$$\begin{array}{ccc} X & \to & Z \\ \downarrow & & \downarrow \\ Y & \to & W \end{array}$$

on W

X

$*\frac{(1,1)}{4}$ $z^2 + (uv + w^2)$

$\downarrow 2{:}1$ A_1

$\downarrow 2{:}1$ A_3

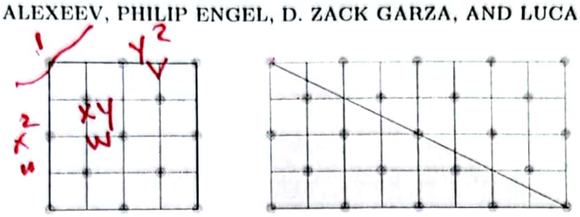


FIGURE 1. Polytopes Q and lattices for the toric surfaces Y and W

does not contain any of the four torus-fixed points. Let $P \subset \mathbb{R}^3$ be the pyramid over $Q \subset 0 \times \mathbb{R}^2$ with the vertex $(2,2,2)$, to which we associate the monomial z^2. Then the K3 surface X is a hypersurface defined by the equation $z^2 + f(x,y)$ in the projective toric variety V_P associated with P. The polynomials $f(x,y)$ invariant under τ are linear combinations of 13 monomials marked by gray dots. Thus, f defines a point in an open subset $U \subset \mathbb{P}^{12}$ and in its quotient $U/D_4 \ltimes (\mathbb{C}^*)^2$, of dimension 10. There are three commuting involutions on X:

$$\begin{array}{lll} \text{del Pezzo} & \iota_{\mathrm{dP}}\colon (x,y,z) \to (x,y,-z) \\ \text{Enriques} & \iota_{\mathrm{En}}\colon (x,y,z) \to (-x,-y,-z) \\ \text{Nikulin} & \iota_{\mathrm{Nik}}\colon (x,y,z) \to (-x,-y,z) \end{array}$$

which together with the identity form a Klein-four group. Both ι_{En} and ι_{Nik} are lifts of τ. On an affine subset of X a nonvanishing 2-form is given by

$$\omega = \operatorname{Res}_X \frac{dx \wedge dy \wedge dz}{z^2 + f}.$$

One has $\iota_{\mathrm{dP}}^*\omega = \iota_{\mathrm{En}}^*\omega = -\omega$ and $\iota_{\mathrm{Nik}}^*\omega = \omega$. So ι_{dP} and ι_{En} are nonsymplectic and ι_{Nik} is symplectic. The Enriques surface Z is then a hypersurface in the toric variety for the polytope P but for the even sublattice $\mathbb{Z}^3_{\mathrm{ev}} = \{(a,b,c) \mid a+b+c \in 2\mathbb{Z}\}$. It is defined by the same polynomial $z^2 + f(x,y)$ whose monomials lie in $\mathbb{Z}^3_{\mathrm{ev}}$.

Let R be the ramification divisor of π. The involution ι_{dP} on X descends to an involution τ_{dP} on Z, and $W = Z/\tau_{\mathrm{dP}}$. Let R_Z and B_W be the ramification and branch divisors of ρ. Then $R = \psi^*(R_Z)$ and $R_Z = \frac{1}{2}\psi_*(R)$. Since $R = \frac{1}{2}\pi^*(B)$ is an ample divisor, R_Z is ample as well. One has $\mathcal{O}(R_Z) = \mathcal{L}_Z^{\otimes 2} \in \operatorname{Pic} Z$.

Horikawa [Hor78b] analyzed in some detail the sets of possible equations $f(x,y)$ and the maps from various opens subsets of $\mathbb{P}^{12}$ to the period domain $\mathbb{D}/\Gamma$ and its Baily-Borel compactification, introduced in the next section. In particular, he showed that certain mildly singular $f(x,y)$ vanishing at a torus-fixed point correspond to Coble surfaces, which are S_2-quotients of nodal K3 surfaces.

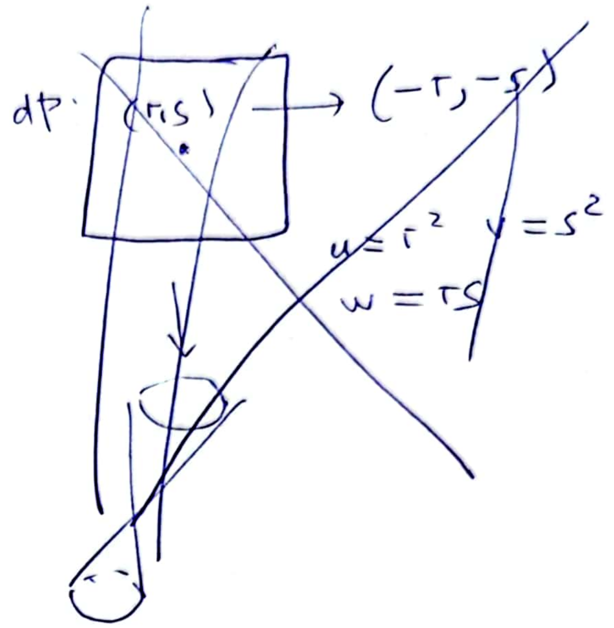
The GIT compactification $\mathbb{P}^{12}/\!/D_4 \ltimes (\mathbb{C}^*)^2$ was described by Shah [Sha81], who gave normal forms for polystable orbits. As usual for the moduli of K3 surfaces with a projective construction, the relation between the GIT and the Baily-Borel compactifications is not straightforward, cf. [Loo86] for K3 surfaces of degree 2 and [LO21] for degree 4 K3 surfaces which are double covers of $\mathbb{P}^1 \times \mathbb{P}^1$.

