $w_1, \ldots, w_{10}$ is a basis of $H^2(Y, \mathbb{Z})_f$ and hence $E_1, \ldots, E_{10}$ is also. Now put

$$F = \tfrac{1}{3}(E_1 + \cdots + E_{10});$$

then we have $\langle F, E_i \rangle = 3$ and hence $F \in H^2(Y, \mathbb{Z})_f^* = H^2(Y, \mathbb{Z})_f$. Obviously, $F^2 = 10$. The complete linear system $|F|$ gives a map

$$\varphi_{|F|} \colon Y \longrightarrow \mathbb{P}^5,$$

which is birational onto its image. The image has degree 10 and is called a Fano model. For Fano models, we refer the reader to Cossec, Dolgachev [CD].

9.4.4 Hessians of cubic surfaces and Enriques surfaces. Let $(x_1, x_2, x_3, x_4, x_5)$ be homogeneous coordinates of $\mathbb{P}^4$. Let S be the surface defined by the equations

$$\lambda_1 x_1^3 + \cdots + \lambda_5 x_5^3 = 0, \quad x_1 + \cdots + x_5 = 0. \tag{9.17}$$

Here λ_i is a non-zero constant. Note that S is a cubic surface in the hyperplane $\mathbb{P}^3$ defined by the second linear equation. In general, for a non-singular cubic surface in $\mathbb{P}^3$ defined by a homogeneous polynomial $F(z_1, z_2, z_3, z_4)$ of degree 3, one can define the *Hessian* of F by

$$\det\left(\frac{\partial^2}{\partial z_i \partial z_j}(F)\right) = 0, \tag{9.18}$$

which is the determinant of a 4×4 matrix with linear forms in $z_1, \ldots, z_4$ as its entries. Therefore, if it is not identically zero, then it defines a quartic surface H in $\mathbb{P}^3$. For a cubic surface defined by equation (9.17), a direct calculation shows that the defining equation of H is given by

$$\frac{1}{\lambda_1 x_1} + \cdots + \frac{1}{\lambda_5 x_5} = 0, \quad x_1 + \cdots + x_5 = 0. \tag{9.19}$$

We can easily see that H has 10 rational double points

$$p_{ijk}\colon x_i = x_j = x_k = 0$$

of type A_1 and contains 10 lines

$$\ell_{mn}\colon x_m = x_n = 0.$$

Note that the minimal non-singular model X of H is a $K3$ surface (Remark 4.22). Let E_{ijk} be the exceptional curves over the rational double points p_{ijk} and let L_{mn} be the proper transform of the line ℓ_{mn}. Then X contains 20 non-singular rational curves $\{E_{ijk}, L_{mn}\}$. Each E_{ijk} meets exactly 3 L_{ij}, L_{ik}, L_{jk} at one point transversally, and does not meet the remaining L_{mn}. On the other hand, L_{ij} meets exactly 3 E_{ijk} $(k \neq i, j)$ at 1 point transversally and does not meet the remaining E_{kmn}. Thus there exist two sets $\{E_{ijk}\}$, $\{L_{mn}\}$ of 10 mutually disjoint non-singular rational curves on X such that each member in one set meets exactly 3 members in another set with intersection multiplicity 1. This is reminiscent of the configuration, called the (16_6)-configuration, of 32 non-singular rational curves on the Kummer surface $\mathrm{Km}(C)$ associated with a curve of genus 2 mentioned in Section 4.4.

Now, the Cremona transformation of $\mathbb{P}^4$ of order 2,

$$\iota\colon (x_1, \ldots, x_5) \to \left(\frac{1}{\lambda_1 x_1}, \ldots, \frac{1}{\lambda_5 x_5}\right) \tag{9.20}$$

preserves equation (9.19) and hence induces an automorphism σ of X of order 2.

Exercise 9.51. Show that σ interchanges E_{ijk} and L_{mn} ($\{i, j, k, m, n\} = \{1, \ldots, 5\}$).

Lemma 9.52. *If $\lambda_1, \ldots, \lambda_5$ are general, then σ has no fixed points.*

Proof. Note that the intersection of $x_i = 0$ and H consists of 4 lines ℓ_{ij} $(j \neq i)$. The fact that σ interchanges E_{ijk} and L_{mn} implies that there are no fixed points on these 20 curves. Hence it suffices to consider the case that $x_i \neq 0$ for any i. In this case, the set of fixed points of ι is given by

$$\lambda_1 x_1^2 = \lambda_2 x_2^2 = \cdots = \lambda_5 x_5^2,$$

that is,

$$(x_1 : x_2 : \cdots : x_5) = \left(\sqrt{\frac{1}{\lambda_1}} : \pm\sqrt{\frac{1}{\lambda_2}} : \cdots : \pm\sqrt{\frac{1}{\lambda_5}}\right).$$

Therefore, if we choose the constants $\lambda_1, \ldots, \lambda_5$ such that these points do not satisfy equations (9.19), then σ has no fixed points. □

It follows from Lemma 9.52 that the quotient surface $Y = X/\langle\sigma\rangle$ is an Enriques surface for a general $\lambda_1, \dots, \lambda_5$. By this method we obtain a 4-dimensional family of Enriques surfaces because the cubic surfaces given by equations (9.17) form a 4-dimensional family. Recall that σ interchanges E_{ijk} and L_{mn} ($\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$). We denote by $\bar{L}_{mn}$ the image of E_{ijk} and L_{mn} on Y. Then $\bar{L}_{ij}$ meets exactly 3 $\bar{L}_{km}$, $\bar{L}_{kn}$, and $\bar{L}_{mn}$. The dual graph of the 10 non-singular rational curves $\{\bar{L}_{ij}\}$ on Y coincides with the *Petersen graph* given in Figure 9.2 and the group of symmetries of this graph is isomorphic to the symmetric group $\mathfrak{S}_5$ of degree 5. Recall that S, H are defined in $\mathbb{P}^4$. The group $\mathfrak{S}_5$ acts on $\mathbb{P}^4$ as permutations of the coordinates. This action induces the one on the set of 10 lines $\{\ell_{mn}\}$, which is nothing but the group of symmetries of the Petersen graph.

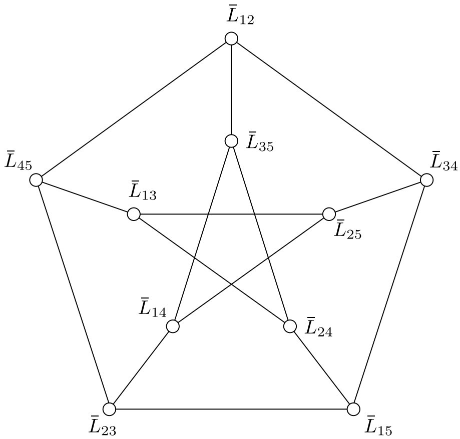


Figure 9.2. Petersen graph.

The Enriques surfaces Y form a 4-dimensional family and hence the rank of the Néron–Severi lattice of the covering $K3$ surface might be at least rank 16 (= 20 − 4). This will be confirmed as follows. The dual graph of 10 non-singular rational curves $\bar{L}_{ij}$ contains a Dynkin diagram of type E_6. For example,

$$\bar{L}_{12}, \ \bar{L}_{35}, \ \bar{L}_{14}, \ \bar{L}_{25}, \ \bar{L}_{23}, \ \bar{L}_{15}$$

is one of them. By considering the pullback of these curves on X, we can see that L_X^- contains $E_6(2)$. Thus the Picard number of X is at least 16.

Now we assume that the Picard number of X is the minimum 16 and calculate the root invariant (R, K). In this case it is known that the transcendental lattice T_X of X is isomorphic to $T_X \cong U(2) \oplus U \oplus A_2(2)$ (Dolgachev, Keum [DK]). On the other hand, since $\mathrm{rank}(E_6) + \mathrm{rank}(T_X) = 12$ and there are no root lattices containing E_6 of finite index, we can conclude $R = E_6$. Obviously, $\mathrm{Ker}(d) = 0$ and hence $(R, K) \cong (E_6, \{0\})$.

Note that X can be realized in $\mathbb{P}^3 \times \mathbb{P}^3$ as the intersection of 4 hypersurfaces of bidegree $(1,1)$. In fact, X is the image of the rational map

$$H \to \mathbb{P}^3 \times \mathbb{P}^3, \quad x \to (x, \iota(x)), \tag{9.21}$$

and if we denote by $(x_1, \dots, x_4)$ homogeneous coordinates of the first factor $\mathbb{P}^3$ and by $(y_1, \dots, y_4)$ those of the second factor $\mathbb{P}^3$, then 4 hypersurfaces are given by

$$\lambda_1 x_1 y_1 - \lambda_i x_i y_i = 0 \ \ (i = 2,3,4), \quad \lambda_1 x_1 y_1 - \lambda_5 \left(\sum_{i=1}^{4} x_i\right)\left(\sum_{i=0}^{4} y_i\right) = 0. \tag{9.22}$$

This is a special case of an example of Enriques surfaces mentioned later (see equation (9.25), Lemma 9.58).

Remark 9.53. A hyperplane section of a cubic surface is called a *tritangent plane* if the cubic curve obtained by the hyperplane section decomposes into 3 lines. It is classically known that a general cubic surface has 45 tritangent planes. If 3 lines on a tritangent plane meet at one point, then the intersection point is called an *Eckardt point* (see Figure 9.3).

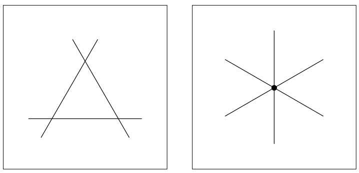


Figure 9.3. Tritangent planes and an Eckardt point.

Now, a cubic surface defined by equations (9.17) has an Eckardt point if and only if

$$\lambda_i = \lambda_j \quad (i \neq j),$$

and the corresponding tritangent plane is given by $x_i + x_j = 0$ and p_{kmn} is the Eckardt point, where $\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$. The intersection of the hyperplane $x_i + x_j = 0$ and H consists of $2\ell_{ij}$ and two lines passing to p_{kmn}. The proper transforms of these lines give 2 new non-singular rational curves N_{ij}^+, N_{ij}^- and σ interchanges N_{ij}^+ and N_{ij}^-. Note that $N_{ij}^{\pm}$ meets L_{ij}, E_{kmn} with the intersection multiplicity 1 and does not meet the other 18 curves.

In the case that all λ_i coincide in equation (9.17), S is called the *Clebsch diagonal cubic surface*, which has 10 Eckardt points. Moreover, the symmetric group $\mathfrak{S}_5$ of degree 5 acts on S as automorphisms. The corresponding Enriques surface Y contains

20 non-singular rational curves and $\mathfrak{S}_5$ acts on Y as automorphisms (Dardanelli, van Geemen [DvG]). It is known that this Enriques surface is one of those with finite automorphism group mentioned in Remark 9.43 and has $\mathrm{Aut}(Y) \cong \mathfrak{S}_5$. Moreover, Y contains exactly 20 non-singular rational curves as mentioned above and its root invariant is $(E_6 \oplus A_4, \{0\})$ (Kondo [Kon1]).

Remark 9.54. In this subsection, we constructed Enriques surfaces associated with quartic surfaces in $\mathbb{P}^4$ defined by equation (9.17) and a Cremona transformation (9.20), and obtained an Enriques surface with finite automorphism group as a special case (Remark 9.53). Recently, Hisanori Ohashi succeeded in obtaining another Enriques surface with the automorphism group $\mathfrak{S}_5$ as an analogue of this example (see Mukai, Ohashi [MuO]). Let $(x_1, \ldots, x_5)$ be homogeneous coordinates of $\mathbb{P}^4$, and let us consider the surface F defined by

$$\sum_{i<j} x_i x_j = 0, \quad \sum_{i<j} \frac{1}{x_i x_j} = 0. \tag{9.23}$$

Note that the Cremona transformation

$$c\colon (x_i) \to \left(\frac{1}{x_i}\right)$$

acts on F. The singularities of F are 5 rational double points of type A_1 and the minimal non-singular model X of F is a $K3$ surface. The Cremona transformation c induces an fixed-point-free automorphism σ of X of order 2 and hence as its quotient we obtain an Enriques surface Y. The symmetric group $\mathfrak{S}_5$ of degree 5 acts on F which induces automorphisms of Y. By construction we can check that there are 20 non-singular rational curves.

Actually, this Enriques surface Y is isomorphic to one of those with finite automorphism group mentioned in Remark 9.43. It is known that Y contains exactly 20 non-singular rational curves and satisfies $\mathrm{Aut}(Y) \cong \mathfrak{S}_5$. This Enriques surface was discovered by Fano [Fa] from the viewpoint of Reye congruence and was reconstructed by Kondo [Kon1]. Fano considered the two cubic curves

$$C_\pm\colon \left(1 \pm \sqrt{-1}\right)\left(x_2^3 - x_3^3 - x_1x_2x_3\right) + \left(x_1^2x_2 + x_2^2x_3 + x_3^2x_1\right) = 0,$$

where (x_1, x_2, x_3) are homogeneous coordinates of $\mathbb{P}^2$. Cubic curves C_+ and C_- meet at $(1 : 0 : 0)$ with multiplicity 9. The minimal non-singular model of the double covering of $\mathbb{P}^2$ branched along the sextic $C_+ + C_-$ is a $K3$ surface which has a fixed-point-free automorphism of order 2. The quotient surface is isomorphic to the above Enriques surface Y obtained by Ohashi and its root invariant is $(A_9 \oplus A_1, \mathbb{Z}/2\mathbb{Z})$. Contrary to Ohashi's example, it is not so easy to find 20 non-singular rational curves and the action of $\mathfrak{S}_5$ in the case of Fano–Kondo's example.

9.4.5 Reye congruence. We introduce a 9-dimensional family, called a Reye congruence, which is a generalization of the example given earlier (equation (9.22)). Let (x_1, x_2, x_3, x_4) be homogeneous coordinates of $\mathbb{P}^3$. A quadric surface Q in $\mathbb{P}^3$ is given by

$$Q\colon\ q(x) = \sum_{i,j} a_{ij} x_i x_j = 0,$$

where (a_{ij}) is a symmetric matrix of degree 4. For simplicity, we denote by the same symbol $q(x, y)$ the associated symmetric bilinear form $\sum_{i,j} a_{ij} x_i y_j$ with the quadratic form $q(x)$. Then $q(x + y) - q(x) - q(y) = 2q(x, y)$. The rank of a quadric surface Q is defined by the rank of the corresponding matrix (a_{ij}). A quadric surface of rank 4 is a non-singular surface, of rank 3 is a cone over a conic, and of rank 2 is the union of two projective planes. We sometimes denote by q the quadric surface Q given by $q(x) = 0$.

Definition 9.55. Let $W \subset \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)))$ be a 3-dimensional subspace ($\cong \mathbb{P}^3$). In the following we assume that W satisfies the following conditions:

(i) W has no base points as a linear system of quadric surfaces in $\mathbb{P}^3$. In other words, there are no points $x \in \mathbb{P}^3$ such that $q(x) = 0$ for any $q \in W$.

(ii) Let $q \in W$ be a quadric surface of rank 2 and let ℓ be the double line appearing as the singularities of q. Then there exists no $q' \in W$ $(q' \neq q)$ with $\ell \subset q'$.

For W, we denote by $R(W)$ the set of lines ℓ in $\mathbb{P}^3$ contained in two quadric surfaces belonging to W:

$$R(W) = \left\{\ell \subset \mathbb{P}^3 : \ell \subset q \cap q',\ q, q' \in W,\ q \neq q'\right\}. \tag{9.24}$$

By the Plücker embedding

$$G(1, 3) \subset \mathbb{P}^5$$

of the Grassmann manifold $G(1, 3)$ consisting of lines in $\mathbb{P}^3$ (see relation (4.7)), we can consider $R(W)$ as a subset of $\mathbb{P}^5$. We call $R(W)$ a *Reye congruence*.

To prove that $R(W)$ is an Enriques surface, we first construct the covering $K3$ surface.

Definition 9.56. For W, we set

$$\tilde{S}(W) = \left\{(x, y) \in \mathbb{P}^3 \times \mathbb{P}^3 : q(x, y) = 0\ \forall\, q \in W\right\} \tag{9.25}$$

(where $\tilde{S}(W)$ is the minimal non-singular model of the Steiner surface mentioned later in Definition 9.61). We remark that by Definition 9.55(i), if $(x, y) \in \tilde{S}(W)$, then $x \neq y$. This implies that an automorphism σ of order 2 defined by $\sigma(x, y) = (y, x)$ acts on $\tilde{S}(W)$ fixed-point-freely.

By using the following lemma we will show the smoothness of $\tilde{S}(W)$.

Lemma 9.57. *Let $x = (x_i) \in \mathbb{P}^m$, $y = (y_j) \in \mathbb{P}^n$ be homogeneous coordinates, $f_k(x, y)$ a homogeneous polynomial of degree 1 with respect to each of the coordinates x and y ($k = 1, \dots, l$; $l \leq m + n$), and define*

$$X = \{(x, y) \in \mathbb{P}^m \times \mathbb{P}^n : f_k(x, y) = 0,\ k = 1, \dots, l\}.$$

Then X is non-singular at (x_0, y_0) if and only if there are no $\lambda = (\lambda_i) \in \mathbb{P}^l$ satisfying

$$f(\lambda)(x, y_0) = f(\lambda)(x_0, y) = 0 \quad (\forall\, x \in \mathbb{P}^m,\ \forall\, y \in \mathbb{P}^n),$$

where

$$f(\lambda)(x, y) = \sum_{k=1}^{l} \lambda_k f_k(x, y).$$

Proof. If (x_0, y_0) is a singular point of X, then the rank of the Jacobian matrix

$$\frac{\partial(f_1, \dots, f_l)}{\partial(x_1, \dots, x_{m+1}, y_1, \dots, y_{n+1})}(x_0, y_0)$$

is less than l, and hence there exists a $(\lambda) = (\lambda_1, \dots, \lambda_l) \in \mathbb{P}^l$ satisfying

$$\sum_{k=1}^{l} \lambda_k \cdot \frac{\partial f_k}{\partial x_i}(x_0, y_0) = \sum_{k=1}^{l} \lambda_k \cdot \frac{\partial f_k}{\partial y_j}(x_0, y_0) = 0 \quad (1 \leq i \leq m + 1,\ 1 \leq j \leq n + 1).$$

Now we set $x_0 = (x_1^0, \dots, x_{m+1}^0)$, $y_0 = (y_1^0, \dots, y_{n+1}^0)$, $f_k(x, y) = \sum_{i,j} a_{i,j}^k x_i y_j$. Then we obtain

$$\sum_{k,j} a_{i,j}^k \lambda_k y_j^0 = \sum_{k,i} a_{i,j}^k \lambda_k x_i^0 = 0.$$

Therefore we have

$$f(\lambda)(x, y_0) = \sum_i \left(\sum_{j,k} \lambda_k a_{i,j}^k y_j^0 \right) x_i = 0 \quad (\forall\, x \in \mathbb{P}^m).$$

Similarly we have proved that $f(\lambda)(x_0, y) = 0$ ($\forall\, y \in \mathbb{P}^n$). □

Lemma 9.58. *The surface $\tilde{S}(W)$ is a $K3$ surface and σ has no fixed points. In particular, $\tilde{S}(W)$ is the covering $K3$ surface of an Enriques surface.*

Proof. We have mentioned that σ has no fixed points (Definition 9.56). We take a basis $q_1, \ldots, q_4$ of W and write an element q of W as

$$q = q(\lambda) = \sum_{i=1}^{4} \lambda_i q_i.$$

Now assume that $\tilde{S}(W)$ has a singular point at (x_0, y_0). Recall that $x_0 \neq y_0$. Then it follows from Lemma 9.57 that there exists a $\lambda^0 = (\lambda_1^0, \ldots, \lambda_4^0) \in \mathbb{P}^3$ satisfying

$$q(\lambda^0)(x_0, y) = 0 \,\forall\, y, \quad q(\lambda^0)(x, y_0) = 0 \,\forall\, x. \tag{9.26}$$

Note that x_0 and y_0 are perpendicular with respect to $q(\lambda)(x, y)$ and equations (9.26) imply that x_0, y_0 are contained in the quadric surface $q(\lambda^0)$. Therefore, if we denote by ℓ the line passing through x_0 and y_0, then ℓ is contained in $q(\lambda^0)$. Moreover, equations (9.26) imply

$$\frac{\partial q(\lambda^0)}{\partial x_i}(x_0) = \frac{\partial q(\lambda^0)}{\partial x_i}(y_0) = 0 \,\forall\, i$$

(see the proof of Lemma 9.57), which means that $q(\lambda^0)$ has singularities along the line ℓ. Again, by the fact that x_0 and y_0 are perpendicular with respect to $q(\lambda)(x, y)$, we can say that $q(\lambda)$ contains the line ℓ if and only if $q(\lambda)(x_0) = q(\lambda)(y_0) = 0$. By considering this as simultaneous equations with respect to $\lambda_1, \ldots, \lambda_4$, there exists a $\lambda \neq \lambda_0$ such that $q(\lambda)$ contains ℓ. This contradicts the assumption about W (Definition 9.55(ii)). Thus $\tilde{S}(W)$ is non-singular. Since $\tilde{S}(W)$ is the intersection of 4 divisors $\mathbb{P}^3 \times \mathbb{P}^3$ of type $(1, 1)$, it is a $K3$ surface by the adjunction formula (Theorem 3.3) and Lefschetz hyperplane theorem (Theorem 3.8). □

Theorem 9.59. *The surface $R(W)$ is isomorphic to the Enriques surface $\tilde{S}(W)/\langle\sigma\rangle$.*

Proof. For any $(x, y) \in \tilde{S}(W)$, let ℓ be the line passing through two points $x, y \in \mathbb{P}^3$. For $q \in W$, we denote it by $q = \sum_i \lambda_i q_i$ with respect to a basis $\{q_1, \ldots, q_4\}$ of W. Then as mentioned in the proof of Lemma 9.58, it follows from the equation

$$q(x + y) - q(x) - q(y) = 2q(x, y)$$

that the quadric surface defined by $q = 0$ contains ℓ if and only if $q(x) = q(y) = 0$. By considering this as simultaneous equations with respect to the variables $\lambda_1, \ldots, \lambda_4$, we can prove that there exist at least two quadric surfaces containing ℓ, and hence have obtained $\ell \in R(W)$.

Conversely, assume that two quadric surfaces contain a line $\ell \subset \mathbb{P}^3$. Let N be a 1-dimensional subspace of W such that N and the two quadric surfaces generate W.

Then the restriction $N|\ell$ of N to ℓ defines two hypersurfaces in $\ell \times \ell$ of type $(1,1)$. Their intersection is exactly two points (x, y), (y, x) satisfying $q(x, y) = 0$ $(\forall\, q \in N)$. Then we have $(x, y) \in \tilde{S}(W)$, which gives the inverse correspondence. □

Let $G(3,9)$ be the Grassmann manifold parametrizing 3-dimensional subspaces in $\mathbb{P}^9$. Then by

$$\dim G(3,9) - \dim \mathrm{PGL}(3) = 24 - 15 = 9,$$

we know that the $R(W)$ form a 9-dimensional family of Enriques surfaces.

Finally, we introduce another three surfaces related to the Enriques surface $R(W)$ and its covering $K3$ surface $\tilde{S}(W)$.

Definition 9.60. For any W, we define

$$H(W) = \{q \in W : \det(q) = 0\},$$

and call it the *Hessian* or the *symmetroid*. Note that $H(W)$ is a quartic surface in W.

It is known that $H(W)$ has 10 rational double points of type A_1. These 10 points correspond to the quadric surfaces of rank 2, that is, a union of 2 planes. Let q_0 be one of them and let (x_1, x_2, x_3, x_4) be homogeneous coordinates of $\mathbb{P}^3$ $(= W)$. Assume that $q_0 = (1,0,0,0)$. Then the quartic surface $H(W)$ in $\mathbb{P}^3$ is given by the equation

$$x_1^2 A_2(x_2, x_3, x_4) + 2x_1 B_3(x_2, x_3, x_4) + C_4(x_2, x_3, x_4) = 0.$$

Here A_2, B_3, C_4 are homogeneous polynomials in variables x_2, x_3, x_4 of degrees 2, 3, 4 respectively. The projection

$$\pi\colon H(W) \to \mathbb{P}^2$$

from q_0 defines a double covering branched along the sextic curve given by $B_3^2 - A_2C_4 = 0$. This sextic curve decomposes into 2 cubic curves. Moreover, the conic defined by $A_2 = 0$ is tangent to each of 2 cubic curves at 3 points. The construction of the Enriques surface with finite automorphism by Fano–Kondo mentioned in Remark 9.54 is nothing but the above one for $H(W)$.

Definition 9.61. For each $q \in W$ we denote by $\mathrm{sing}(q)$ the set of singular points of the quadric surface given by $q = 0$, and define

$$\tilde{H}(W) = \left\{(x, q) \in \mathbb{P}^3 \times W : x \in \mathrm{sing}(q)\right\}. \tag{9.27}$$

Let $p_1 : \tilde{H}(W) \to \mathbb{P}^3$, $p_2 : \tilde{H}(W) \to W$ be the projections. Then we have $p_2(\tilde{H}(W)) = H(W)$. The surface $p_1(\tilde{H}(W))$ is denoted by $S(W)$ and is called the *Steiner surface*.

By definition we have

$$S(W) = \left\{x \in \mathbb{P}^3 : \exists\, \lambda \in W,\ \frac{\partial q(\lambda)}{\partial x_i}(x) = 0\ (i = 1, \dots, 4)\right\}$$
$$= \left\{x \in \mathbb{P}^3 : \det\left(\frac{\partial(q_1, \dots, q_4)}{\partial(x_1, \dots, x_4)}(x)\right) = 0\right\}.$$

It follows that $S(W)$ is also a quartic surface in $\mathbb{P}^3$. And it might be obvious that $\tilde{H}(W) \cong \tilde{S}(W)$.

The relation between surfaces mentioned in this section is given by the following diagram:

$$\begin{array}{ccccc} \tilde{H}(W) & \cong & \tilde{S}(W) & & \\ \downarrow p_2 & \searrow^{p_1} & \downarrow & \searrow^{2:1} & \\ H(W) & \xrightarrow{\sigma} & S(W) & & R(W). \end{array}$$

Here σ is a birational map.

As mentioned before, the $R(W)$ form a 9-dimensional family of Enriques surfaces. Also it is known that $R(W)$ contains a non-singular rational curve. Hence the minimum of the Picard numbers of their covering $K3$ surfaces is 11. In the case that the Picard number of the covering $K3$ surface is 11, it should be that $R(2) = A_1(2)$ and hence the root invariant is $(R, K) = (A_1, \{0\})$.

Remark 9.62. G. Fano studied Reye congruences and Enriques surfaces. A modern treatment of them is given by Cossec [Co]. We follow the latter. Reye congruences form a 9-dimensional family of Enriques surfaces in $\mathbb{P}^5$ of degree 10, which is a special case of the Fano models mentioned in Remark 9.50.

Remark 9.63. Artin, Mumford [AM] used a Hessian quartic surface $H(W)$ to construct a 3-dimensional algebraic variety V which is unirational but not rational. They proved the non-rationality of V to show the existence of a torsion in $H^3(V, \mathbb{Z})$. It is known that there is a natural correspondence between this torsion and a torsion in the Néron–Severi group of the Enriques surface $R(W)$ (Beauville [Be2]).

10

Application to the moduli space of plane quartic curves

In Chapter 9 we studied the periods of Enriques surfaces by associating an Enriques surface with the pair of a $K3$ surface and a fixed-point-free automorphism of order 2, and then applying the Torelli-type theorem for $K3$ surfaces. We can generalize this method by considering a pair of a $K3$ surface and its automorphism of finite order. As one of the examples in this chapter, we consider plane quartic curves. We show that the moduli space of plane quartics can be described as the quotient space of a complex ball by a discrete group by associating it with a pair of a $K3$ surface and an automorphism of order 4. We also introduce del Pezzo surfaces related to plane quartic curves.

10.1 Plane quartics and del Pezzo surfaces of degree 2

10.1.1 Plane quartics. Let (x, y, z) be homogeneous coordinates of the projective plane $\mathbb{P}^2$ and let $f(x, y, z)$ be a homogeneous polynomial of degree 4. We set

$$C = \{(x, y, z) \in \mathbb{P}^2 : f(x, y, z) = 0\}.$$

We assume that the curve C is non-singular. Then the canonical divisor $K_{\mathbb{P}^2}$ of $\mathbb{P}^2$ is linearly equivalent to -3ℓ (ℓ is a line), and we have

$$g(C) = \tfrac{1}{2}(K_{\mathbb{P}^2} \cdot C + C^2) + 1 = 3.$$

It is known that the moduli space of curves of genus 3 has dimension $3g(C) - 3 = 6$. On the other hand, the vector space V_4 of homogeneous polynomials of degree 4 in 3 variables has dimension $\binom{6}{2} = 15$. Therefore the dimension of the moduli space of plane quartic curves is equal to

$$\dim \mathbb{P}(V_4) - \dim \mathrm{PGL}_2(\mathbb{C}) = 14 - 8 = 6.$$

Now assume that a curve C of genus 3 is not hyperelliptic. Then the map

$$\Phi_{|K_C|} : C \to \mathbb{P}^2$$

defined by the linear system associated with the canonical line bundle K_C gives an embedding whose image is a plane quartic curve. Let $\mathcal{M}_3$ be the moduli space of curves of genus 3 and $\mathcal{H}_3$ the moduli space of hyperelliptic curves of genus 3. Then $\mathcal{M}_3 \setminus \mathcal{H}_3$ is the moduli space of plane quartic curves. Consider the Siegel upper half-space of degree 3,

$$\mathfrak{H}_3 = \{Z : Z \text{ is a complex symmetric matrix of degree 3 with } \operatorname{Im}(Z) > 0\}$$

which is a generalization of the upper half-plane. Let I_3 be the identity matrix and let $J = \left(\begin{smallmatrix} 0 & I_3 \\ -I_3 & 0 \end{smallmatrix}\right)$. Define

$$\mathrm{Sp}_6(\mathbb{Z}) = \{X \in M_6(\mathbb{Z}) : {}^tXJX = J\},$$

where $M_6(\mathbb{Z})$ is the set of square matrices of degree 6 with integral coefficients. The space $\mathfrak{H}_3$ is nothing but a bounded symmetric domain of type III mentioned in Remark 5.4. The group $\mathrm{Sp}_6(\mathbb{Z})$ acts on $\mathfrak{H}_3$ by

$$Z \longrightarrow (AZ+B)(CZ+D)^{-1}, \quad Z \in \mathfrak{H}_3, \quad X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_6(\mathbb{Z})$$

properly discontinuously. By associating a curve of genus 3 with its Jacobian $J(C)$ we have an injection

$$j\colon \mathcal{M}_3 \longrightarrow \mathfrak{H}_3/\mathrm{Sp}_6(\mathbb{Z}).$$

Note that $\dim \mathcal{M}_3 = \dim \mathfrak{H}_3/\mathrm{Sp}_6(\mathbb{Z}) = 6$. Moreover, it is known that there exist automorphic forms χ_{18}, χ_{140} of weights 18, 140 such that the set defined by $\chi_{18} = \chi_{140} = 0$ is the complement of the image of j and the set $\chi_{18} = 0$, $\chi_{140} \neq 0$ coincides with $j(\mathcal{H}_3)$ (Igusa [I, Lem. 11]), and hence $j(\mathcal{M}_3 \setminus \mathcal{H}_3)$ and $j(\mathcal{M}_3)$ are Zariski open sets in $\mathfrak{H}_3/\mathrm{Sp}_6(\mathbb{Z})$.

Now let C be a non-singular plane quartic curve. The intersection number of a line ℓ in $\mathbb{P}^2$ and C is 4. A line ℓ is said to be a *bitangent line* of C if ℓ touches C at 2 points. It is classically known that C has 28 bitangent lines. We will introduce this fact in view of del Pezzo surfaces.

10.1.2 Del Pezzo surfaces.

Definition 10.1. Let Y be a non-singular algebraic surface. Then Y is said to be a *del Pezzo surface* if the anti-canonical divisor is ample. We call $(-K_Y)^2$ the *degree* of Y.

By definition, del Pezzo surfaces are rational. In fact, it is known that any del Pezzo surface is isomorphic to $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or the surface obtained from $\mathbb{P}^2$ by blowing up n points $p_1, \ldots, p_n$ ($n \leq 8$) in general position. Here n points $p_1, \ldots, p_n$ are *in general position* if the following three conditions are satisfied:

(i) There are no lines passing through any 3 points p_i, p_j, p_k ($i \neq j \neq k \neq i$).

(ii) If $n \geq 6$, then there are no conics passing through any 6 points among them.

(iii) If $n = 8$, then there are no cubic curves passing through the 8 points and with a singularity at one of the 8 points.

These conditions prohibit the existence of a curve on a del Pezzo surface such that its intersection number with the anti-canonical divisor is non-positive. The projective plane $\mathbb{P}^2$ is a del Pezzo surface of degree 9, $\mathbb{P}^1 \times \mathbb{P}^1$ one of degree 8, and if Y is obtained by blowing up n points in general position, then its degree is $9 - n$. A non-singular rational curve C on Y is called a *line* on Y if it satisfies $(-K_Y) \cdot C = 1$. However, in the case of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, a line is defined as a non-singular rational curve with the intersection number 1 with the divisor $\frac{1}{3}(-K_{\mathbb{P}^2})$ or $\frac{1}{2}(-K_{\mathbb{P}^1 \times \mathbb{P}^1})$, respectively. In these cases, infinitely many lines exist.

Let $\pi\colon Y_n \to \mathbb{P}^2$ be the surface obtained by blowing up at n points $p_1, \ldots, p_n$ of $\mathbb{P}^2$. Let C be an irreducible curve on Y_n. It follows from

$$g(C) = \frac{1}{2}(K_{Y_n} \cdot C + C^2) + 1$$

that $C \cong \mathbb{P}^1$ is a line if and only if $C^2 = -1$. Moreover, $\mathrm{Pic}(Y_n) \cong H^2(Y_n, \mathbb{Z})$ is a unimodular lattice of signature $(1, n)$. Since this lattice is odd by the existence of an exceptional curve, we have obtained

$$H^2(Y_n, \mathbb{Z}) \cong I_+ \oplus I_-^{\oplus n} \tag{10.1}$$

by Theorem 1.22. Let e_0 be the class of the pullback of a line in $\mathbb{P}^2$ and let e_i $(1 \leq i \leq n)$ be that of the exceptional curve over p_i. Then we can easily prove that

$$e_0^2 = 1, \quad e_i^2 = -1 \ \ (1 \leq i \leq n), \quad \langle e_i, e_j \rangle = 0 \ \ (i \neq j),$$

and $e_0, e_1, \ldots, e_n$ is a basis of the lattice $H^2(Y_n, \mathbb{Z})$. The canonical divisor of Y_n is given by

$$-K_{Y_n} = 3e_0 - e_1 - \cdots - e_n.$$

Moreover, it is known that lines on Y_n are given as follows. For simplicity, we state only the cases $n \geq 5$ (in the next subsection we will give a proof in the case of $n = 7$):

(1) in the case $n = 5$,

$$e_i \ (1 \leq i \leq 5),\ e_0 - e_i - e_j \ (1 \leq i < j \leq 5),$$
$$2e_0 - e_1 - e_2 - e_3 - e_4 - e_5;$$

(2) in the case $n = 6$,

$$e_i \ (1 \le i \le 6), \quad e_0 - e_i - e_j \ (1 \le i < j \le 6),$$
$$2e_0 - e_1 - \cdots - e_6 + e_i \ (1 \le i \le 6);$$

(3) in the case $n = 7$,

$$e_i \ (1 \le i \le 7), \quad e_0 - e_i - e_j \ (1 \le i < j \le 7),$$
$$2e_0 - e_1 - \cdots - e_7 + e_i + e_j \ (1 \le i < j \le 7),$$
$$3e_0 - e_1 - \cdots - e_7 - e_i \ (1 \le i \le 7);$$

(4) in the case $n = 8$,

$$e_i \ (1 \le i \le 8), \quad e_0 - e_i - e_j \ (1 \le i < j \le 8),$$
$$2e_0 - e_1 - \cdots - e_8 + e_i + e_j + e_k \ (1 \le i < j < k \le 8),$$
$$3e_0 - e_1 - \cdots - e_8 - e_i + e_j \ (1 \le i \ne j \le 8),$$
$$4e_0 - e_1 - \cdots - e_8 - e_i - e_j - e_k \ (1 \le i < j < k \le 8),$$
$$5e_0 - 2e_1 - \cdots - 2e_8 + e_i + e_j \ (1 \le i < j \le 8),$$
$$6e_0 - 2e_1 - \cdots - 2e_8 - e_i \ (1 \le i \le 8).$$

If we denote by ℓ_n the number of lines on Y_n, then we obtain Table 10.1.

Table 10.1. The number of lines

n	1	2	3	4	5	6	7	8
ℓ_n	1	3	6	10	16	27	56	240

Exercise 10.2. (1) Make the list of lines in the cases $n = 1, \ldots, 4$.

(2) Show that the dual graph of 10 lines on a del Pezzo surface Y_4 of degree 5 is the Petersen graph (Figure 9.2).

It is important to study the anti-canonical models of del Pezzo surfaces. The results are the following (in the next subsection we will give a proof of the case of del Pezzo surfaces of degree 2).

Proposition 10.3. *Let Y_n be a del Pezzo surface of degree* $9 - n$:

(1) *In the case $n \le 6$, $-K_{Y_n}$ is very ample and the anti-canonical map*

$$\Phi_{|-K_{Y_n}|} : Y_n \to \mathbb{P}^{9-n}$$

gives an embedding whose image is as in the following:

- *Y_4 is a quintic surface in* $\mathbb{P}^5$;
- *Y_5 is a complete intersection of two quadric hypersurfaces in* $\mathbb{P}^4$;
- *Y_6 is a cubic surface in* $\mathbb{P}^3$.

(2) *In the cases $n = 7, 8$, the following hold*:

- $\Phi_{|-K_{Y_7}|} : Y_7 \longrightarrow \mathbb{P}^2$ *is a double covering branched along a non-singular quartic curve*;
- $\Phi_{|-K_{Y_8}|}$ *has a base point. After blowing up the base point, we obtain a rational elliptic surface. The image of the anti-bicanonical map*

$$\Phi_{|-2K_{Y_8}|} : Y_8 \longrightarrow \mathbb{P}^3$$

is a quadric cone Q and $\Phi_{|-2K_{Y_8}|}$ is a double covering branched along a curve belonging to $|\mathcal{O}_Q(3)|$ (for a quadric cone, see Section 9.4.5).

Now we calculate the dimension of the moduli space of del Pezzo surfaces Y_n. To do this it suffices to determine the dimension of the moduli space of n points. Recall that any distinct 4 points on $\mathbb{P}^2$ can be transformed to

$$(1:0:0), \quad (0:1:0), \quad (0:0:1), \quad (1:1:1)$$

by a projective transformation. Thus the isomorphism class of Y_n $(1 \leq n \leq 4)$ is unique. For $p_5, \ldots, p_n$ $(5 \leq n \leq 8)$, we can choose them from an open set of $\mathbb{P}^2$ successively. Thus the dimension of the moduli space is $2(n-4)$.

Note that $(-K_{Y_n})^2 = 9 - n$. The orthogonal complement of K_{Y_n} in $H^2(Y_n, \mathbb{Z})$ is denoted by Q_n. Then we can prove that

$$Q_4 \cong A_4, \quad Q_5 \cong D_5, \quad Q_6 \cong E_6, \quad Q_7 \cong E_7, \quad Q_8 \cong E_8.$$

It is surprising that root lattices appear suddenly in this situation. The number r_n of roots (an element of norm -2) in each case is given in Table 10.2.

Table 10.2. The number of roots

n	1	2	3	4	5	6	7	8
r_n	0	2	8	20	40	72	126	240

The roots up to ± are given as follows:

(1) in the cases $n = 4, 5$,

$$e_i - e_j \ (1 \le i < j \le n), \quad e_0 - e_i - e_j - e_k \ (1 \le i < j < k \le n);$$

(2) in the case $n = 6$,

$$e_i - e_j \ (1 \le i < j \le 6), \quad e_0 - e_i - e_j - e_k \ (1 \le i < j < k \le 6),$$
$$2e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6;$$

(3) in the case $n = 7$,

$$e_i - e_j \ (1 \le i < j \le 7), \quad e_0 - e_i - e_j - e_k \ (1 \le i < j \le 7),$$
$$2e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_i \ (1 \le i \le 7);$$

(4) in the case $n = 8$,

$$e_i - e_j \ (1 \le i < j \le 8), \quad e_0 - e_i - e_j - e_k \ (1 \le i < j < k \le 8),$$
$$2e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8 + e_i + e_j \ (1 \le i < j \le 8),$$
$$3e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8 - e_i \ (1 \le i \le 8).$$

The fact that n points $p_1, \ldots, p_n$ are in general position is equivalent to the condition that no root can be represented by an effective divisor

Exercise 10.4. Among the 27 lines on a del Pezzo surface Y_3 of degree 3, there exist 12 lines

$$\ell_1, \ldots, \ell_6, m_1, \ldots, m_6$$

satisfying the following property: 6 lines $\ell_1, \ldots, \ell_6$ do not meet, 6 lines $m_1, \ldots, m_6$ do not meet, and ℓ_i meets m_j if and only if $i \ne j$. Such a set of lines is called Schläfli's double six. For example,

$$\{e_i \ (1 \le i \le 6),\ 2e_0 - e_1 - \cdots - e_6 + e_i \ (1 \le i \le 6)\}$$

is a Schläfli double six. Then show the following:

(1) For a set of 6 lines $\{\ell_1, \ldots, \ell_6\}$ not meeting each other, there exists a unique root α satisfying $\langle \alpha, \ell_i \rangle = 1 \ \forall i$. Moreover, if we denote by s_α the reflection associated with α, then $\{\ell_1, \ldots, \ell_6\}$ and $\{s_\alpha(\ell_1), \ldots, s_\alpha(\ell_6)\}$ is a Schläfli double six.

(2) Show that there exist 36 Schläfli double sixes.

There are interesting subjects, for example cubic surfaces and 27 lines etc., but we will restrict ourselves to the topic of del Pezzo surfaces of degree 2 in the next subsection.

10.1.3 Plane quartics and del Pezzo surfaces of degree 2. In the following we denote by Y a del Pezzo surface of degree 2. First of all, we show that the lines on Y are exactly the 56 given in the previous subsection.

Proposition 10.5. *The lines on* Y *are*

$$e_i \ (1 \leq i \leq 7), \quad e_0 - e_i - e_j \ (1 \leq i < j \leq 7),$$

$$2e_0 - e_1 - \cdots - e_7 + e_i + e_j \ (1 \leq i < j \leq 7),$$

$$3e_0 - e_1 - \cdots - e_7 - e_i \ (1 \leq i \leq 7).$$

Proof. Recall that a line on Y is a non-singular rational curve E with $-K_Y \cdot E = 1$. Then $E^2 = -1$. The 56 classes above are exceptional curves over the 7 blowing-up points $p_1, \ldots, p_7$ in general position in $\mathbb{P}^2$, the proper transform of the line passing through p_i and p_j, that of the conic passing through 5 points among $p_1, \ldots, p_7$, and that of the cubic curve passing through the 7 points $p_1, \ldots, p_7$ and having a node at one of them. Obviously, these are lines on Y.

Conversely, let E be a line and let

$$E = ae_0 - \sum_{i=1}^{7} b_i e_i \quad (a, b_i \in \mathbb{Z}).$$

If $a = 0$, then $E = e_i$ for some i and hence we may assume that now $a \neq 0$. Then a is the degree of the image E_0 of E by the blowing down $\pi\colon Y \to \mathbb{P}^2$ and hence $a > 0$. On the other hand, since b_i is the multiplicity of E_0 at p_i, we have $b_i \geq 0$. Moreover, we have

$$-K_Y \cdot E = 3a - \sum_i b_i = 1, \quad E^2 = a^2 - \sum_i b_i^2 = -1.$$

By the Cauchy–Schwarz inequality, we obtain

$$\left(\sum_i b_i\right)^2 \leq 7 \sum_i b_i^2.$$

Thus we have

$$(3a - 1)^2 \leq 7(a^2 + 1), \quad \text{that is, } a^2 - 3a - 3 \leq 0,$$

and hence $a = 1, 2, 3$. If $a = 1$, then by $\sum b_i = 2$ and $\sum b_i^2 = 2$ we have $E = e_0 - e_i - e_j$. If $a = 2$, then by $\sum b_i = 5$ and $\sum b_i^2 = 5$ we have $E = 2e_0 - e_1 - \cdots - e_7 + e_i + e_j$. Finally, if $a = 3$, then by $\sum b_i = 8$ and $\sum b_i^2 = 10$ we obtain $E = 3e_0 - e_1 - \cdots - e_7 - e_i$. □

Next we give a proof of Proposition 10.3.

Proposition 10.6. *The rational map $\Phi_{|-K_Y|}$ is holomorphic and gives a double covering $\Phi_{|-K_Y|}\colon Y \to \mathbb{P}^2$ of $\mathbb{P}^2$ branched along a non-singular quartic curve C. Conversely, the double covering of $\mathbb{P}^2$ branched along a non-singular quartic curve is a del Pezzo surface of degree 2.*

Proof. The linear system $|-K_Y|$ is the pullback of the linear system on $\mathbb{P}^2$ consisting of cubic curves passing through $p_1, \dots, p_7$ and hence has dimension 2. Moreover, the following hold:

- $|-K_Y|$ has no base points.
- For any $p \in \mathbb{P}^2$, the linear system $|3\ell - p_1 - \cdots - p_7 - p|$ has dimension 1.
- Let $E_1, E_2 \in |3\ell - p_1 - \cdots - p_7 - p|$ be independent elements. Since $E_1 \cap E_2$ consists of 9 points with multiplicities, there exists a point $p' \in \mathbb{P}^2$ with $E_1 \cap E_2 = \{p, p', p_1, \dots, p_7\}$.

The map $\Phi_{|-K_Y|}$ sends $p \in \mathbb{P}^2$ to the 1-dimensional subspace $|-K_Y - p| \subset |-K_Y|$. It follows that $\Phi_{|-K_Y|}\colon Y \to \mathbb{P}^2$ is a holomorphic map of degree 2. Let C be the branch curve of $\Phi_{|-K_Y|}$. The pullback of a general line $\ell \subset \mathbb{P}^2$ by $\Phi_{|-K_Y|}$ is an elliptic curve. It follows from the Hurwitz formula (e.g., Griffiths, Harris [GH, Chap. 2]) that the intersection number of ℓ and C is 4, and hence C is a quartic curve. If C has a singular point, then Y has a singularity at its inverse image, which is a contradiction. Thus C is non-singular.

Conversely, let $\pi\colon Y \to \mathbb{P}^2$ be the double covering branched along a non-singular quartic curve C. Then we have

$$K_Y = \pi^*(K_{\mathbb{P}^2} + 2\ell) = \pi^*(-\ell).$$

This implies that $-K_Y$ has positive intersection number with any irreducible curve and hence is ample. By $(-K_Y)^2 = (\pi^*(\ell))^2 = 2\ell^2 = 2$, we prove that Y is of degree 2. □

Let Y be a del Pezzo surface of degree 2. Recall that $H^2(Y, \mathbb{Z}) \cong I_+ \oplus I_-^{\oplus 7}$ (see equation (10.1)). Let ι be the covering transformation of the double covering $\Phi = \Phi_{|-K_Y|}$. Let ℓ be a line in $\mathbb{P}^2$. Then we have $\Phi^*(\ell) = -K_Y$. Since $H^2(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}\ell$, the invariant sublattice of $H^2(Y, \mathbb{Z})$ under the action of ι^* is generated by $-K_Y$ and ι^* acts on its orthogonal complement $(-K_Y)^\perp$ by multiplication by -1.

Lemma 10.7. *The ι^*-invariant sublattice of $H^2(Y, \mathbb{Z})$ is $\langle -K_Y \rangle = \langle 2 \rangle$ and its orthogonal complement is isomorphic to E_7.*

Proof. Since $(-K_Y)^2 = 2$, $-K_Y$ is primitive in $H^2(Y, \mathbb{Z})$. In the previous Section 10.1.2, we gave 126 roots ((-2)-elements) contained in the orthogonal complement

Q_7 of $-K_Y$ in $H^2(Y,\mathbb{Z})$. There are 7 elements among them generating E_7. For example,

$$e_0 - e_1 - e_2 - e_3, \quad e_i - e_{i+1} \ (1 \le i \le 6)$$

are such elements. Therefore $E_7 \subset Q_7$. On the other hand, since Q_7 is the orthogonal complement of the primitive sublattice $\langle -K_Y \rangle$ in the unimodular lattice $H^2(Y,\mathbb{Z})$, we have

$$d(Q_7) = d(\langle -K_Y \rangle) = 2$$

(Lemma 1.31 for even lattices holds in this case). The fact $d(Q_7) = d(E_7) = 2$ implies $Q_7 = E_7$. □

The automorphism ι of Y fixes $-K_Y$ and hence preserves lines. By the equations

$$-K_Y = e_i + \left(3e_0 - \sum_{k=1}^{7} e_k - e_i\right) = e_0 - e_i - e_j + \left(2e_0 - \sum_{k=1}^{7} e_k + e_i + e_j\right),$$

we obtain

$$\begin{cases} \iota^*(e_i) = 3e_0 - \sum_{k=1}^{7} e_k - e_i, \\ \iota^*(e_0 - e_i - e_j) = 2e_0 - \sum_{k=1}^{7} e_k + e_i + e_j. \end{cases} \tag{10.2}$$

In other words, 56 lines on Y are divided into pairs of two lines such that the intersection number of two lines in each pair is 2 and two lines are interchanged by ι. Thus the image of two lines by Φ is a line in $\mathbb{P}^2$. Since the pullback of this line to Y splits into two lines, it is a bitangent line of the branch curve. Thus we conclude the following.

Proposition 10.8. *Let C be a non-singular plane quartic curve. Then C has exactly* 28 *bitangents.*

Exercise 10.9. We assume the fact that any del Pezzo surface of degree 3 is a non-singular cubic surface S in $\mathbb{P}^3$. Take a general point $p_0 \in S$ and let $\tilde{S}$ be the blowing up of S at p_0. Assume that $\tilde{S}$ is a del Pezzo surface (of degree 2). Then show that the projection $\mathbb{P}^3 \setminus \{p_0\} \to \mathbb{P}^2$ from p_0 induces a double covering $\pi\colon \tilde{S} \to \mathbb{P}^2$. Moreover, show that the branch divisor $C \subset \mathbb{P}^2$ is a non-singular quartic curve.

Remark 10.10. The references for del Pezzo surfaces are Demazure [De], Dolgachev, Ortland [DO], Hartshorne [Har].

10.2 $K3$ surfaces associated with plane quartics

Let C be a plane quartic curve given by

$$C = \{(x, y, z) \in \mathbb{P}^2 : f(x, y, z) = 0\}.$$

Let t be a new variable and let (x, y, z, t) be homogeneous coordinates of $\mathbb{P}^3$. Now we define

$$X = \{(x, y, z, t) \in \mathbb{P}^3 : t^4 = f(x, y, z)\}.$$

Since C is non-singular, X is also a non-singular quartic surface and, in particular, a $K3$ surface. We call X the $K3$ *surface associated with a plane quartic curve*. The projection

$$\mathbb{P}^3 \longrightarrow \mathbb{P}^2, \quad (x, y, z, t) \longrightarrow (x, y, z)$$

from the point $(0, 0, 0, 1)$ induces a holomorphic map

$$\pi \colon X \longrightarrow \mathbb{P}^2$$

which is a 4-cyclic covering branched along C. The covering transformation σ is given by

$$\sigma \colon (x, y, z, t) \longrightarrow (x, y, z, \zeta t)$$

where ζ is a primitive 4th root of unity. The quotient surface $X/\langle\tau\rangle$ of X by an automorphism $\tau = \sigma^2$ of order 2 is a double covering of $\mathbb{P}^2$ branched along C, which is nothing but a del Pezzo surface Y of degree 2. Thus we have the following diagram:

$$\begin{array}{ccc} X & & \\ \downarrow \pi_2 & \searrow^{\pi} & \\ Y & \xrightarrow{\pi_1} & \mathbb{P}^2. \end{array}$$

If $\sigma^*(\omega_X) = -\omega_X$, then τ is a symplectic automorphism and has a fixed curve. This contradicts Lemma 8.25. Thus we have the following.

Lemma 10.11. *The fixed point set of σ is the inverse image of C which is a non-singular curve of genus 3. Moreover, we have $\sigma^*(\omega_X) = \pm\sqrt{-1} \cdot \omega_X$.*

We study the action of σ^* on $H^2(X, \mathbb{Z})$. To do this, we first consider the action of τ^*. We know $H^2(X, \mathbb{Z})^{\tau^*} = \pi_2^*(H^2(Y, \mathbb{Z}))$ (see equation (8.7) for $H^2(X, \mathbb{Z})^{\tau^*}$), and $\langle\pi_2^*(a), \pi_2^*(b)\rangle = 2\langle a, b\rangle$ for $a, b \in H^2(Y, \mathbb{Z})$, because π_2 is a double covering. Hence, by (10.1), we have

$$H^2(X, \mathbb{Z})^{\tau^*} \cong H^2(Y, \mathbb{Z})(2) = \langle 2\rangle \oplus \langle -2\rangle^{\oplus 7}.$$

Now we define

$$L(X)_+ = H^2(X,\mathbb{Z})^{\tau^*}, \quad L(X)_- = \left\{x \in H^2(X,\mathbb{Z}) : \tau^*(x) = -x\right\}. \tag{10.3}$$

Then $L(X)_+$ and $L(X)_-$ are orthogonal complements of each other in $H^2(X,\mathbb{Z})$. It follows from Theorem 1.32 that $A_{L(X)_+} \cong A_{L(X)_-}$. Thus $L(X)_-$ is a 2-elementary lattice with signature $(2,12)$, $\ell = 2^8$, $\delta = 1$, and hence by Proposition 1.39 we have

$$L(X)_- \cong U \oplus U(2) \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}.$$

By applying Lemma 10.7 we have obtained the following.

Lemma 10.12. (1) $L(X)_+ \cong \langle 2\rangle \oplus \langle -2\rangle^{\oplus 7}$, $L(X)_- \cong U \oplus U(2) \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}$.

(2) *There exists an isomorphism $H^2(X,\mathbb{Z})^{\langle\sigma^*\rangle} \cong \langle 4\rangle$ of lattices such that the orthogonal complement of $H^2(X,\mathbb{Z})^{\langle\sigma^*\rangle}$ in $L(X)_+$ is isomorphic to $E_7(2)$.*

Remark 10.13. The lattice $L(X)_+$ corresponds to Proposition 8.29(3), $(r,\ell,\delta) = (8,8,1)$.

Definition 10.14. We set $L_+ = \langle 2\rangle \oplus \langle -2\rangle^{\oplus 7}$, $L_- = U \oplus U(2) \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}$.

Next we study the action of σ^* on $H^2(X,\mathbb{Z})$. We denote by $\tilde{e}_0, \tilde{e}_1, \dots, \tilde{e}_7$ the pullback of the basis $e_0, e_1, \dots, e_7$ of $H^2(Y,\mathbb{Z})$ by π_2. Then they give a basis of $L(X)_+$. The action of σ^* on $L(X)_+$ is determined by that of ι^* on $H^2(Y,\mathbb{Z})$. By setting

$$\tilde{\kappa} = 3\tilde{e}_0 - \tilde{e}_1 - \tilde{e}_2 - \tilde{e}_3 - \tilde{e}_4 - \tilde{e}_5 - \tilde{e}_6 - \tilde{e}_7,$$

and using formula (10.2), we have

$$\sigma^*(\tilde{\kappa}) = \tilde{\kappa}, \quad \sigma^*(\tilde{e}_i) = \tilde{\kappa} - \tilde{e}_i, \quad \sigma^*(\tilde{e}_0 - \tilde{e}_i - \tilde{e}_j) = \tilde{\kappa} - \tilde{e}_0 + \tilde{e}_i + \tilde{e}_j.$$

The group $A_{L(X)_+} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 8}$ is generated by $\{\frac{1}{2}\tilde{e}_i \bmod L(X)_+ \ (0 \le i \le 7)\}$. Moreover, the action of σ^* on $A_{L(X)_+}$, by omitting $\bmod L(X)_+$, is given by

$$\sigma^*\left(\tfrac{1}{2}\tilde{e}_i\right) = \tfrac{1}{2}\tilde{e}_i + \tfrac{1}{2}\tilde{\kappa}. \tag{10.4}$$

Definition 10.15. We fix an isomorphism between L_+ and $L(X)_+$ and define an automorphism ρ_+ of L_+ by $\sigma^*|L(X)_+$.

On the other hand, the action of σ^* on $L(X)_-$ is of order 4 and its fixed point is only 0 (Lemma 8.12). We give this action concretely in the following (Lemma 10.16). We fix an orthogonal decomposition

$$L_- = U \oplus U(2) \oplus D_4 \oplus D_4 \oplus A_1 \oplus A_1$$

of L_- and define an action of an automorphism of order 4. First of all, let e, f be a basis of U with $e^2 = f^2 = 0$, $\langle e, f\rangle = 1$, and let e', f' be a basis of $U(2)$ with $(e')^2 = (f')^2 = 0$, $\langle e', f'\rangle = 2$. An automorphism ρ_0 of $U \oplus U(2)$ is defined by

$$\rho_0(e) = -e - e', \quad \rho_0(f) = f - f', \quad \rho_0(e') = e' + 2e, \quad \rho_0(f') = 2f - f'$$

which fixes only 0 by $\rho_0^2 = -1$. Moreover, the action of ρ_0 on $A_{U\oplus U(2)}$ is trivial. Next recall that

$$D_4 \cong \left\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 : x_1 + x_2 + x_3 + x_4 \equiv 0 \,(\mathrm{mod} 2)\right\}$$

(see Exercise 1.7). We define an automorphism ρ_1 of D_4 by

$$\rho_1(x_1, x_2, x_3, x_4) = (x_2, -x_1, x_4, -x_3)$$

which satisfies $\rho_1^2 = -1$ and hence does not fix any non-zero vector in D_4. It is easily checked that the action of ρ_1 on A_{D_4} is trivial. Finally, let u_1, u_2 respectively be a basis of the first and the second factor of $A_1 \oplus A_1$. Then define an automorphism ρ_2 by

$$\rho_2(u_1) = u_2, \quad \rho_2(u_2) = -u_1$$

which also satisfies $\rho_2^2 = -1$. Let ρ_- be the automorphism of $L_- = U \oplus U(2) \oplus D_4 \oplus D_4 \oplus A_1 \oplus A_1$ defined by

$$\rho_- = \rho_0 \oplus \rho_1 \oplus \rho_1 \oplus \rho_2.$$

Then it satisfies $(\rho_-)^2 = -1$ and does not fix any non-zero vector in L_-.

Now we choose a generator of each direct summand of

$$A_{L_-} \cong A_{U(2)} \oplus A_{D_4} \oplus A_{D_4} \oplus A_{A_1 \oplus A_1}$$

as follows. As a generator α_1, α_2 of $A_{U(2)}$ we take

$$\alpha_1 = e'/2 \bmod U(2), \quad \alpha_2 = f'/2 \bmod U(2);$$

as a generator $\beta_1, \beta_2, \beta_3, \beta_4$ of $A_{D_4} \oplus A_{D_4}$ we take

$$\beta_1 = \tfrac{1}{2}(1,1,1,1) \bmod D_4, \quad \beta_2 = (-1,0,0,0) \bmod D_4,$$

and $\{\beta_3, \beta_4\}$ is a copy of $\{\beta_1, \beta_2\}$; as a generator δ_1, δ_2 of $A_{A_1\oplus A_1}$ we take

$$\delta_1 = u_1/2 \bmod A_1 \oplus A_1, \quad \delta_2 = u_2/2 \bmod A_1 \oplus A_1.$$

Let $\bar{\rho}_{\pm}$ be the automorphisms of $A_{L_\pm}$ induced by $\rho_{\pm}$, respectively. It follows that $\bar{\rho}_-$ fixes α_i, β_j and $\bar{\rho}_-(\delta_1) = \delta_2$. Moreover, we define an orthogonal basis $v_0, v_1, \dots, v_7$ of A_{L_-} with respect to b_{L_-} by

$$v_0 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \delta_1, \quad v_1 = \beta_1 + \delta_1, \quad v_2 = \beta_2 + \delta_1,$$
$$v_3 = \alpha_1 + \beta_1 + \beta_2 + \delta_1, \quad v_4 = \alpha_2 + \beta_1 + \beta_2 + \delta_1,$$
$$v_5 = \beta_3 + \delta_2, \quad v_6 = \beta_4 + \delta_2, \quad v_7 = \beta_3 + \beta_4 + \delta_2.$$

Then we can check that $q_{L_-}(v_0) = -\frac{1}{2}$, $q_{L_-}(v_i) = \frac{1}{2}$ $(i = 1, \dots, 7)$, and

$$\bar{\rho}_-(v_i) = v_i + \delta_1 + \delta_2. \tag{10.5}$$

Combining (10.4), (10.5), we can get an isomorphism $\gamma\colon A_{L_+} \cong A_{L_-}$ satisfying

$$q_{L_+} = -q_{L_-} \circ \gamma, \quad \gamma \circ \bar{\rho}_+ = \bar{\rho}_- \circ \gamma. \tag{10.6}$$

Thus we have obtained the following.

Lemma 10.16. *The isomorphism $\rho_+ \oplus \rho_- \in \mathrm{O}(L_+ \oplus L_-)$ can be extended to an isomorphism ρ of L.*

Proof. The assertion follows from equality (10.6) and Corollary 1.33. □

Since ρ_- and $\rho_-^2 = -1$ do not fix any non-zero vector in L_-, the eigenspace decomposition of ρ_- is given by

$$L_- \otimes \mathbb{C} = V_{\sqrt{-1}} \oplus V_{-\sqrt{-1}}, \tag{10.7}$$

where $V_{\pm\sqrt{-1}}$ is the eigenspace of ρ_- for the eigenvalue $\pm\sqrt{-1}$. Since ρ_- is defined over $\mathbb{Z}$, $V_{\sqrt{-1}}$ and $V_{-\sqrt{-1}}$ are interchanged by the complex conjugation, and hence we have

$$\dim V_{\sqrt{-1}} = \dim V_{-\sqrt{-1}} = 6.$$

Lemma 10.17. *The action of σ^* on $H^2(X, \mathbb{Z})$ is conjugate to that of ρ on L.*

Proof. We take $\omega \in V_{\sqrt{-1}}$ satisfying the following conditions:

(1) $\langle \omega, \bar{\omega} \rangle > 0$.

(2) $\langle \omega, \delta \rangle \neq 0$ for any $\delta \in L_-$, $\delta \neq 0$.

Then it follows from

$$\langle \omega, \omega \rangle = \langle \rho(\omega), \rho(\omega) \rangle = \left\langle \sqrt{-1}\,\omega, \sqrt{-1}\,\omega \right\rangle = -\langle \omega, \omega \rangle$$

that $\langle \omega, \omega \rangle = 0$. By the surjectivity of the period map of $K3$ surfaces, there exists a marked $K3$ surface $(X', \alpha_{X'})$ with $\alpha_{X'}(\omega_{X'}) = \omega$. Here $\omega_{X'}$ is a holomorphic 2-form on X'. By definition,

$$\phi = \alpha_{X'}^{-1} \circ \rho \circ \alpha_{X'}$$

preserves $\mathbb{C}\cdot\omega_{X'}$. By the condition of ω, we have $S_{X'} \cong L_+$. The ϕ-invariant sublattice of $S_{X'}$ is $\langle 4 \rangle$, and its orthogonal complement in $S_{X'}$ is $E_7(2)$ (Lemma 10.12) which does not contain elements of norm -2 (Lemma 10.12). Thus the group $\langle \phi \rangle$ satisfies the assumption in Lemma 8.24. Therefore there exists a $w \in W(X')$ such that $w^{-1} \circ \phi \circ w$ is represented by an automorphism g of X'. The sublattice fixed by $(g^*)^2$ is isomorphic to L_+ and has rank 8, $\ell = 8$, $\delta = 1$ (for l, δ, see Proposition 1.39 and the definition before it). It follows from Proposition 8.29 that the set of fixed points of g^2 is a non-singular curve C of genus 3, and hence the quotient surface $R = X'/\langle g^2 \rangle$ is a non-singular rational surface. Let f be the automorphism of R induced by g and let $\bar{C}$ be the image of C. Since the set of fixed points of g is contained in C, that of f is a subset of $\bar{C}$. By applying the Lefschetz fixed point formula (Ueno [U]) to f, we obtain

$$\mathrm{trace}(f^*|H^*(R, \mathbb{Z})) = 2 + (1 - 7) = -4$$

and hence the set of fixed points of f is not finite. Thus the set of fixed points of f coincides with the non-singular curve $\bar{C}$ of genus 3. Since $\mathrm{Pic}(R/\langle f \rangle) = \mathbb{Z}$, we have $R/\langle f \rangle \cong \mathbb{P}^2$ and the image of C is a plane quartic curve. Thus we have proved that X' is a 4-cyclic covering of $\mathbb{P}^2$ branched along a plane quartic curve. Since any two plane quartic curves can be deformed, the covering $X \to \mathbb{P}^2$ can deform to $X' \to \mathbb{P}^2$. Now we have finished the proof. □

10.3 The moduli space of plane quartics and a complex ball

We define the period domain for a pair of a $K3$ surface X associated with a plane quartic curve and the covering transformation σ by

$$\mathcal{B} = \left\{ \omega \in \mathbb{P}(V_{\sqrt{-1}}) : \langle \omega, \bar{\omega} \rangle > 0 \right\}. \tag{10.8}$$

Here $V_{\sqrt{-1}}$ is the eigenspace defined by formula (10.7). Since the signature of L_- is $(2, 12)$, the hermitian form $\langle \omega, \bar{\omega} \rangle$ has the signature $(1, 6)$. Therefore, by choosing suitable coordinates $(z_0, z_1, \ldots, z_6) \in V_{\sqrt{-1}}$ we have

$$\langle \omega, \bar{\omega} \rangle = |z_0|^2 - |z_1|^2 - \cdots - |z_6|^2.$$

Since $z_0 \neq 0$ we have

$$\mathcal{B} \cong \left\{(z_1, \ldots, z_6) \in \mathbb{C}^6 : |z_1|^2 + \cdots + |z_6|^2 < 1\right\}$$

which is a 6-dimensional complex ball. The complex ball $\mathcal{B}$ is nothing but a bounded symmetric domain of type $\mathrm{I}_{1,6}$ (see Remark 5.4). Now define

$$\mathcal{H} = \bigcup_{\delta \in L_-,\, \delta^2 = -2} \mathcal{H}_\delta, \quad \mathcal{H}_\delta = \{\omega \in \mathcal{B} : \langle \omega, \delta \rangle = 0\}. \tag{10.9}$$

Then for the same reason as in the case of Lemma 9.18, $\mathcal{B} \setminus \mathcal{H}$ is the period domain for $K3$ surfaces associated with non-singular plane quartic curves. Moreover, the following holds.

Lemma 10.18. *For $r \in L$, $r^2 = -2$, we set $\mathcal{H}_r = \{\omega \in \mathcal{B} : \langle \omega, r \rangle = 0\}$. If $r \in (L^{\langle \rho \rangle})^\perp$ and $\mathcal{H}_r \neq \emptyset$, then there exists a $\delta \in L_-$, $\delta^2 = -2$ satisfying $\mathcal{H}_r = \mathcal{H}_\delta$.*

Proof. Since the orthogonal complement of $(L^{\langle \rho \rangle})^\perp$ in L_+ is isomorphic to $E_7(2)$, it contains no vectors of norm -2. Hence $r \notin L_+$. Now we can set

$$r = r_+ + r_-, \quad r_+ \in L_+^*, \quad r_- \in L_-^*$$

with $r_- \neq 0$. Since $L^{\langle \rho \rangle} \cong \langle 4 \rangle$ is positive definite and the signature of L_+ is $(1, 7)$, we have $r_+^2 \leq 0$. If $r_-^2 \geq 0$, then $\mathcal{H}_r = \emptyset$ and hence we have $r_-^2 < 0$. Note that $L_\pm$ is 2-elementary, that is, $L_+^*/L_+ \cong L_-^*/L_-$ is a 2-elementary abelian group, and hence $2r_+, 2r_- \in L$. Thus the following cases occur:

$$(r_+^2, r_-^2) = (0, -2), \ (-1, -1), \ \left(-\tfrac{3}{2}, -\tfrac{1}{2}\right), \ \left(-\tfrac{1}{2}, -\tfrac{3}{2}\right).$$

If $r_-^2 = -2$, then $r = r_- \in L_-$ and hence we can take $\delta = r$. If $r_-^2 = -\frac{1}{2}$, then $\delta = 2r_- \in L_-$ is the desired one. If $r_-^2 = -\frac{3}{2}$, then we have $r_+^2 = -\frac{1}{2}$ and hence $2r_+ \in E_7(2)$, which contradicts the fact that $E_7(2)$ contains no elements of norm -2. Finally, consider the case $r_-^2 = -1$. By the equation

$$\langle r_-, \rho(r_-) \rangle = \langle \rho(r_-), \rho^2(r_-) \rangle = -\langle r_-, \rho(r_-) \rangle,$$

we obtain $\langle r_-, \rho(r_-) \rangle = 0$. Since $\rho(r_+) = -r_+$, $r + \rho(r) = r_- + \rho(r_-)$ is an element in L_- of norm -2. For $\omega \in \mathcal{B}$, we have

$$\langle \omega, r_- \rangle = \langle \rho(\omega), \rho(r_-) \rangle = \sqrt{-1}\, \langle \omega, \rho(r_-) \rangle,$$

$$\langle \omega, r_- + \rho(r_-) \rangle = \langle \rho(\omega), \rho(r_-) + \rho^2(r_-) \rangle = \sqrt{-1}\, \langle \omega, \rho(r_-) - r_- \rangle,$$

and hence $\mathcal{H}_r = \mathcal{H}_{\rho(r)} = \mathcal{H}_{r+\rho(r)}$. Therefore we can take $\delta = r + \rho(r)$. □

Now we define

$$\Gamma = \{\gamma \in \mathrm{O}(L_-) : \gamma \circ \rho_- = \rho_- \circ \gamma\}. \tag{10.10}$$

Then Γ acts on $\mathcal{B}$ properly discontinuously.

Theorem 10.19. $\mathcal{M}_3 \setminus \mathcal{H}_3 \cong (\mathcal{B} \setminus \mathcal{H})/\Gamma$.

Proof. To each non-singular plane quartic curve C, we associate a $K3$ surface and an automorphism σ of order 4. If necessary by replacing σ by σ^3, we may assume that $\sigma^*(\omega_X) = \sqrt{-1} \cdot \omega_X$. It follows from Lemma 10.17 that there exists an isomorphism $\alpha_X : H^2(X,\mathbb{Z}) \to L$ of lattices satisfying $\alpha_X \circ \sigma^* = \rho \circ \alpha_X$. Thus we have a map

$$p\colon \mathcal{M}_3 \setminus \mathcal{H}_3 \to (\mathcal{B} \setminus \mathcal{H})/\Gamma.$$

For two plane quartic curves C, C', let X, X' be the corresponding $K3$ surfaces and let σ, σ' be the automorphisms of order 4, respectively. If

$$\alpha_X(\omega_X) \bmod \Gamma = \alpha_{X'}(\omega_{X'}) \bmod \Gamma,$$

then there exists an isomorphism $\phi\colon H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ of lattices which preserves holomorphic 2-forms and satisfies $\phi \circ \sigma^* = (\sigma')^* \circ \phi$. It follows from Lemma 10.11 that ϕ sends an ample divisor C to an ample divisor C'. It now follows from the Torelli-type theorem for $K3$ surfaces that there exists an isomorphism $\varphi\colon X' \to X$ with $\varphi \circ \sigma^* = (\sigma')^* \circ \varphi$. Obviously, φ induces an isomorphism $\mathbb{P}^2 \to \mathbb{P}^2$ which sends C' to C. Thus we have proved the injectivity of p.

Let us prove the surjectivity of p. Let $\omega \in \mathcal{B} \setminus \mathcal{H}$. First, it follows from the surjectivity of the period map of $K3$ surfaces (Theorem 7.5) that there exists a marked $K3$ surface (X, α_X) with $\alpha_X(\omega_X) = \omega$. By definition,

$$\phi = \alpha_X^{-1} \circ \rho \circ \alpha_X$$

preserves $\mathbb{C} \cdot \omega_X$. By Lemma 10.18, we have

$$\langle \delta, H^2(X,\mathbb{Z})^{\langle \phi \rangle} \rangle \neq 0$$

for $\delta \in S_X$, $\delta^2 = -2$. Then we can prove the surjectivity of p by the same argument as in the proof of Lemma 10.17. □

Remark 10.20. The description of the moduli space of plane quartic curves as the quotient of a complex ball is due to Kondo [Kon5]. The divisor $\mathcal{H}/\Gamma$ consists of two irreducible components. A general point of a component (resp. the other component) corresponds to a plane quartic curve with a node (resp. a hyperelliptic curve of genus 3).

Remark 10.21. As in the case of non-hyperelliptic curves of genus 3, we can give a similar description of the moduli space of non-hyperelliptic curves of genus 4 (Kondo [Kon6]). Recall that the canonical model of a non-hyperelliptic curve of genus 4 is the intersection

$$C = Q \cap D$$

of a quadric surface Q and a cubic surface D in $\mathbb{P}^3$. The triple covering of Q branched along C is a $K3$ surface with an automorphism order 3. In this case a 9-dimensional complex ball appears. In the case of genus 3, the complex ball is induced from a hermitian form defined over the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$. In the case of genus 4, a hermitian form is defined over the Eisenstein integers $\mathbb{Z}[\zeta_3]$ where ζ_3 is a primitive cube root of unity. When Q is a quadric cone, the double covering of Q branched along C is a del Pezzo surface of degree 1 (see Proposition 10.3). This correspondence gives a description of the moduli space of del Pezzo surfaces of degree 1 as the quotient of a complex ball.

Remark 10.22. Recall that a del Pezzo surface of degree 3 is a cubic surface S in $\mathbb{P}^3$. Let S be defined by a homogeneous polynomial $f_3(x, y, z, t)$ of degree 3. Then, by setting

$$X\colon s^3 = f_3(x, y, z, t) \subset \mathbb{P}^4,$$

we get a cubic hypersurface X in $\mathbb{P}^4$ which is the triple covering of $\mathbb{P}^3$ branched along S. Consider the intermediate Jacobian $J(X) = H^{2,1}(X)/H_3(X, \mathbb{Z})$ of X and the automorphism of order 3 induced from the covering transformation. By associating X with the pair of $J(X)$ and the automorphism, one can obtain a description of the moduli space of cubic surfaces as the quotient of a 4-dimensional complex ball (Allcock, Carlson, Toledo [ACT]). In the case of plane quartic curves, the complex ball is embedded in a bounded symmetric domain of type IV. In the case of cubic surfaces, the complex ball is embedded in a Siegel upper half-space (a bounded symmetric domain of type III). The description of the moduli of cubic surfaces by Allcock and others is interpreted in terms of the periods of $K3$ surfaces and a bounded symmetric domain of type IV (Dolgachev, van Geemen, Kondo [DGK]).

In these examples, every complex ball is naturally embedded in a bounded symmetric domain of type IV, and one can apply the theory of automorphic forms on bounded symmetric domains of type IV due to Borcherds [Bor4] to study the moduli spaces of cubic surfaces, plane quartic curves, and Enriques surfaces (Allcock, Freitag [AF], Kondo [Kon7], [Kon8]).

Remark 10.23. A plane quartic curve with a node corresponds to an interior point of $\mathcal{B}/\Gamma$, but in the description of the moduli space by using the Jacobian of curves of genus 3, it corresponds to a boundary point of $\mathfrak{H}_3/\mathrm{Sp}_6(\mathbb{Z})$. Thus the two descriptions of the moduli space are different. As an example of such moduli spaces with two different descriptions, the moduli space of 6 ordered points on $\mathbb{P}^1$ is famous. The double covering of $\mathbb{P}^1$ branched along 6 points is a hyperelliptic curve of genus 2 and an order of 6 points corresponds to a level 2-structure of its Jacobian. It follows that the quotient space $\mathfrak{H}_2/\Gamma(2)$ of the Siegel upper half-plane $\mathfrak{H}_2$ by the principal 2-congruence subgroup $\Gamma(2)$ is the moduli space of 6 ordered distinct points, and its Satake compactification is isomorphic to a projective variety $\mathcal{I}_4$, called the *Igusa quartic*, which is a quartic hypersurface in $\mathbb{P}^4$. Let $(x_1, \dots, x_6)$ be homogeneous coordinates of $\mathbb{P}^5$. Then $\mathcal{I}_4$ is given by

$$\sum_i x_i = \Big(\sum_i x_i^2\Big)^2 - 4\Big(\sum_i x_i^4\Big) = 0 \subset \mathbb{P}^5.$$

The symmetric group $\mathfrak{S}_6$ of degree 6 naturally acts on the moduli space of 6 ordered points on $\mathbb{P}^1$. This action corresponds to that of $Sp_4(\mathbb{Z})/\Gamma(2)$ ($\cong \mathfrak{S}_6$) on $\mathfrak{H}_2/\Gamma(2)$. It is known that $\mathcal{I}_4$ contains 15 lines corresponding to 6 points on $\mathbb{P}^1$ whose 2 points coincide. These lines are the boundary of Satake compactification of $\mathfrak{H}_2/\mathrm{Sp}_4(\mathbb{Z})$. Let $T_p(\mathcal{I}_4)$ be the tangent space of $\mathcal{I}_4$ at a point p not lying on 15 lines. Then $\mathcal{I}_4 \cap T_p(\mathcal{I}_4)$ is a Kummer quartic surface with 16 rational double points of type A_1 at p and the intersection of the tangent space with 15 lines (see Section 4.4). This Kummer surface is nothing but the Kummer surface associated with the curve of genus 2 corresponding to p, and thus $\mathcal{I}_4$ is worthy of being called the moduli space of curves of genus 2.

On the other hand, the triple covering of $\mathbb{P}^1$ branched along 6 points is a trigonal curve of genus 4 and its moduli space can be described as the quotient of a 3-dimensional complex ball. Its Baily–Borel compactification is a projective variety $\mathcal{S}_3$, called a *Segre cubic*, which is a cubic hypersurface in $\mathbb{P}^4$. It is known that $\mathcal{S}_3$ is given by

$$\sum_i x_i = \sum_i x_i^3 = 0 \subset \mathbb{P}^5.$$

At the beginning of the 20th century, it was already known that $\mathcal{I}_4$ and $\mathcal{S}_3$ were projective dual to each other (Baker [Ba]). In this story, a symmetric group of degree 6 appears naturally. Let $\mathfrak{S}_6$ be the symmetry group of the letters $\{1, \dots, 6\}$. The group $\mathfrak{S}_6$ contains 15 transpositions $(12), \dots, (56)$ and 15 elements $(12)(34)(56)$, $(12)(35)(46), \dots$ of order 2. The former was named the Sylvester duad and the latter was named the Sylvester syntheme (Baker [Ba]). Each duad appears in exactly 3 synthemes and each syntheme contains exactly 3 duads. For example, (12) appears in (12)(34)(56), (12)(35)(46), (12)(36)(45), and (12)(34)(56) contains (12), (34), (56). We can identify 15 lines on $\mathcal{I}_4$ mentioned above with 15 duads and the 15 intersection points of 15 lines with 15 synthemes, and the incidence relation between 15 lines and 15 intersection points is nothing but the one between duads and synthemes. For example, the 3 lines corresponding to the duads (12), (34), (56) meet at the point corresponding to (12)(34)(56). For the Igusa quartic and the Segre cubic, we refer the reader to Dolgachev [Do2], Dolgachev, Ortland [DO], van der Geer [vG], Hunt [Hun], Yoshida [Yo].

11

Finite groups of symplectic automorphisms of $K3$ surfaces and the Mathieu group

The main purpose of this chapter is to study finite groups of symplectic automorphisms of $K3$ surfaces and to show that such groups can be embedded in the Mathieu group M_{23} by using the classification of Niemeier lattices.

11.1 Niemeier lattices and the Mathieu group

Recall that the rank of an even unimodular negative definite lattice is $8m$, $m \in \mathbb{N}$ (Corollary 1.26). Contrary to the case of even unimodular indefinite lattices (Theorem 1.27), the isomorphism class is not determined by its signature (its rank in this case). The classification is known only in the case of $m \leq 3$. The lattice E_8 is the unique such lattice of rank 8 and the set of isomorphism classes of such lattices of rank 16 consists of $E_8 \oplus E_8$ and the overlattice of the root lattice D_{16} (see Remark 1.28).

Definition 11.1. A *Niemeier lattice* is an even unimodular negative definite lattice of rank 24.

We call an element of norm -2 of a lattice a *root*. Let N be a Niemeier lattice. Let $R(N)$ be the sublattice of N generated by all roots in N. We set $R(N) = \emptyset$ if N contains no roots. If $R(N) \neq \emptyset$, then $R(N)$ is nothing but a root lattice.

Theorem 11.2. *There are exactly* 24 *isomorphism classes of Niemeier lattices. Each isomorphism class is uniquely determined by* $R(N)$*. The following is the list of* $R(N)$: $A_1^{\oplus 24}$, $A_2^{\oplus 12}$, $A_3^{\oplus 8}$, $A_4^{\oplus 6}$, $A_6^{\oplus 4}$, $A_8^{\oplus 3}$, $A_{12}^{\oplus 2}$, A_{24}, $D_4^{\oplus 6}$, $D_6^{\oplus 4}$, $D_8^{\oplus 3}$, $D_{12}^{\oplus 2}$, D_{24}, $E_6^{\oplus 4}$, $E_8^{\oplus 3}$, $A_5^{\oplus 4} \oplus D_4$, $A_7^{\oplus 2} \oplus D_5^{\oplus 2}$, $A_9^{\oplus 2} \oplus D_6$, $A_{15} \oplus D_9$, $A_{11} \oplus D_7 \oplus E_6$, $A_{17} \oplus E_7$, $D_{10} \oplus E_7^{\oplus 2}$, $D_{16} \oplus E_8$, $\emptyset$.

This theorem is due to Niemeier [Nie]. His proof depends on case-by-case analysis. On the other hand, Venkov [Venk] gave a characterization of Niemeier lattices as follows. Let R be an irreducible root lattice of rank r. Let m be the number of all roots in R (e.g., see Table 10.2). The ratio m/r is called the *Coxeter number* of R. The Coxeter numbers of irreducible root lattices are given in Table 11.1.

Table 11.1. Coxeter number

R	A_n	D_n	E_6	E_7	E_8
# of roots	$n(n+1)$	$2n(n-1)$	72	126	240
$h(R)$	$n+1$	$2(n-1)$	12	18	30

The following theorem gives the meaning of the root lattices appearing in Theorem 11.2. Moreover, by using this theorem we can easily obtain the list of $R(N)$ in Theorem 11.2.

Theorem 11.3. *Let N be a Niemeier lattice with $N(R) \neq \emptyset$. The following hold*:

(1) $\text{rank}(R(N)) = 24$.

(2) *Any irreducible component of R has the same Coxeter number h.*

(3) *The number of roots of N is equal to $24h$.*

To prove Theorem 11.3, Venkov applied the theory of theta functions of lattices due to Hecke and the theory of modular forms. For the proof, we refer the reader to Venkov [Venk] and Ebeling [E]. The existence and the uniqueness of N are proved by an explicit construction as an overlattice of $R(N)$.

In the following we will construct the Niemeier lattices N with $R(N) = A_1^{\oplus 24}$ and $\emptyset$. We first consider the case of $R(N) = A_1^{\oplus 24}$.

Let $R = A_1^{\oplus 24}$. Note that the discriminant group A_R is $(\mathbb{Z}/2\mathbb{Z})^{24}$. Thus we need to find an isotropic subgroup of $(\mathbb{Z}/2\mathbb{Z})^{24}$ of order 2^{12} to construct N (Theorem 1.19). Consider a 24-dimensional vector space $\mathbb{F}_2^{24}$ over the finite field $\mathbb{F}_2$. For $x = (x_i)$, $x_i \in \mathbb{F}_2$, we set $w(x) = |\{i : x_i = 1\}|$ and call it the *weight* of x.

Theorem 11.4. *There exists a subspace $\mathcal{G}$ in $\mathbb{F}_2^{24}$ of dimension 12 satisfying the following conditions*:

(1) *$\mathcal{G}$ contains the element $(1, \ldots, 1)$ of weight 24.*

(2) *For any non-zero element $x \in \mathcal{G}$, $w(x)$ is a multiple of 4 and $w(x) \geq 8$.*

For the proof of this theorem we refer the reader to Conway [Co1], Ebeling [E]. Also Milnor, Husemoller [MH, App. 5] presented the 12×24 matrix whose rows give a generator of $\mathcal{G}$. We call $\mathcal{G}$ the *extended binary Golay code*.

Example 11.5 (Niemeier lattice N with $N(R) = A_1^{\oplus 24}$). Now we give the Niemeier lattice N with $R(N) = A_1^{\oplus 24}$. Consider the discriminant quadratic form q_R. By property (2) in Theorem 11.4, $\mathcal{G}$ is isotropic with respect to q_R. Thus we get an overlattice N of R with $N/R \cong \mathcal{G}$. Since $\dim(\mathcal{G}) = 12$, N is unimodular. Finally, again by property (2), $w(x) \geq 8$, no new roots are added, and hence $R(N) = R$. Thus we have obtained the Niemeier lattice N with $R(N) = A_1^{\oplus 24}$.

The extended binary Golay code is related to the following Steiner system in combinatorial mathematics. Let Ω be a set of 24 elements and $P(\Omega)$ the power set of Ω.

Definition 11.6. A subset $\mathcal{S}$ in $P(\Omega)$ is called the *Steiner system* if $\mathcal{S}$ satisfies the following:

(1) If $A \in \mathcal{S}$, then $|A| = 8$.

(2) For any $B \in P(\Omega)$ with $|B| = 5$, there exists a unique element $A \in \mathcal{S}$ with $B \subset A$.

Each element of $\mathcal{S}$ is called an *octad*.

By property (2) in Definition 11.6, the number of octads ($= |\mathcal{S}|$) is equal to

$$\binom{24}{5} \Big/ \binom{8}{5} = 759.$$

One can find a list of all octads in Todd [Todd, Table I]. Similarly we have the following lemma.

Lemma 11.7. *Let $B \in P(\Omega)$ and let $n_B = |\{A \in \mathcal{S} : B \subset A\}|$. Then*

$$n_B = \begin{cases} \binom{24-k}{5-k} \Big/ \binom{8-k}{5-k} & \text{if } k = |B| \leq 5, \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases}$$

Consider $P(\Omega) \cong \mathbb{F}_2^{24}$ as a vector space defined by the symmetric difference $A + B = A \setminus B \cup B \setminus A$. We have a non-degenerate symmetric bilinear form

$$f \colon P(\Omega) \times P(\Omega) \to \mathbb{F}_2, \quad f(A, B) = |(A \cap B)| \bmod 2.$$

Now define $\mathcal{G}$ as the subspace of $\mathbb{F}_2^{24}$ generated by the Steiner system $\mathcal{S}$. It is known that $\mathcal{G}$ is nothing but the extended binary Golay code. For the proof we refer the reader to Conway [Co1].

Exercise 11.8. Show that $\mathcal{G}$ is isotropic with respect to f.

Definition 11.9. The Niemeier lattice N with $R(N) = \emptyset$ is called the *Leech lattice*. In the following we denote the Leech lattice by Λ. The Leech lattice is given as follows. Let N be the Niemeier lattice with $R(N) = A_1^{\oplus 24}$. Let $u \in A_1^*$ be a generator with $u^2 = -1/2$. Let ϵ_i ($i \in \Omega$) be the element of $(A_1^*)^{\oplus 24}$ given by

$$\epsilon_i = (\delta_{ij}u)_{j\in\Omega},$$

where δ_{ij} is the Kronecker delta. Then $\{2\epsilon_i\}_{i\in\Omega}$ is an orthogonal basis of $R(N) = A_1^{24}$. Note that $\epsilon_i^2 = -1/2$. We set

$$\Omega = \{\infty, 0, 1, 2, \dots, 22\} = \mathbb{P}^1(\mathbb{F}_{23}), \quad \nu_i = \tfrac{1}{2}\epsilon_i, \ i \in \Omega.$$

Thus $\nu_i^2 = -1/8$. We use $\{\nu_i\}$ instead of $\{\epsilon_i\}$ because we keep the same notation as that in Conway [Co1], Conway, Sloane [CS], Todd [Todd]. Let $(\xi_\infty, \xi_0, \dots, \xi_{23})$ be coordinates of $\mathbb{R}^{24} = \Lambda \otimes \mathbb{R}$ with respect to the basis $\{\nu_i\}_{i\in\Omega}$, namely, $x = \sum_{i\in\Omega} x_i\nu_i$. Consider the map

$$\phi\colon N \to \mathbb{F}_2, \quad \phi(x) = \sum_i x_i/4 \bmod 2.$$

This is well defined by Theorem 11.4. Let $\Lambda_0 = \mathrm{Ker}(\phi)$ and $\Lambda_1 = \phi^{-1}(1)$. Then we have $[N : \Lambda_0] = 2$ and Λ_1 contains $A_1^{\oplus 24}$. Now we define

$$\Lambda = \Lambda_0 \cup (\Lambda_1 + (1, \dots, 1)) = \langle \Lambda_0, (5, 1, \dots, 1)\rangle.$$

Then we can easily check that Λ is an even lattice. Moreover, Λ is unimodular because $[\Lambda : \Lambda_0] = 2$. Since $(5, 1, \dots, 1)^2 = -6$, Λ contains no roots. Obviously, minimal vectors are norm -4. In the next chapter we will study more details of the Leech lattice.

Definition 11.10. Note that the symmetric group $\mathfrak{S}_{24}$ of degree 24 acts on $A_1^{\oplus 24}$ as permutations. This action extends to the one on $(A_1^*)^{\oplus 24}$. The stabilizer subgroup of the extended binary Golay code $\mathcal{G}$,

$$M_{24} = \{\sigma \in \mathfrak{S}_{24} : \sigma(\mathcal{G}) = \mathcal{G}\},$$

is called the *Mathieu group of degree* 24. The subgroup M_{23} (resp. M_{22}) of M_{24} fixing the first coordinate (resp. the first and second coordinates) is also called the *Mathieu group of degree* 23 (resp. *of degree* 22). There are another two Mathieu groups M_{12}, M_{11}. These five groups are the first examples of finite sporadic simple groups.

Let N be the Niemeier lattice with $R(N) = A_1^{\oplus 24}$ (Example 11.5). Note that N contains 48 roots $\pm 2\epsilon_i$ which define 24 reflections. Since $\mathrm{O}(A_1^{\oplus 24}) \cong (\mathbb{Z}/2\mathbb{Z})^{24} \rtimes \mathfrak{S}_{24}$, we have

$$\mathrm{O}(N)/W(R(N)) \cong M_{24}, \tag{11.1}$$

where $W(R(N)) = (\mathbb{Z}/2\mathbb{Z})^{24}$ is the reflection group generated by 24 reflections. It is known that M_{24} acts on Ω 5-ply transitively. Thus its order is $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 48$.

Remark 11.11. The automorphism group $\mathrm{O}(\Lambda)$ of the Leech lattice Λ is denoted by Co_0 and its quotient by the center ± 1 is denoted by Co_1. The stabilizer group of a point in Λ of norm -4, -6 is denoted by Co_2, Co_3, respectively. These groups Co_1, Co_2, Co_3 are called Conway groups, and are also finite sporadic simple groups. The order of Co_1 is $2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$.

11.2 Finite symplectic automorphisms and the Mathieu group

Recall that any finite symplectic automorphism has only a finite number of fixed points (see the proof of Proposition 8.7). We start with the following theorem.

Theorem 11.12. *Let X be a $K3$ surface. Let $g \in \mathrm{Aut}(X)$ be a finite symplectic automorphism of order $m > 1$ and let f_m be the number of fixed points of g on X. Then $m \leq 8$ and f_m depends only on the order m as in Table 11.2.*

Table 11.2. The number of fixed points

m	2	3	4	5	6	7	8
f_m	8	6	4	4	2	3	2

Proof. Let G be the group generated by g and let Y be the minimal resolution of X/G. Then it follows from Proposition 8.7 that Y is also a $K3$ surface. Consider the rational map $\pi\colon X \longrightarrow Y$ of degree m. Let G_i $(i = 1, \ldots, N)$ be non-trivial subgroups of $G \cong \mathbb{Z}/m\mathbb{Z}$. Denote by m_i the order of G_i and by k_i the number of points whose stabilizer subgroup of G is isomorphic to G_i. Put $k = k_1 + \cdots + k_N$ and let $\{P_{ij}\}$ $(j = 1, \ldots, k_i)$ be the set of points whose stabilizer is G_i. Note that for each i, G acts on the set $\{P_{ij}\}$ $(j = 1, \ldots, k_i)$ and hence m/m_i is a divisor of k_i. Let

$$X' = X \setminus \bigcup_{1\leq i\leq N,\, 1\leq j\leq k_i} \{P_{ij}\}, \quad Y' = Y \setminus E,$$

where E is the union of exceptional curves of the minimal resolution $Y \longrightarrow X/G$. Note that

$$e(X') = e(X) - k = 24 - k, \quad e(Y') = e(Y) - \sum_{i=1}^{N} k_i(m_i/m)m_i.$$

The second equation as above follows from the fact that for each i, there exist exactly $k_i m_i/m$ singularities of type A_{m_i-1} on X/G and $(m_i - 1)$ non-singular rational curves appear on Y as the exceptional set of the resolution of a singularity of type A_{m_i-1}. Therefore, by removing the exceptional set of each singularity of type A_{m_i-1}, the Euler number decreases by m_i.

On the other hand, the covering $X' \to Y'$ is an unramified cover of degree m and hence

$$e(X') = m \cdot e(Y').$$

Combining these equations of Euler numbers, we have

$$24 - k = 24m - \sum_{i=1}^{N} k_i m_i^2, \quad (m/m_i)|k_i. \tag{11.2}$$

Now assume that m is a prime number p. Then $N = 1$, $m_i = p$, and $k_i = k$. It follows from equation (11.2) that $24 - k = 24p - kp^2$. Hence

$$k = \frac{24}{p+1} \tag{11.3}$$

and the stabilizer subgroup of each fixed point is G. It follows from equation (11.3) that

$$p = 2, 3, 5, 7, 11, 23.$$

If the case $p = 11$ or 23 happens, then Y contains 20 or 22 non-singular rational curves as the exceptional set which generate a negative definite sublattice of $H^2(Y, \mathbb{Z})$ of rank 20 or 22. This contradicts the fact that the signature of $H^2(Y, \mathbb{Z})$ is $(3, 19)$. Hence $p = 2, 3, 5, 7$. Similarly we can prove that $m \leq 8$. We leave the details to the reader. □

Remark 11.13. The above theorem was first proved by Nikulin [Ni3]. Mukai [Muk1] gave another proof by using the holomorphic Lefschetz fixed point formula.

Now we state a relation between finite groups of symplectic automorphisms and the Mathieu group. Let Ω be a set of 24 letters. By definition, M_{24} is a subgroup of $\mathfrak{S}(\Omega) = \mathfrak{S}_{24}$. Recall that M_{23} is its stabilizer subgroup of a point in Ω. Consider the action of M_{23} on Ω. Let $\sigma \in M_{23}$ be of order m. It is known that the number of fixed points of σ on Ω depends only on the order m. Let g_m be the number of fixed points of $\sigma \in M_{23}$ on Ω and let c_m be the conjugacy class of σ. In Table 11.3, we give c_m, g_m in the case $m \leq 8$.

By comparing Tables 11.2, 11.3, we claim the following main theorem in this chapter.

Table 11.3. The conjugacy classes and the number of fixed points in the case $m \leq 8$

m	c_m	g_m	m	c_m	g_m
1	$(1)^{24}$	24	5	$(1)^4(5)^4$	4
2	$(1)^8(2)^8$	8	6	$(1)^2(2)^2(3)^2(6)^2$	2
3	$(1)^6(3)^6$	6	7	$(1)^3(7)^3$	3
4	$(1)^4(2)^2(4)^4$	4	8	$(1)^2(2)(4)(8)^2$	2

Theorem 11.14. *Let G be a finite group. Then the following are equivalent*:

(1) *G acts on a $K3$ surface symplectically.*

(2) *G can be embedded into M_{23}, whose number of orbits on Ω is greater than or equal to 5.*

Following Mukai [Muk1], we define

$$\epsilon(m) = 24\left(m \prod_{p|m,\ \text{prime}} \left(1 + \frac{1}{p}\right)\right)^{-1}.$$

Then for $2 \leq m \leq 8$, we can check that $\epsilon(m) = f_m = g_m$. By noting that the Euler number of a $K3$ surface is 24 and applying the Lefschetz fixed point formula, we obtain the following.

Lemma 11.15. *Assume that a finite group G acts on a $K3$ surface X symplectically. Then the induced action of G on $H^*(X, \mathbb{Q})$ is a representation of degree 24 defined over $\mathbb{Q}$ with the character ϵ.*

The following lemma explains the meaning of condition (2) in Theorem 11.14 on the number of orbits.

Lemma 11.16. *Assume that a finite group G acts on a $K3$ surface X symplectically. Then we have*

$$\dim H^*(X, \mathbb{Q})^G \geq 5.$$

Proof. Let ω_X be a non-zero holomorphic 2-form on X and let κ be a Kähler class. Then $H^0(X, \mathbb{Q})$, $H^4(X, \mathbb{Q})$, $\mathrm{Re}(\omega_X)$, $\mathrm{Im}(\omega_X)$, and $\sum_{g \in G} g^*(\kappa)$ are G-invariant. □

Now we give a lattice-theoretic proof of assertion (1) $\Longrightarrow$ (2) in Theorem 11.14. We prepare three lemmas. Let G be a finite group acting on a $K3$ surface X

symplectically. We denote $H^2(X,\mathbb{Z})$ by L for simplicity. Let

$$L^G = \{x \in L : g^*(x) = x \ \forall g \in G\}, \quad L_G = (L^G)^\perp \text{ in } L.$$

For an even lattice S we denote by $\ell(S)$ the number of minimal generators of the discriminant group A_S.

Lemma 11.17. (1) *L_G is negative definite; in particular,* $\mathrm{rank}(L_G) \leq 19$.

(2) $\ell(L_G) \leq 22 - \mathrm{rank}(L_G)$.

(3) *L_G contains no roots.*

Proof. (1) Since $L^G \otimes \mathbb{R}$ contains $\mathrm{Re}(\omega_X)$, $\mathrm{Im}(\omega_X)$, $\sum_{g\in G} g^*(\kappa)$, which generate a 3-dimensional positive definite subspace (e.g., see Section 6.5), the assertion follows.

(2) Since L^G and L_G are orthogonal complements of each other in L, we have $A_{L^G} \cong A_{L_G}$ (Theorem 1.32). In particular we have

$$\ell(L_G) = \ell(L^G) \leq \mathrm{rank}(L^G) = 22 - \mathrm{rank}(L_G).$$

(3) Assume that L_G contains a root r. Since $\langle r, \omega_X\rangle = 0$, we have $r \in \mathrm{NS}(X)$. By the Riemann–Roch theorem, we may assume that r is effective. Since any automorphism preserves effective classes, we have $\sum_{g\in G} g^*(r) \neq 0$. Obviously, it is contained in $L_G \cap L^G$. Since L_G is non-degenerate by assertion (1), $L_G \cap L^G = \{0\}$, which is a contradiction. □

Lemma 11.18. *Assume that a finite group G acts on a $K3$ surface symplectically. Then there exists a Niemeier lattice N satisfying the following:*

(1) *$L_G \oplus A_1$ can be primitively embedded into N.*

(2) *The action of G on L_G can be extended to the one on N acting trivially on $(L_G)^\perp$ in N.*

Proof. (1) This follows from Nikulin [Ni4, Thm. 1.12.2].

(2) Since G acts on L^G trivially and hence on $A_{L^G} \cong A_{L_G}$ so, the assertion follows from Corollary 1.33. □

Assume that a finite group G acts on a $K3$ surface symplectically. It follows from Lemma 11.18 that G is a subgroup of $\mathrm{O}(N)$.

Lemma 11.19. *The map*

$$G \to \mathrm{O}(N) \to \mathrm{O}(N)/\{\pm 1\}\cdot W(N)$$

is injective.

Proof. Let C be a fundamental domain of $W(N)$ with respect to the action on $N \otimes \mathbb{R}$ (see Section 1.1.1, the part before Exercise 1.6). We claim $C \cap N^G \otimes \mathbb{R} \neq \emptyset$. Assume that $C \cap N^G \otimes \mathbb{R} = \emptyset$. Since $N^G \otimes \mathbb{R}$ is a subspace of the vector space $N \otimes \mathbb{R}$, there exists a root r in N such that the hyperplane $r^\perp$ contains N^G. This implies that $r \in (N^G)^\perp = N_G$, which contradicts Lemma 11.17(3). Thus there exists a G-invariant element in C. Hence it follows from Theorem 2.9 that G preserves C. Thus we have $G \subset \mathrm{Aut}(C) = \mathrm{O}(N)/\{\pm 1\} \cdot W(R)$. □

Now we finish the proof of the assertion (1) $\Longrightarrow$ (2) in Theorem 11.14. By Lemma 11.18, we have $A_1 \subset N^G$ and hence N is not the Leech lattice. We consider the most difficult case, that is, the case $R(N) = A_1^{\oplus 24}$. By Lemma 11.19 and (11.1), G is a subgroup of M_{24}. Again by Lemma 11.18, G fixes at least one root, that is, G is contained in M_{23}. In the case of other Niemeier lattices, we can prove the assertion similarly by using a description of the group $\mathrm{O}(N)/\{\pm\} \cdot W(R(N))$ in Conway, Sloane [CS, Chap. 16, Table 16.1]. Thus we have finished the proof.

Remark 11.20. Theorem 11.14 was discovered by Mukai. The above proof was given in Kondo [Kon3]. Mukai's original proof in [Muk1] is as follows. By using Lemmas 11.15, 11.16, he determined maximal groups of finite symplectic automorphisms (there are 11 such groups) and then proved that each maximal one can be embedded in M_{23}. His proof depends deeply on finite group theory. Conversely, he gave examples of all 11 maximal groups to show the assertion (2) $\Longrightarrow$ (1) in Theorem 11.14. (The group $\mathfrak{A}_6$ given in Exercise 8.10 is one of 11 maximal groups. For others, see Mukai [Muk1].) Also, a lattice-theoretic proof of the converse is known (see the appendix by Mukai in [Kon3]).

On the other hand, Xiao [X] gave a complete list of all finite groups of symplectic automorphisms of $K3$ surfaces by using the argument of Nikulin in Theorem 11.12. There are 81 types. It is a natural problem to determine the action of a finite symplectic group G on $H^2(X, \mathbb{Z})$ up to conjugacy. Nikulin [Ni3] proved its uniqueness for any abelian group G. In the general case, Hashimoto [Has] determined the actions for all groups and proved the uniqueness except for 5 groups.

Huybrechts [Huy2] extended the notion of symplectic automorphisms to auto-equivalences of the bounded derived category of coherent sheaves on a $K3$ surface and gave a relation to the Conway group $Co_0 = \mathrm{O}(\Lambda)$. For the invariant sublattices of elements in Co_0, we refer the reader to Höhn, Mason [HM]. Mukai, Ohashi [MuO] studied finite groups of semi-symplectic automorphisms of Enriques surfaces and the Mathieu group M_{12} of degree 12. Eguchi, Ooguri, Tachikawa [EOT] presented "Mathieu moonshine" concerning a relation between the elliptic genus of a $K3$ surface and the Mathieu group M_{24}.

Remark 11.21. Scattone [Sc] was the first to use Niemeier lattices for studying the geometry of $K3$ surfaces. In fact, he determined the boundary components of the Satake–Baily–Borel compactifications of the moduli spaces of polarized $K3$ surfaces of degrees 2 and 4 by using the classification of Niemeier lattices (see Section 5.1.2).

12

Automorphism group of the Kummer surface associated with a curve of genus 2

At the end of the 19th century, geometers discovered many involutions of the Kummer surface $X = \mathrm{Km}(C)$ associated with a curve C of genus 2. We show that these classical involutions generate the automorphism group of a generic X by applying the Torelli-type theorem for $K3$ surfaces and lattice theory.

12.1 The Leech lattice and the even unimodular lattice of signature $(1, 25)$

We use the same notation as in Definition 11.9. Recall that $\{v_i\}_{i \in \Omega}$ is an orthogonal basis of $\mathbb{R}^{24} = \Lambda \otimes \mathbb{R}$, where $\Omega = \{\infty, 0, 1, 2, \ldots, 22\}$ and $v_i^2 = -1/8$. Let $(\xi_\infty, \xi_0, \ldots, \xi_{22})$ be coordinates of $\mathbb{R}^{24} = \Lambda \otimes \mathbb{R}$ with respect to the basis $\{v_i\}_{i \in \Omega}$.

Proposition 12.1. *The vector $(\xi_\infty, \xi_0, \ldots, \xi_{22})$ is in Λ if and only if*

(i) *the coordinates ξ_i are all congruent modulo 2, to some m,*

(ii) *the set of i for which ξ_i takes any given value modulo 4 is in the extended binary Golay code $\mathcal{G}$,*

(iii) *the coordinate sum is congruent to $4m$ mod 8.*

For the proof we refer the reader to Conway [Co1, §4, Thm. 2].

Proposition 12.2. *Let Λ_{2n} be the set of all vectors in Λ with norm $-2n$ ($n \geq 2$). Then the complete lists of Λ_4, Λ_6 are*

$$\Lambda_4 = \left\{(\pm 2^8, 0^{16}), (\pm 3, \pm 1^{23}), (\pm 4^2, 0^{22})\right\},$$
$$\Lambda_6 = \left\{(\pm 2^{12}, 0^{12}), (\pm 3^3, \pm 1^{21}), (\pm 4, \pm 2^8, 0^{15}), (\pm 5, \pm 1^{23})\right\},$$

where the signs are taken to satisfy the conditions in Proposition 12.1.

For the proof see Conway, Sloane [CS, p. 133, Table 4.13].

For a subset S of Ω, we set $\nu_S = \sum_{i\in S} \nu_i$. Then some elements (not all) in Λ_4, Λ_6 can be written as

$$\Lambda_4 \ni 2\nu_K \ (K \in \mathcal{S}), \quad \nu_\Omega - 4\nu_i \ (i \in \Omega), \quad 4\nu_i \pm 4\nu_j \ (i, j \in \Omega);$$
$$\Lambda_6 \ni 4\nu_i + \nu_\Omega \ (i \in \Omega),$$

where $\mathcal{S}$ is the Steiner system, namely, K is an octad.

Let L be an even unimodular lattice of signature $(1, 25)$. Note that such a lattice is unique up to isomorphisms (Theorem 1.27) and in particular L is isomorphic to $U \oplus \Lambda$. We fix an isomorphism

$$L \cong U \oplus \Lambda$$

and identify L and $U \oplus \Lambda$. Let $x = (m, n, \lambda) \in L = U \oplus \Lambda$ such that $m, n \in \mathbb{Z}$, $\lambda \in \Lambda$, and the norm of x is given by $x^2 = 2mn + \lambda^2$. Let $\rho = (1, 0, 0) \in L$. Then $\rho^2 = 0$ and the orthogonal complement $\rho^\perp$ of ρ does not contain any roots because Λ does not. A root $r \in L$ is called a *Leech root* if $\langle r, \rho\rangle = 1$, in other words, $r = (-1 - \lambda^2/2, 1, \lambda)$. We denote by Δ the set of all Leech roots. Then Δ can be identified with Λ as

$$\Lambda \ni \lambda \longleftrightarrow \left(-1 - \tfrac{\lambda^2}{2}, 1, \lambda\right) \in \Delta.$$

Note that

$$(r - r')^2 = (\lambda - \lambda')^2 \tag{12.1}$$

for any Leech roots $r = (-1 - \lambda^2/2, 1, \lambda)$, $r' = (-1 - \lambda'^2/2, 1, \lambda')$. The next lemma follows from (12.1).

Lemma 12.3.

(1) $\langle r, r'\rangle = 0 \Longleftrightarrow (\lambda - \lambda')^2 = -4$.

(2) $\langle r, r'\rangle = 1 \Longleftrightarrow (\lambda - \lambda')^2 = -6$.

Definition 12.4. We now extend the notion of reflections (see Definition 2.1). Let S be an even lattice of signature $(1, r-1)$. Let $\delta \in S$ be a primitive vector with norm $-2m$. Then we define the *reflection* associated with δ by

$$s_\delta \colon S \otimes \mathbb{Q} \to S \otimes \mathbb{Q}, \quad x \to x - \frac{2\langle x, \delta\rangle}{\langle \delta, \delta\rangle}\delta.$$

When $m = 1$, s_δ is nothing but the reflection with respect to a root δ which is contained in $\mathrm{O}(S)$, that is, s_δ is an automorphism of the lattice S. Note that if $2\langle x, \delta\rangle/\langle \delta, \delta\rangle \in \mathbb{Z}$ for any $x \in S$, then s_δ is contained in $\mathrm{O}(S)$. Assume that S is unimodular. Since δ is primitive, there exists an element $x \in S$ with $\langle x, \delta\rangle = 1$ (e.g., see the proof of Theorem 1.22, Step (1)). Thus $s_\delta \in \mathrm{O}(S)$ if and only if $m = 1$, namely, $\delta^2 = -2$. We will give examples with $m \geq 2$ in Lemmas 12.23, 12.24.

Remark 12.5. Let $\widetilde{W}(S)$ be the subgroup of $O(S)$ generated by all reflections contained in $O(S)$. Then S is called *reflective* if $\widetilde{W}(S)$ is of finite index in $O(S)$. For example, the Picard lattice of a $K3$ surface with finite automorphism group is reflective (see Corollary 8.2). It is known that any reflective lattice has rank r with $1 \leq r \leq 20$ or $r = 22$ (Esselmann [Ess]). In the case of rank 22, only one example of such a lattice (up to scale) is known, that is, $S = U \oplus D_{20}$, which was found by Borcherds [Bor1]. Since the Picard numbers of complex $K3$ surfaces are at most 20, the lattice $U \oplus D_{20}$ cannot be realized as the Picard lattice of a $K3$ surface. However, it is known that $U \oplus D_{20}$ is isomorphic to the Picard lattice of a $K3$ surface defined over an algebraically closed field in characteristic 2 (see Remark 12.29).

We now return to the case that L is the even unimodular lattice of signature $(1, 25)$. Let $W(L) = \widetilde{W}(L)$ be the subgroup of the orthogonal group $O(L)$ generated by all reflections. As mentioned above, any reflection is defined by a root because L is even unimodular. Let $P^+(L)$ be a positive cone of L. Recall that $P^+(L)$ is a connected component of the set $\{x \in L \otimes \mathbb{R} : x^2 > 0\}$ and $W(L)$ acts on $P^+(L)$ (see Section 2.2). We define

$$\mathcal{C} = \{x \in P^+(L) : \langle x, r \rangle > 0 \text{ for any } r \in \Delta\}.$$

By Remark 12.5, L is not reflective. However, the following theorem, due to Conway [Co2], tells us that Δ is nothing but a set of “simple roots”.

Theorem 12.6. *$\mathcal{C}$ is a fundamental domain of $W(L)$ with respect to the action on $P^+(L)$.*

The proof of this theorem depends on Vinberg’s algorithm finding a fundamental domain of a reflection group and the facts that the Leech lattice contains no roots and its covering radius is $\sqrt{2}$. For the detail, see Conway [Co2].

Let R be a negative definite sublattice of L generated by a finite number of some Leech roots. Then R is a root lattice. Let S be the orthogonal complement of R in L. Then S has the signature $(1, 25 - \text{rank}(R))$. Let $P(S)^+$ be the positive cone of S contained in $P^+(L)$ under the embedding of S into L. Borcherds [Bor1] studied a hyperbolic lattice S and its reflection subgroup by using the domain obtained by the restriction of $\mathcal{C}$ to $P^+(S)$. The following theorem, due to Borcherds, is fundamental to studying the automorphism group of the Kummer surface.

Theorem 12.7. *Let $\mathcal{D}$ be the restriction of $\mathcal{C}$ to $P^+(S)$ and let w be the projection of ρ into $S \otimes \mathbb{Q}$. Then*

(i) *$\mathcal{D}$ contains w, and in particular, $\mathcal{D}$ is non-empty,*

(ii) *$\mathcal{D}$ is a finite polyhedron.*

Proof. (i) We show that ρ and w are on the same side with respect to any hyperplane defined by $r \in \Delta \setminus R$. Let $\rho = w + w'$, $w' \in R \otimes \mathbb{Q}$. Then $\langle r, \rho\rangle = \langle r, w\rangle + \langle r, w'\rangle = 1$ and hence it suffices to prove that $\langle r, w'\rangle \leq 0$. For any Leech root $r' \in R$, we have $\langle r', w'\rangle = \langle r', \rho\rangle = 1$. This implies that $-w'$ is a Weyl vector of the root lattice R, that is, $-w'$ is the sum of simple roots of R with positive coefficients (e.g., Ebeling [E, Lem. 1.14]). Since $\langle r, r'\rangle \geq 0$ for any two different Leech roots r, r' (e.g., Lemma 2.8), we have $\langle -w', r\rangle \geq 0$ as desired.

(ii) Note that if R and a Leech root r generate a non-negative definite lattice, then the hyperplane $r^{\perp}$ does not cut the positive cone $P^{+}(S)$. Thus we may assume that R and r generate a negative definite lattice. This means that $\langle r, r'\rangle \leq 1$ for any simple root r' of R. Let $r = (-1 - \lambda^2/2, 1, \lambda)$, $r' = (-1 - \lambda'^2/2, 1, \lambda')$. Then it follows from Lemma 12.3 that $(\lambda - \lambda')^2 \geq -6$. Obviously, the number of λ satisfying $(\lambda - \lambda')^2 \geq -6$ for any simple root r' in R is finite. Thus we have proved the assertion. □

12.2 Néron–Severi lattice of the Kummer surface

Let $X = \mathrm{Km}(C)$ be the Kummer surface associated with a curve of genus 2 (see Section 4.4). Let S_X be the Néron–Severi lattice of X.

In this chapter we always assume that the Néron–Severi lattice of the Jacobian surface $J(C)$ is generated by the theta divisor, that is, $S_{J(C)} \cong \langle 2\rangle$ (see Definition 12.27 of a generic Kummer surface).

It follows from Corollary 6.26 that the transcendental lattice T_X is isomorphic to $U(2) \oplus U(2) \oplus \langle -4\rangle$. Thus $q_{S_X} \cong q_{U(2)} \oplus q_{U(2)} \oplus q_{\langle 4\rangle}$ (Theorem 1.32).

Let $\{N_\alpha, T_\alpha\}$ be 32 non-singular rational curves on X forming the Kummer (16_6)-configuration given in Section 4.4. Let H be the total transform of the hyperplane section of the Kummer quartic surface $\bar{X}$. It is known that the plane containing a conic T_α tangents to $\bar{X}$ along T_α. We know that $H^2 = 4$, $\langle H, N_\alpha\rangle = 0$, $\langle H, T_\alpha\rangle = 2$, and $H = 2T_\alpha + N_{\alpha_1} + \cdots + N_{\alpha_6}$, where $N_{\alpha_1}, \ldots, N_{\alpha_6}$ are 6 curves meeting T_α. The following describes the (16_6)-configuration between $\{N_\alpha\}$ and $\{T_\alpha\}$:

$$2T_0 = H - N_0 - N_1 - N_2 - N_3 - N_4 - N_5,$$
$$2T_1 = H - N_0 - N_1 - N_{12} - N_{13} - N_{14} - N_{15},$$
$$2T_2 = H - N_0 - N_2 - N_{12} - N_{23} - N_{24} - N_{25},$$
$$2T_3 = H - N_0 - N_3 - N_{13} - N_{23} - N_{34} - N_{35},$$
$$2T_4 = H - N_0 - N_4 - N_{14} - N_{24} - N_{34} - N_{45},$$
$$2T_5 = H - N_0 - N_5 - N_{15} - N_{25} - N_{35} - N_{45},$$

$$2T_{12} = H - N_1 - N_2 - N_{12} - N_{34} - N_{35} - N_{45},$$
$$2T_{13} = H - N_1 - N_3 - N_{13} - N_{24} - N_{25} - N_{45},$$
$$2T_{14} = H - N_1 - N_4 - N_{14} - N_{23} - N_{25} - N_{35},$$
$$2T_{15} = H - N_1 - N_5 - N_{15} - N_{23} - N_{24} - N_{34},$$
$$2T_{23} = H - N_2 - N_3 - N_{14} - N_{15} - N_{23} - N_{45},$$
$$2T_{24} = H - N_2 - N_4 - N_{13} - N_{15} - N_{24} - N_{35},$$
$$2T_{25} = H - N_2 - N_5 - N_{13} - N_{14} - N_{25} - N_{34},$$
$$2T_{34} = H - N_3 - N_4 - N_{12} - N_{15} - N_{25} - N_{34},$$
$$2T_{35} = H - N_3 - N_5 - N_{12} - N_{14} - N_{24} - N_{35},$$
$$2T_{45} = H - N_4 - N_5 - N_{12} - N_{13} - N_{23} - N_{45}.$$

Lemma 12.8. (1) *S_X is generated by $\{N_\alpha, T_\alpha\}$.*

(2) $\mathrm{O}(q_{S_X}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathrm{Sp}_4(\mathbb{F}_2)$, *where* $\mathrm{Sp}_4(\mathbb{F}_2)$ *is the symplectic group of the 4-dimensional symplectic space over* $\mathbb{F}_2$.

Proof. (1) We know that $d(S_X) = d(T_X) = 2^6$. Let S be the sublattice of S_X generated by H and $\{N_\alpha\}$. Since these classes are perpendicular to each other, S is isomorphic to $\langle 4 \rangle \oplus A_1^{\oplus 16}$ and S_X is an overlattice of S of index 2^6. Consider the following vectors:

$$2T_0 = H - N_0 - N_1 - N_2 - N_3 - N_4 - N_5,$$
$$2T_1 = H - N_0 - N_1 - N_{12} - N_{13} - N_{14} - N_{15},$$
$$2T_2 = H - N_0 - N_2 - N_{12} - N_{23} - N_{24} - N_{25},$$
$$2T_3 = H - N_0 - N_3 - N_{13} - N_{23} - N_{34} - N_{35},$$
$$2T_4 = H - N_0 - N_4 - N_{14} - N_{24} - N_{34} - N_{45},$$
$$2T_{12} = H - N_1 - N_2 - N_{12} - N_{34} - N_{35} - N_{45}.$$

By considering N_5, N_{i5}, we can prove that T_0, T_1, T_2, T_3, T_4, T_{12} give 6 linearly independent vectors in S_X/S. Thus S_X is obtained from S by adding T_0, T_1, T_2, T_3, T_4, T_{12}, and hence S_X is generated by $\{N_\alpha, T_\alpha\}$.

(2) Next consider the 2-elementary subgroup F of A_{S_X} ($\cong \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^4$) which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^5$. By restricting q_{S_X} to F, we have a quadratic form q on F. Then q has a 1-dimensional radical and the quotient group $(\mathbb{Z}/2\mathbb{Z})^4$ by the radical is a symplectic space over $\mathbb{F}_2$. Only the inversion -1 of A_{S_X} acts trivially on F. Thus we have the assertion. □

Lemma 12.9. *The natural map* $\mathrm{O}(S_X) \to \mathrm{O}(q_{S_X})$ *is surjective.*

Proof. The assertion follows from Proposition 1.37. □

Remark 12.10. We can prove Lemma 12.9 geometrically. First, it is known that $\mathrm{Sp}_4(\mathbb{F}_2)$ is isomorphic to the symmetric group $\mathfrak{S}_6$. This group corresponds to the symmetry group of the 6 branch points on $\mathbb{P}^1$ given in (4.4). Recall that X can be embedded into $\mathbb{P}^5$, whose image is the intersection of the 3 quadrics given in (4.6). This embedding is given by the linear system

$$\left|2H - \sum_{\alpha \in J(C)_2} N_\alpha/2\right| = \left|\sum_{\alpha \in J(C)_2} (N_\alpha + T_\alpha)/4\right|.$$

The 32 curves N_α, T_α are now lines in $\mathbb{P}^5$. The group $(\mathbb{Z}/2\mathbb{Z})^5$ acts on X as projective transformations

$$(X_1, X_2, X_3, X_4, X_5, X_6) \to (X_1, \pm X_2, \pm X_3, \pm X_4, \pm X_5, \pm X_6).$$

We can easily prove that if the number of -1 is even (resp. odd), then it is symplectic (resp. non-symplectic). Any non-symplectic involution acts on A_{S_X} as -1. We conclude that the group $(\mathbb{Z}/2\mathbb{Z})^5 \rtimes \mathfrak{S}_6$ acts on the dual graph of 32 non-singular rational curves on $\mathrm{Km}(C)$.

Exercise 12.11. Show that if the number of -1 is 3, then the transformation as above is fixed-point-free. In particular, the quotient surface is an Enriques surface (see Example 9.4).

Proposition 12.12. *Let* $\Gamma(S_X) = \{\varphi \in \mathrm{O}(S_X) : \varphi|A_{S_X} = \pm 1\}$. *Then*

$$\mathrm{Aut}(X) \cong \Gamma(S_X)/\{\pm 1\} \cdot W(S_X).$$

Proof. Since the rank of the transcendental lattice T_X is 5, by Corollary 8.13 any automorphism acts on T_X of order at most 2. This implies that the image of the natural map $\mathrm{Aut}(X) \to \mathrm{O}(S_X)$ is contained in $\Gamma(S_X)$. As mentioned above, there exists a non-symplectic automorphism and hence the assertion follows from the proof of Theorem 8.1. □

Remark 12.13. Proposition 12.12 is due to Nikulin [Ni1].

12.3 Classical automorphisms of the Kummer surface

The following automorphisms are classically known. We call an automorphism of order 2 an involution.

(i) **Translations**. Let $a \in J(C)$ be a 2-torsion point. The translation of $J(C)$ by a commutes with the inversion ι of $J(C)$ and hence it induces an involution, denoted by t_a, of $\mathrm{Km}(C)$. We call it a translation too. We have 16 translations.

(ii) **Switches**. The Kummer surface $\mathrm{Km}(C)$ and the dual $\mathrm{Km}(C)^*$ are projectively isomorphic (see Section 4.4). Thus we have an involution of $\mathrm{Km}(C)$ which is called a switch. By composing translations, we have 16 switches. In Remark 12.10, we mention that there are 16 non-symplectic involutions induced from projective transformations. These are nothing but the switches. We denote by σ the one with $\sigma(N_\alpha) = T_\alpha$ (see Ohashi [Oh, Sect. 5] and also Kondo [Kon4, Rem. 4.3(ii)] for the existence of σ).

(iii) **Projections**. As mentioned in Section 4.4, the projection from a node n_α of the Kummer quartic surface $\bar{X}$ gives a double covering $\pi_\alpha\colon \mathrm{Km}(C) \to \mathbb{P}^2$ branched along 6 lines in $\mathbb{P}^2$. We denote the covering transformation by p_α and call it a projection too. The 6 lines are the images of the 6 T_β meeting with N_α. These 6 lines touch a conic which is the image of $N_\alpha + p_\alpha(N_\alpha)$. Thus we have 16 projections.

(iv) **Correlations**. This is the dual version of the projections. In other words, $\sigma^{-1} \circ p_\alpha \circ \sigma$ is a correlation. There are 16 correlations.

(v) **Cremona involution associated with a Göpel tetrad**. Recall that a Kummer quartic surface $\bar{X}$ contains 16 nodes and 16 conics (see Section 4.4). In the following we denote a node n_α by α for simplicity. A *Göpel tetrad* is a set of 4 nodes of $\bar{X}$ such that any 3 of them are not on a conic. For example, $\{0, 3, 14, 25\}$ is a Göpel tetrad. It is known that there are 60 Göpel tetrads (Hudson [Hud, §51]). We remark that for each 2 nodes in a Göpel tetrad there exist exactly 2 conics passing to the 2 nodes. Now we take a Göpel tetrad and fix it. Let (x, y, z, t) be homogeneous coordinates. We may assume that the 4 vertices of the Göpel tetrad are given by those of the tetrahedron $xyzt = 0$. With respect to these coordinates, the Kummer quartic surface can be given by

$$\begin{aligned} &A(x^2t^2 + y^2z^2) + B(y^2t^2 + z^2x^2) + C(z^2t^2 + x^2y^2) + Dxyzt \\ &\quad + F(yt + zx)(zt + xy) + G(zt + xy)(xt + yz) + H(xt + yz)(yt + zx) = 0 \end{aligned} \tag{12.2}$$

(Hutchinson [Hut2], Hudson [Hud, §120]). The standard Cremona transformation

$$(x, y, z, t) \to (yzt, xzt, xyt, xyz)$$

preserves the above equation of the Kummer quartic surface and hence induces an involution c_G of $\mathrm{Km}(C)$. We call this involution c_G a Cremona involution associated with the Göpel tetrad G. Thus we have 60 such involutions.

(vi) **Cremona involution associated with a Weber hexad**. A tetrad consisting of 4 nodes is called a *Rosenhain tetrad* if each 3 nodes are contained in a conic. For example, $\{3, 15, 23, 34\}$ is a Rosenhain tetrad. It is known that there are 80 Rosenhain tetrads (Hudson [Hud, §50]). A hexad consisting of 6 nodes is called a *Weber hexad* if it is the symmetric difference of a Göpel tetrad and a Rosenhain tetrad. There are 192 Weber hexads (Hudson [Hud, §52]). Let W be a Weber hexad. Then the linear system

$$\left| \mathcal{O}_{\bar{X}}(2) - \sum_{\alpha \in W} n_\alpha \right|$$

gives another model of X as a quartic surface

$$\bar{X}_W : \sum_{i=1}^{5} t_i = \sum_{i=1}^{5} \frac{\mu_i}{t_i} = 0, \tag{12.3}$$

where $(t_1, \ldots, t_5)$ are homogeneous coordinates of $\mathbb{P}^4$, and $\mu_1, \ldots, \mu_5$ are non-zero constants. Note that this quartic model is nothing but the Hessian of a cubic surface (see equation (9.19)). The Cremona transformation given in (9.20) defines an involution c_W of X. We call c_W a Cremona involution associated with a Weber hexad W. Thus we have 192 such involutions (Hutchinson [Hut1], Dolgachev, Keum [DK]). It is known that a Hessian quartic surface is birationally isomorphic to a Kummer surface if and only if the coefficients $\mu_1, \ldots, \mu_5$ in (12.3) satisfy the cubic relation

$$\sum_{i=1}^{5} \mu_i^3 - \sum_{i \neq j} \mu_i^2 \mu_j + 2 \sum_{i \neq j \neq k} \mu_i \mu_j \mu_k = 0 \tag{12.4}$$

(Rosenberg [Ro]). Combining the proof of Lemma 9.52 and equation (12.4), c_W is a fixed-point-free involution of X if X is general, and hence $X/\langle c_W \rangle$ is an Enriques surface.

Remark 12.14. There is a non-degenerate symplectic form

$$\langle\ ,\ \rangle : J(C)_2 \times J(C)_2 \to \mathbb{F}_2, \quad (\alpha, \beta) \to |\alpha \cap \beta| \bmod 2$$

called the Weil pairing (e.g., Arbarello, Cornalba, Griffiths, Harris [ACGH, p. 210]). A Göpel tetrad can be defined as a maximal isotropic subspace in $J(C)_2$ with respect to the Weil pairing and its translations by 2-torsions. The Rosenhain tetrads are 2-dimensional non-isotropic subspaces and their translations (e.g., see Dolgachev, Keum [DK, §4]).

Remark 12.15. In 1885, Klein [Kl] gave automorphisms (i), (ii), (iii), (iv) of the Kummer surface $\mathrm{Km}(C)$ associated with a curve C of genus 2, and raised a question about whether they generate the automorphism group of $\mathrm{Km}(C)$ or not. Later, Hutchinson [Hut1], [Hut2] presented two new automorphisms (v), (vi).

12.4 The Kummer surface and the Leech roots

We now apply the results in Section 12.1 to the Kummer surface. Consider the following points in the Leech lattice Λ:

$$X = 4\nu_\infty + \nu_\Omega, \quad Y = 4\nu_0 + \nu_\Omega, \quad Z = 0, \quad P = 4\nu_\infty + 4\nu_0, \quad Q_i = 2\nu_{K_i} \ (i = 1, \dots, 5),$$

where $K_1, \dots, K_5$ are 5 octads containing 4 points ∞, 0, 1, 2. By definition we have

$$X^2 = Y^2 = -6, \quad P^2 = Q_i^2 = -4.$$

Consider the Leech roots

$$x = (2, 1, X), \quad y = (2, 1, Y), \quad z = (-1, 1, Z), \quad x_0 = (1, 1, P), \quad x_i = (1, 1, Q_i)$$

and let R be the root lattice generated by these Leech roots. We can easily calculate that

$$\langle X, Y\rangle = -4, \quad \langle X, P\rangle = \langle Y, P\rangle = \langle X, Q_i\rangle = \langle Y, Q_i\rangle = -3,$$
$$\langle P, Q_i\rangle = \langle Q_i, Q_j\rangle = -2 \quad (i \neq j),$$

which implies that R is isomorphic to $A_3 \oplus A_1^{\oplus 6}$ by equation (12.1). The discriminant group A_R is generated by

$$t = \frac{(x + 2z + 3y)}{4}, \quad t_i = \frac{x_i}{2} \quad (i = 0, 1, \dots, 5).$$

Note that R is not primitive in L. Let T be an overlattice of R in L given by an isotropic vector

$$2t + t_0 + t_1 + \cdots + t_5 \tag{12.5}$$

(see Theorem 1.19).

Proposition 12.16. *The lattice T is primitive in L and R is the maximal root lattice contained in T.*

For the proof of Proposition 12.16 we refer the reader to Kondo [Kon4, Lem. 4.1]. The discriminant group of A_T is generated by

$$t_0 + t_1, \quad t_0 + t_2, \quad t_3 + t_4, \quad t_3 + t_5, \quad t + t_0 + t_1 + t_2$$

and q_T is isomorphic to $q_{U(2)} \oplus q_{U(2)} \oplus q_{\langle -4 \rangle}$. This implies that $q_T \cong q_{T_X} \cong -q_{S_X}$. It follows from Theorem 1.34 that S_X can be primitively embedded in L with the orthogonal complement T.

Now we may assume that the positive cone $P^+(S_X)$ is embedded into $P^+(L)$. We denote by $\mathcal{C}(X)$ the restriction of $\mathcal{C}$ to $P^+(S_X)$ under this embedding. Since any root in S_X is a root in L, $\mathcal{C}(X)$ is contained in the ample cone $A(X)$. For a Leech root r, we set $r = r' + r''$ with $r' \in S_X^*$ and $r'' \in R^*$. Then each face of $\mathcal{C}(X)$ is defined by the hyperplane perpendicular to r' for some Leech root r. If a face of $\mathcal{C}(X)$ is defined by a hyperplane perpendicular to a vector r' of norm $-k$, we call it a $(-k)$-face. Note that the hyperplane $(r')^\perp$ meets $P^+(S_X)$ if and only if $(r')^2 < 0$. This is equivalent to the condition that the lattice $\tilde{R}$ generated by R and r is negative definite, that is, $\tilde{R}$ is a root lattice. Such an $\tilde{R}$ is one of the following:

$$A_3 \oplus A_1^{\oplus 7}, \quad D_4 \oplus A_1^{\oplus 6}, \quad A_3^{\oplus 2} \oplus A_1^{\oplus 4}, \quad A_5 \oplus A_1^{\oplus 5}, \quad A_4 \oplus A_1^{\oplus 6},$$
$$A_3 \oplus A_2 \oplus A_1^{\oplus 5}, \quad D_5 \oplus A_1^{\oplus 5}, \quad D_6 \oplus A_1^{\oplus 4}, \quad D_4 \oplus A_3 \oplus A_1^{\oplus 3}.$$

Lemma 12.17. *The faces of $\mathcal{C}(X)$ consist of* 32 (-2)*-faces,* 32 (-1)*-faces,* 60 (-1)*-faces, and* 192 $(-3/4)$*-faces.*

Proof. **Case** (1): $\tilde{R} = A_3 \oplus A_1^{\oplus 7}$. Then $r = (m, n, \lambda)$ is perpendicular to all Leech roots in R. Since $\langle r, z \rangle = \langle r, (-1, 1, 0) \rangle = 0$, we have $r = (1, 1, \lambda)$ with $\lambda^2 = -4$. Since r is perpendicular to x, y, x_i $(i = 0, 1, \ldots, 5)$, it follows from Proposition 12.2 that $\lambda = 2\nu_K$ where K is an octad containing $\infty, 0$ and satisfying $|K \cap K_i| = 4$ $(i = 1, \ldots, 5)$. Now we count the number of such octads. Take a point A in $(K_1 \cap K_2) \setminus \{\infty, 0\}$ and a point $B \in K_1 \setminus K_2$. Then there exist exactly 5 octads K containing $\{\infty, 0, A, B\}$ (Lemma 11.7). One of the 5 octads is K_1. Since $K \cap K_i$ contains ∞, 0, A, we have $|K \cap K_i| = 4$ $(i = 1, \ldots, 5)$ for $K \neq K_1$. Thus the number of the desired octads is

$$\binom{2}{1} \times \binom{4}{1} \times 4.$$

We have two sets of 16 octads according to the choice of A. Any 2 octads in the same set have 4 common points and hence the corresponding Leech roots are perpendicular. Assume that K contains $\{\infty, 0, A, B\}$. Then an octad K' in the other set can be written as $K' = \{\infty, 0, A', B, \ldots\}$ or $K' = \{\infty, 0, A', B', \ldots\}$ where $\{A, A'\} = (K_1 \cap K_2) \setminus \{\infty, 0\}$ and $K_1 \setminus K_2 = \{B, B', \ldots\}$. If $K' = \{\infty, 0, A', B, \ldots\}$, then $|K \cap K'| = 4$. If we take the 4 points $\{\infty, 0, A', B'\}$, then there are 4 octads other than K_1 containing these 4 points in which 2 octads have 4 common points with K and the other 2 octads have 2 common points with K. Thus there exist exactly 3×2 octads in the other set having 2 common points with K. Later we will list the 32 such octads (see Remark 12.18). We can check that the incidence relation between the 32 corresponding roots coincides with (16_6)-configuration.

Case (2): $\tilde{R} = D_4 \oplus A_1^{\oplus 6}$. In this case, $r = (m, n, \lambda)$ meets only z and is perpendicular to x, y, x_i. This implies $r = (2, 1, \lambda)$ with $\lambda^2 = -6$. It follows from Proposition 12.2 that $\lambda = (\xi_\infty, \xi_0, \xi_{j_1}, \ldots, \xi_{j_6}, \xi_{j_7}, \ldots, \xi_{j_{22}}) = (3, 3, 3, -1, \ldots, -1, 1, \ldots, 1)$ where $K = \{\infty, 0, j_1, \ldots, j_6\}$ is an octad satisfying $|K \cap K_i| = 4$ and $K_i \ni j_1$ for any i. Such octads coincide with the ones obtained in Case (1), and hence the number of desired Leech roots is equal to 32. Since $r = r' - (x + 2z + y)/2$, we have $(r')^2 = -1$.

Case (3): $\tilde{R} = A_3^{\oplus 2} \oplus A_1^{\oplus 4}$. We may assume that r meets x_0 and x_i. Then $r = (1, 1, \lambda)$ and

$$\lambda = \nu_\Omega - 4\nu_k \quad (k \neq \infty, 0), \quad k \in K_i, \quad k \notin K_j \quad (j \neq i).$$

Thus the number of such Leech roots is

$$\binom{6}{2} \times 4 = 60.$$

Since $r = r' - x_0/2 - x_i/2$,, we have $(r')^2 = -1$.

Case (4): $\tilde{R} = A_5 \oplus A_1^{\oplus 5}$. We may assume that r meets y and x_0. Then $r = (1, 1, \lambda)$ and

$$\lambda = 2\nu_K, \quad K \text{ is an octad with } K \ni \infty, \quad K \not\ni 0, \quad |K \cap K_i| = 4 \quad (i = 1, \ldots, 5).$$

Such a K contains three points $(K_1 \cap K_2) \setminus \{0\}$. By choosing A in $K_1 \setminus K_2$, we obtain 4 such octads other than K_1. By considering the choice of x, y and x_i $(i = 0, 1, \ldots, 5)$, we thus have 192 $(= 2 \times 6 \times 4 \times 4)$ such Leech roots. Since $r = r' - (x + 2z + 3y)/4 - x_0/2$, we have $(r')^2 = -3/4$.

Case (5): other cases. In the remaining cases, using case-by-case arguments we can prove that there are no Leech roots r such that r and R generate $\tilde{R}$. □

Remark 12.18. In [Todd, Table I], Todd gave a list of all octads. In Table 12.1 we give a list of the 77 octads containing ∞, 0. In the following we use the notation in the table. We assume that $K_1, \ldots, K_5$ are

$$K_1 = \{\infty, 0, 1, 2, 3, 5, 14, 17\}, \quad K_2 = \{\infty, 0, 1, 2, 4, 13, 16, 22\},$$
$$K_3 = \{\infty, 0, 1, 2, 6, 7, 19, 21\}, \quad K_4 = \{\infty, 0, 1, 2, 8, 11, 12, 18\},$$
$$K_5 = \{\infty, 0, 1, 2, 9, 10, 15, 20\}.$$

Then the 32 (-2)-faces obtained in Lemma 12.17 are defined by the Leech roots corresponding to the following octads. We also give the correspondence between these 32 octads and the 32 non-singular rational curves $\{N_\alpha, T_\alpha\}$ on $X = \mathrm{Km}(C)$:

$$\begin{aligned}
N_{45} &= \{\infty, 0, 1, 3, 4, 11, 19, 20\}, & N_5 &= \{\infty, 0, 1, 3, 6, 8, 10, 13\},\\
N_{24} &= \{\infty, 0, 1, 3, 7, 9, 16, 18\}, & N_2 &= \{\infty, 0, 1, 3, 12, 15, 21, 22\},\\
N_3 &= \{\infty, 0, 1, 4, 5, 7, 8, 15\}, & N_{35} &= \{\infty, 0, 1, 4, 6, 9, 12, 17\},\\
N_4 &= \{\infty, 0, 1, 4, 10, 14, 18, 21\}, & N_1 &= \{\infty, 0, 1, 5, 6, 18, 20, 22\},\\
N_{34} &= \{\infty, 0, 1, 5, 9, 11, 13, 21\}, & N_{14} &= \{\infty, 0, 1, 5, 10, 12, 16, 19\},\\
N_{13} &= \{\infty, 0, 1, 6, 11, 14, 15, 16\}, & N_{15} &= \{\infty, 0, 1, 7, 10, 11, 17, 22\},\\
N_0 &= \{\infty, 0, 1, 7, 12, 13, 14, 20\}, & N_{25} &= \{\infty, 0, 1, 8, 9, 14, 19, 22\},\\
N_{23} &= \{\infty, 0, 1, 8, 16, 17, 20, 21\}, & N_{12} &= \{\infty, 0, 1, 13, 15, 17, 18, 19\},\\
T_1 &= \{\infty, 0, 2, 3, 4, 8, 9, 21\}, & T_{34} &= \{\infty, 0, 2, 3, 6, 12, 16, 20\},\\
T_{14} &= \{\infty, 0, 2, 3, 7, 11, 13, 15\}, & T_3 &= \{\infty, 0, 2, 3, 10, 18, 19, 22\},\\
T_2 &= \{\infty, 0, 2, 4, 5, 6, 10, 11\}, & T_{25} &= \{\infty, 0, 2, 4, 7, 17, 18, 20\},\\
T_{15} &= \{\infty, 0, 2, 4, 12, 14, 15, 19\}, & T_{45} &= \{\infty, 0, 2, 5, 7, 9, 12, 22\},\\
T_{24} &= \{\infty, 0, 2, 5, 8, 13, 19, 20\}, & T_5 &= \{\infty, 0, 2, 5, 15, 16, 18, 21\},\\
T_4 &= \{\infty, 0, 2, 6, 8, 15, 17, 22\}, & T_{23} &= \{\infty, 0, 2, 6, 9, 13, 14, 18\},\\
T_{12} &= \{\infty, 0, 2, 7, 8, 10, 14, 16\}, & T_0 &= \{\infty, 0, 2, 9, 11, 16, 17, 19\},\\
T_{13} &= \{\infty, 0, 2, 10, 12, 13, 17, 21\}, & T_{35} &= \{\infty, 0, 2, 11, 14, 20, 21, 22\}.
\end{aligned}$$

In the following we identify 32 non-singular rational curves $\{N_\alpha, T_\alpha\}$ and 32 Leech roots defining 32 (-2)-faces of $\mathcal{C}(X)$.

Lemma 12.19. *The automorphism group of $\mathcal{C}(X)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^5 \rtimes \mathfrak{S}_6$. In particular the subgroup $(\mathbb{Z}/2\mathbb{Z})^5$ consists of 16 translations and 16 switches.*

Proof. Recall that $\mathrm{O}(q_T) \cong \mathbb{Z}/2\mathbb{Z} \times \mathrm{Sp}_4(\mathbb{F}_2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_6$ (Lemma 12.8). Note that the symmetry group of the Dynkin diagram of $R = A_3 \oplus A_1^{\oplus 6}$ is isomorphic to the same group. Moreover, it preserves the isotropic vector given by (12.5), and hence the map $\mathrm{O}(T) \to \mathrm{O}(q_T)$ is surjective. It follows from Corollary 1.33 that any isomorphism in $\mathrm{Aut}(\mathcal{C}(X))$ can be extended to an isomorphism of L. Thus $\mathrm{Aut}(\mathcal{C}(X))$ is a subgroup of $\mathrm{Aut}(\mathcal{C})$ preserving R. The restriction of $\mathrm{Aut}(\mathcal{C})$ to R is a subgroup of $\mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_6$. Now let G be the subgroup $\mathrm{Aut}(\mathcal{C})$ acting on R trivially. Since G fixes X, Y, Z, and P, so G fixes two points ∞, 0, that is, G is a subgroup of the Mathieu group M_{22}. By using the classification table of the maximal subgroups of M_{22} in Conway [Co1], we can prove that G is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. On the other hand, we already know that $(\mathbb{Z}/2\mathbb{Z})^5 \rtimes \mathfrak{S}_6$ is the symmetry group of the Kummer (16_6)-configuration (see Remark 12.10). Thus we have the assertion. □

Lemma 12.20. *The group* $\mathrm{Aut}(\mathcal{C}(X))$ *acts transitively on the set of faces of* $\mathcal{C}(X)$ *of each type.*

Proof. In the cases of 32 (-2)-faces and 32 (-1)-faces, the assertion is obvious.

In the case of 60 (-1)-faces we will show that

$$2r' = H - (N_{\alpha_1} + N_{\alpha_2} + N_{\alpha_3} + N_{\alpha_4}),$$

where $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a Göpel tetrad in Lemma 12.22. Recall that Göpel subspaces are maximal isotropic subspaces in $J(C)_2$ with respect to the Weil pairing $\langle\ ,\ \rangle$ (Remark 12.14). The automorphism group of the symplectic space $(J(C)_2, \langle\ ,\ \rangle)$ is $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathrm{Sp}(4, \mathbb{F}_2) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathfrak{S}_6$, which acts transitively on the set of isotropic subspaces. Thus the assertion follows.

In the case of 192 $(-3/4)$-faces, we will show that the stabilizer group of such a Leech root is $\mathfrak{S}_5$. Let r be a Leech root given in the proof of Lemma 12.17, Case (3). The stabilizer subgroup of $2\nu_K$ is a subgroup of the Mathieu group M_{22} fixing ∞, 0 which preserves the octad K and whose orbits on Ω have length $(1, 1, 2, 5, 15)$. It follows from Conway [CS, Chap. 10, Table 10.3] that the stabilizer subgroup of r' is isomorphic to $\mathfrak{S}_5$. Since $[\mathrm{Aut}(\mathcal{C}(X)) : \mathfrak{S}_5] = 192$, we have the assertion. □

Lemma 12.21. *The projection w of the vector ρ into $S_X \otimes \mathbb{Q}$ is equal to*

$$\frac{1}{4} \sum_{\alpha \in J(C)_2} (N_\alpha + T_\alpha).$$

Proof. Let $\rho = w + w'$ with $w' \in R \otimes \mathbb{Q}$. Then

$$w' = -(3x + 3y + 4z)/2 - x_0/2 - \cdots - x_5/2,$$

which has norm -8. Since $\rho^2 = 0$, we have $w^2 = 8$. Moreover, $\langle \rho, r \rangle = 1$ for any Leech root r, and hence w has the intersection number 1 with any N_α, T_α. As mentioned in Remark 12.10, the divisor

$$\sum_{\alpha \in J(C)_2} (N_\alpha + T_\alpha)/4$$

is the hyperplane section of X in $\mathbb{P}^5$ and N_α, T_α are lines in $\mathbb{P}^5$. Therefore it has the same norm 8 and the same intersection number 1 with any N_α, T_α. Since $\{N_\alpha, T_\alpha\}$ generate S_X (Lemma 12.8(1)), we have proved the assertion. □

Now we study a geometric meaning of the remaining faces of $\mathcal{C}(X)$.

Lemma 12.22. (1) *In the case $\tilde{R} = D_4 \oplus A_1^{\oplus 6}$,*

$$2r' = H - 2N_\alpha \quad \text{or} \quad \sigma(H - 2N_\alpha)$$

for some α. In particular, the reflection $s_{r'}$ coincides with p_α^* or $(\sigma^{-1} \circ p_\alpha \circ \sigma)^*$.

(2) *In the case $\tilde{R} = A_3^{\oplus 2} \oplus A_1^{\oplus 4}$,*

$$2r' = H - (N_\alpha + N_\beta + N_\gamma + N_\delta),$$

where $G = \{\alpha, \beta, \gamma, \delta\}$ is a Göpel tetrad.

(3) *In the case $\tilde{R} = A_5 \oplus A_1^{\oplus 5}$,*

$$4r' = 3H - 2\sum_{\alpha \in W} N_\alpha,$$

where W is a Weber hexad.

Proof. (1) Recall that r is defined by an octad K containing ∞, 0 and satisfying $|K \cap K_i| = 4$ (see Lemma 12.17, Case (2)). This octad also defines, by another correspondence, a non-singular rational curve N_α, T_α (in Lemma 12.17, Case (1)). We consider the case $(1, 1, 2\nu_K) = N_\alpha$ and $r = (2, 1, \nu_\Omega + 4\nu_\infty + 4\nu_0 + 4\nu_{j_1} - 2\nu_K)$ in the notation in Lemma 12.17. It now follows from equation (12.1) that

$$(r, N_\beta) = 2\delta_{\alpha,\beta}.$$

Since $(r')^2 = -1$, we have $2r' = H - 2N_\alpha$. The proof of the case $(1, 1, 2\nu_K) = T_\alpha$ is similar.

(2) We first consider the case that

$$r = (1, 1, \nu_\Omega - 4\nu_k)$$

(see the proof of Lemma 12.17, Case (3)). Then $\langle r, N_\alpha \rangle = 1$ (resp. $\langle r, N_\alpha \rangle = 0$) if and only if $k \in N_\alpha$ (resp. $k \notin N_\alpha$), where we denote by the same symbol N_α the corresponding octad to the curve N_α. The number of N_α with $k \in N_\alpha$ is 4 by the proof of Lemma 12.17, Case (1). Denote by $N_{\alpha_1}, N_{\alpha_2}, N_{\alpha_3}, N_{\alpha_4}$ such N_α. Combining with $(r')^2 = -1$, we have

$$2r' = H - (N_{\alpha_1} + N_{\alpha_2} + N_{\alpha_3} + N_{\alpha_4}).$$

For example, if $k = 3$, then $2r' = H - (N_2 + N_5 + N_{24} + N_{45})$. We can check that $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a Göpel tetrad by using the fact that 2 distinct octads meet at 0, 2, or 4 points.

Next we consider the case that r meets x_i and x_j $(i, j \neq 0)$. Then

$$r = (1, 1, 2\nu_K),$$

where K is an octad satisfying

$$K \ni \infty, 0, \quad |K \cap K_i| = |K \cap K_j| = 2, \quad |K \cap K_k| = 4 \quad (k \neq i, j).$$

By a similar argument as before, we can prove that

$$2r' = H - (N_{\beta_1} + N_{\beta_2} + N_{\beta_3} + N_{\beta_4}),$$

where $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ is a Göpel tetrad.

(3) Since $\mathrm{Aut}(\mathcal{C}(X))$ acts transitively on 192 Leech roots (Lemma 12.20), we may assume that $\{N_\alpha, T_\alpha\}$ are defined by the 32 octads given in Remark 12.18, $K = \{\infty, 1, 2, 3, 4, 6, 15, 18\}$ ([Todd, Table I]) and $r = (1, 1, 2\nu_K)$. Then $\langle N_\alpha, r\rangle = 1$ if and only if $\alpha \in W = \{0, 14, 15, 23, 25, 34\}$, and otherwise $\langle N_\alpha, r\rangle = 0$. Note that W is a Weber hexad which is the symmetric difference of a Göpel tetrad $\{0, 3, 14, 25\}$ and a Rosenhain tetrad $\{3, 15, 23, 34\}$. Since $4r'$ and $3H - 2\sum_{\alpha \in W} N_\alpha$ have the same norm -12 and the same intersection number with N_α $(\alpha \in J(C)_2)$, we have the assertion. □

12.5 Automorphism group of a generic Kummer surface

First, we study the action of classical involutions on the Néron–Severi lattice S_X of $X = \mathrm{Km}(C)$. Recall that $\{N_\alpha, T_\alpha\}$ generate S_X and $\{H, N_\alpha\}$ generate a sublattice of S_X of finite index (Lemma 12.8 and its proof). Hence the action of any automorphism on S_X is determined by the one on $\{H, N_\alpha\}$.

We give a remark on reflections associated with (-4)-vectors (see Definition 12.4). Consider the case $\delta = H - 2N_\alpha$. Note that $\delta^2 = -4$ and $\langle \delta, N_\beta\rangle$, $\langle \delta, T_\beta\rangle$ are even integers. It follows from Lemma 12.8(1) that $2\langle x, \delta\rangle/\langle \delta, \delta\rangle \in \mathbb{Z}$ for any $x \in S_X$. Thus $s_\delta \in \mathrm{O}(S_X)$. Next consider $\delta = H - N_\alpha - N_\beta - N_\gamma - N_\delta$, where $\{\alpha, \beta, \gamma, \delta\}$ is a Göpel tetrad. Then $\delta^2 = -4$ and any $T_{\alpha'}$ meets 0 or 2 members in $\{N_\alpha, N_\beta, N_\gamma, N_\delta\}$. This implies that $2\langle x, \delta\rangle/\langle \delta, \delta\rangle \in \mathbb{Z}$ for any $x \in S_X$. Thus $s_\delta \in \mathrm{O}(S_X)$.

Lemma 12.23. *The action of a translation t_α, the switch σ, and a projection p_α are as follows*:

(1) $t_\alpha^*(H) = H,\ t_\alpha^*(N_\beta) = N_{\alpha+\beta}$.

(2) $\sigma^*(H) = 3H - \sum_{\alpha\in J(C)_2} N_\alpha,\ \sigma^*(N_\alpha) = T_\alpha$.

(3) *p_α^* acts on S_X as the reflection s_{H-2N_α}.*

Proof. (1) The translation is induced from a projective transformation of $\mathbb{P}^3$ containing the Kummer quartic surface and hence $t_\alpha^*(H) = H$. The remaining assertion is obvious.

(2) As mentioned in the definition of switches, σ is induced from a projective transformation of $\mathbb{P}^5$ (see Remark 12.10) and hence σ^* fixes the hyperplane section

$$2H - \sum_{\alpha\in J(C)_2} N_\alpha/2 = \sum_{\alpha\in J(C)_2}(N_\alpha + T_\alpha)/4.$$

By combining this with $\sigma(N_\alpha) = T_\alpha$, we have the assertion.

(3) Since the line passing through n_α and n_β meets $\bar{X}$ only at these 2 points, the projection fixes N_β ($\beta \neq \alpha$). The 6 curves T_β meeting with N_α are the ramification of the double covering π_α mentioned in Section 4.4, and hence p_α fixes these curves. Finally, $N_\alpha + p_\alpha(N_\alpha)$ is the inverse image of the conic tangent to the 6 lines. This implies that $\langle N_\alpha, p_\alpha(N_\alpha)\rangle = 6$, $\langle N_\beta, p_\alpha(N_\alpha)\rangle = 0$ ($\beta \neq \alpha$) and hence $p_\alpha(N_\alpha) = kH - 3N_\alpha$. Since $p_\alpha(N_\alpha)^2 = -2$, we obtain $p_\alpha(N_\alpha) = 2H - 3N_\alpha = s_{H-2N_\alpha}(N_\alpha)$. Thus we have the assertion. □

Next we consider the involution c_G associated with a Göpel tetrad $G = \{\alpha,\beta,\gamma,\delta\}$. Let r_G be the reflection of S_X with respect to the vector $H - N_\alpha - N_\beta - N_\gamma - N_\delta$ of norm -4. Let t_G be an isomorphism of S_X defined as follows: t_G fixes H, N_α, N_β, N_γ, N_δ and t_G changes two $T_{\alpha'}$, $T_{\beta'}$ containing 2 nodes among $\{\alpha,\beta,\gamma,\delta\}$. This defines the action of t_G on the remaining 12 $N_{\alpha'}$. Then we have the following.

Lemma 12.24. *The Cremona involution c_G associated with a Göpel tetrad G coincides with $r_G \circ t_G$.*

Proof. Since c_G is induced by the linear system $|3H - 2(n_\alpha + n_\beta + n_\gamma + n_\delta)|$ on $\mathbb{P}^3$, it sends H to $3H - 2(N_\alpha + N_\beta + N_\gamma + N_\delta)$. Since c_G sends the node n_α to the plane containing 3 nodes n_β, n_γ, n_δ, we have $c_G(N_\alpha) = H - (N_\beta + N_\gamma + N_\delta)$. Assume that $n_\alpha = (1,0,0,0)$, $n_\beta = (0,1,0,0)$, $n_\gamma = (0,0,1,0)$, $n_\delta = (0,0,0,1)$ and consider equation (12.2) of the Kummer quartic surface. By an elementary, but long, calculation, we can see that if a conic passing through n_α, n_β is defined by the hyperplane section $az + bt = 0$, then c_G sends $az + bt = 0$ to $bz + at = 0$. This implies that c_G interchanges two conics passing through n_α, n_β. Now we can easily check the assertion. □

Note that t_G fixes the hyperplane perpendicular to $H - N_\alpha - N_\beta - N_\gamma - N_\delta$. Thus c_G is similar to the reflection r_G, but the difference is that c_G acts non-trivially on the reflective hyperplane.

Finally, we consider the Cremona involution c_W associated with a Weber hexad W. In this case we assume that the Kummer surface is general in the sense that c_W is a fixed-point-free involution.

Let W be a Weber hexad. For each $\alpha \in W$ there exists exactly one $\beta \in J(C)_2$ satisfying $\langle N_\alpha, T_\beta \rangle = 1$ and $\langle N_{\alpha'}, T_\beta \rangle = 0$ for $\alpha' \in W$, $\alpha' \neq \alpha$. We denote this β by $\mu(\alpha)$. We now have two sets

$$\mathcal{A} = \{N_\alpha : \alpha \notin W\}, \quad \mathcal{B} = \{T_\beta : \beta \neq \mu(\alpha),\ \alpha \in W\},$$

each of which consists of 10 disjoint curves. Each member of one set meets exactly 3 members in another set. The dual graph of 20 curves in $\mathcal{A} \cup \mathcal{B}$ coincides with the one of 20 curves $\{E_{ijk}, L_{mn}\}$ on the minimal resolution of the Hessian quartic surface associated with a cubic surface (see Section 9.4.4; also see [DK, Fig. 2] for their incidence relation). The Cremona involution c_W switches $\mathcal{A}$ and $\mathcal{B}$. We denote $c_W(N_\alpha)$ by $T_{\mu'(\alpha)}$. When we consider the dual graph of 20 curves, the vertex $T_{\mu'(\alpha)}$ has the "longest distance" from the vertex N_α (see [DK, Fig. 2]). We also remark that for $\alpha \notin W$, denoting by $T_{\beta_1}, T_{\beta_2}, T_{\beta_3}$ the 3 curves in $\mathcal{B}$ meeting with N_α, $\mu'(\alpha) = \beta_1 + \beta_2 + \beta_3$.

Lemma 12.25. *Assume that c_W is a fixed-point-free involution. Then the action of the Cremona involution c_W associated with a Weber hexad W is given as*

$$c_W(H) = 9H - \sum_{\alpha \in J(C)_2} N_\alpha - 4 \sum_{\beta \in W} N_\beta,$$

$$c_W(N_\alpha) = 3H - \frac{1}{2} \sum_{\beta \in J(C)_2} N_\beta - \sum_{\beta' \in W} N_{\beta'} - T_{\mu(\alpha)} \quad (\alpha \in W),$$

$$c_W(N_\alpha) = T_{\mu'(\alpha)} \qquad (\alpha \notin W).$$

Proof. Since $(\mathbb{Z}/2\mathbb{Z})^5 \rtimes \mathfrak{S}_6$ acts on the set of Weber hexads (Lemma 12.20), it suffices to prove the assertion for a Weber hexad. Let $W = \{0, 14, 15, 23, 25, 34\}$ be the Weber hexad obtained as the symmetric difference of a Göpel tetrad $\{0, 25, 14, 3\}$ and a Rosenhain tetrad $\{34, 23, 15, 3\}$. Then N_α $(\alpha \notin W)$ are contracted to 10 rational double points on $\bar{X}_W$, and 10 curves T_α meeting 3 members N_α, $\alpha \in W$ map to 10 lines on $\bar{X}_W$ (see Section 12.3(vi)). These 20 curves are interchanged by c_W as follows:

$$(N_1, T_{23}), \quad (N_2, T_{14}), \quad (N_3, T_{15}), \quad (N_4, T_{25}), \quad (N_5, T_{34}),$$
$$(N_{12}, T_5), \quad (N_{13}, T_4), \quad (N_{24}, T_3), \quad (N_{35}, T_2), \quad (N_{45}, T_1).$$

Since c_W is a fixed-point-free involution and hence the quotient surface $X/\langle c_W \rangle$ is an Enriques surface, it follows that the c_W^*-invariant sublattice L_X^+ has rank 10 (see Lemma 9.11) and $\{N_\alpha + c_W(N_\alpha)\}_{\alpha \notin W}$ is a $\mathbb{Q}$-basis of the invariant sublattice. On the other hand,

$$N_1 - T_{23}, \quad N_3 - T_{15}, \quad N_4 - T_{25}, \quad N_5 - T_{34}, \quad N_{24} - T_3, \quad N_{35} - T_2$$

generate a lattice isomorphic to $E_6(2)$ (see the Hessian quartic surface in Section 9.4.4). This lattice and the vector of norm (-4)

$$N_0 - N_2 - N_3 - N_4 - T_{23} - T_{34},$$

generate a lattice isomorphic to $E_7(2)$, which is perpendicular to the invariant sublattice. Since $\rho(X) = 17$, these 17 vectors are a $\mathbb{Q}$-basis of $S_X \otimes \mathbb{Q}$. We now know the action of c_W on N_α, T_α appearing in these 17 vectors, except for N_0. For N_0, since

$$c_W(N_0 - N_2 - N_3 - N_4 - T_{23} - T_{34}) = -(N_0 - N_2 - N_3 - N_4 - T_{23} - T_{34}),$$

we have

$$c_W(N_0) = -N_0 + N_1 + N_2 + N_3 + N_4 + N_5 + T_{14} + T_{15} + T_{23} + T_{25} + T_{34}.$$

Now, by calculating the intersection numbers of H, N_α (resp. $c_W(H)$, $c_W(N_\alpha)$) and the above 17 vectors (resp. the images of 17 vectors by c_W), we have proved the assertion. □

As a corollary we have the following.

Lemma 12.26. *Let*

$$w = 2H - \frac{1}{2} \sum_{\alpha \in J(C)_2} N_\alpha, \quad r_W = \frac{3}{4} H - \frac{1}{2} \sum_{\alpha \in W} N_\alpha.$$

Then

$$c_W(w) = w + 8r_W, \quad c_W(r_W) = -r_W.$$

We now state the main result of this chapter.

Definition 12.27. We call a Kummer surface $X = \mathrm{Km}(C)$ *generic* if the Néron–Severi lattice of the Jacobian of C is generated by the theta divisor, and any Cremona involution c_W associated with a Weber hexad W is a fixed-point-free involution.

Theorem 12.28. *Let $X = \mathrm{Km}(C)$ be a generic Kummer surface associated with a curve C of genus 2. Then the automorphism group* $\mathrm{Aut}(X)$ *is generated by* 16 *translations,* 16 *switches,* 16 *projections,* 16 *correlations,* 60 *Cremona involutions associated with* 60 *Göpel tetrads, and* 192 *Cremona involutions associated with* 192 *Weber hexads.*

Proof. Let G be a subgroup of $\mathrm{Aut}(X)$ generated by involutions stated in the assertion of Theorem 12.28. Let w be the projection of ρ in S_X which is the hyperplane section of $X \subset \mathbb{P}^5$ (Lemma 12.21). Recall that $w \in \mathcal{C}(X) \subset A(X)$ (Theorem 12.7), where $A(X)$ is the ample cone of X. Let $g \in \mathrm{Aut}(X)$ and consider $g(w) \in A(X)$. If $g(w) \in \mathcal{C}(X)$, then g is a translation or a switch (Lemma 12.19). Now assume that $g(w) \notin \mathcal{C}(X)$. It suffices to show that there exists a $\phi \in G$ such that $\phi \circ g \in \mathcal{C}(X)$. For each face of $\mathcal{C}(X)$ defined by r, we denote by ι_r the involution with $\iota_r(r) = -r$ (one of projections, correlations, Cremona involutions associated with Göpel tetrads or Weber hexads). We take $\phi \in G$ as the one which attains the minimum value of $\{\langle w, \phi(g(w))\rangle : \phi \in G\}$. Then for any ι_r we have

$$\langle \phi(g(w)), w\rangle \leq \langle \iota_r(\phi(g(w))), w\rangle = \langle \phi(g(w)), \iota_r(w)\rangle.$$

If ι_r is a projection or a correlation, then we have

$$\iota_r(w) = w + 2\langle w, r\rangle r$$

by Lemma 12.23(3) and hence we have obtained

$$\langle \phi(g(w)), w\rangle \leq \langle \phi(g(w)), w\rangle + 2\langle w, r\rangle\langle \phi(g(w)), r\rangle.$$

Since $\langle w, r\rangle > 0$ and $\langle r, \phi(g(w))\rangle \neq 0$, we have proved $\langle r, \phi(g(w))\rangle > 0$. Similarly, in the case that ι_r is a Cremona involution, by Lemmas 12.24, 12.26 we have $\langle r, \phi(g(w))\rangle > 0$. Thus we have obtained $\phi(g(w)) \in \mathcal{C}(X)$ and finished the proof. □

Remark 12.29. In the last section of Hudson's book [Hud] published in 1905, and in Foreword XV by W. Barth in the 1990 reissued version in the Cambridge Library series, they mention the problem considered in this chapter. In 1997, Keum [Keu] found 192 new automorphisms of the Kummer surface by using the Torelli-type theorem. He also studied the action of classical involutions except for the one associated with a Weber hexad. Right after that, by the method mentioned in this chapter, Kondo [Kon4] proved that the automorphism group is generated by 192 Keum automorphisms and classical involutions except for the 192 Cremona involutions associated with Weber hexads. At that time, Keum and Kondo did not know the Hutchinson [Hut1] paper. Later, Ohashi [Oh] pointed out that the 192 Cremona involutions associated with Weber hexads work well instead of Keum's. However, Keum's automorphism is still interesting because it has infinite order but works like a reflection.

Since then, there have been several works using this method (see Shimada [Shim] and its references for related papers, and also see the footnote of Remark 9.43). We mention here one example which is related to the reflective lattice $U \oplus D_{20}$

(see Remark 12.5). The lattice $U \oplus D_{20}$ is isomorphic to the Picard lattice of the supersingular $K3$ surface X with Artin invariant 1 in characteristic 2. It is known that the Picard number ρ of an algebraic $K3$ surface in any characteristic satisfies $1 \leq \rho \leq 20$ or $\rho = 22$ (this is the same as the range of the ranks of reflective lattices). The surface X is constructed as the minimal resolution of a purely inseparable double cover $\bar{X}$ of $\mathbb{P}^2$. Note that the projective plane $\mathbb{P}^2(\mathbb{F}_4)$ over the finite field $\mathbb{F}_4$ contains 21 points and 21 lines which form a (21_5)-configuration (i.e., each line contains 5 points and each point lies on 5 lines). The surface $\bar{X}$ has 21 nodes over 21 points in $\mathbb{P}^2(\mathbb{F}_4)$. Thus X contains 42 non-singular rational curves (21 exceptional curves and 21 proper transforms of 21 lines). If we take $R = D_4$ instead of $A_3 \oplus A_1^{\oplus 6}$ in the case of the Kummer surface, the Picard lattice S_X of X is isomorphic to the orthogonal complement of R. By restricting $\mathcal{C}$ to the positive cone of X we obtain a finite polyhedron $\mathcal{C}(X)$ with 42 (-2)-faces and 168 (-4)-faces. We can identify 42 non-singular rational curves on X as above and 42 (-2)-vectors defining 42 faces of $\mathcal{C}(X)$. Each vector among 168 (-4)-vectors defines a reflection in $\mathrm{O}(S_X)$ which can be realized by an automorphism of X. Thus one can give a generator of $\mathrm{Aut}(X)$ as in the case of the Kummer surface. For more details, we refer the reader to Dolgachev, Kondo [DKon].

Table 12.1. The 77 octads containing ∞, 0

∞ 0 1 2 3 5 14 17	∞ 0 2 4 7 17 18 20	∞ 0 4 6 13 15 20 21
∞ 0 1 2 4 13 16 22	∞ 0 2 4 12 14 15 19	∞ 0 4 7 9 10 13 19
∞ 0 1 2 6 7 19 21	∞ 0 2 5 7 9 12 22	∞ 0 4 7 11 12 16 21
∞ 0 1 2 8 11 12 18	∞ 0 2 5 8 13 19 20	∞ 0 4 8 10 12 20 22
∞ 0 1 2 9 10 15 20	∞ 0 2 5 15 16 18 21	∞ 0 4 8 11 13 14 17
∞ 0 1 3 4 11 19 20	∞ 0 2 6 8 15 17 22	∞ 0 4 9 11 15 18 22
∞ 0 1 3 6 8 10 13	∞ 0 2 6 9 13 14 18	∞ 0 5 6 7 13 16 17
∞ 0 1 3 7 9 16 18	∞ 0 2 7 8 10 14 16	∞ 0 5 6 8 12 14 21
∞ 0 1 3 12 15 21 22	∞ 0 2 9 11 16 17 19	∞ 0 5 7 11 14 18 19
∞ 0 1 4 5 7 8 15	∞ 0 2 10 12 13 17 21	∞ 0 5 8 9 10 17 18
∞ 0 1 4 6 9 12 17	∞ 0 2 11 14 20 21 22	∞ 0 5 10 13 14 15 22
∞ 0 1 4 10 14 18 21	∞ 0 3 4 5 12 13 18	∞ 0 5 11 12 15 17 20
∞ 0 1 5 6 18 20 22	∞ 0 3 4 6 7 14 22	∞ 0 6 7 8 9 11 20
∞ 0 1 5 9 11 13 21	∞ 0 3 4 10 15 16 17	∞ 0 6 7 10 12 15 18
∞ 0 1 5 10 12 16 19	∞ 0 3 5 6 9 15 19	∞ 0 6 9 10 16 21 22
∞ 0 1 6 11 14 15 16	∞ 0 3 5 7 10 20 21	∞ 0 6 10 14 17 19 20
∞ 0 1 7 10 11 17 22	∞ 0 3 5 8 11 16 22	∞ 0 6 11 12 13 19 22
∞ 0 1 7 12 13 14 20	∞ 0 3 6 11 17 18 21	∞ 0 7 8 13 18 21 22
∞ 0 1 8 9 14 19 22	∞ 0 3 7 8 12 17 19	∞ 0 7 9 14 15 17 21
∞ 0 1 8 16 17 20 21	∞ 0 3 8 14 15 18 20	∞ 0 7 15 16 19 20 22
∞ 0 1 13 15 17 18 19	∞ 0 3 9 10 11 12 14	∞ 0 8 9 12 13 15 16
∞ 0 2 3 4 8 9 21	∞ 0 3 9 13 17 20 22	∞ 0 8 10 11 15 19 21
∞ 0 2 3 6 12 16 20	∞ 0 3 13 14 16 19 21	∞ 0 9 12 18 19 20 21
∞ 0 2 3 7 11 13 15	∞ 0 4 5 9 14 16 20	∞ 0 10 11 13 16 18 20
∞ 0 2 3 10 18 19 22	∞ 0 4 5 17 19 21 22	∞ 0 12 14 16 17 18 22
∞ 0 2 4 5 6 10 11	∞ 0 4 6 8 16 18 19	

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Tracts in Mathematics 32

Shigeyuki Kondō

*K*3 Surfaces

*K*3 surfaces are a key piece in the classification of complex analytic or algebraic surfaces. The term was coined by A. Weil in 1958 – a result of the initials Kummer, Kähler, Kodaira, and the mountain K2 found in Karakoram. The most famous example is the Kummer surface discovered in the 19th century.

*K*3 surfaces can be considered as a 2-dimensional analogue of an elliptic curve, and the theory of periods – called the Torelli-type theorem for *K*3 surfaces – was established around 1970. Since then, several pieces of research on *K*3 surfaces have been undertaken and more recently *K*3 surfaces have even become of interest in theoretical physics.

The main purpose of this book is an introduction to the Torelli-type theorem for complex analytic *K*3 surfaces, and its applications. The theory of lattices and their reflection groups is necessary to study *K*3 surfaces, and this book introduces these notions. The book contains, as well as lattices and reflection groups, the classification of complex analytic surfaces, the Torelli-type theorem, the subjectivity of the period map, Enriques surfaces, an application to the moduli space of plane quartics, finite automorphisms of *K*3 surfaces, Niemeier lattices and the Mathieu group, the automorphism group of Kummer surfaces and the Leech lattice.

The author seeks to demonstrate the interplay between several sorts of mathematics and hopes the book will prove helpful to researchers in algebraic geometry and related areas, and to graduate students with a basic grounding in algebraic geometry.

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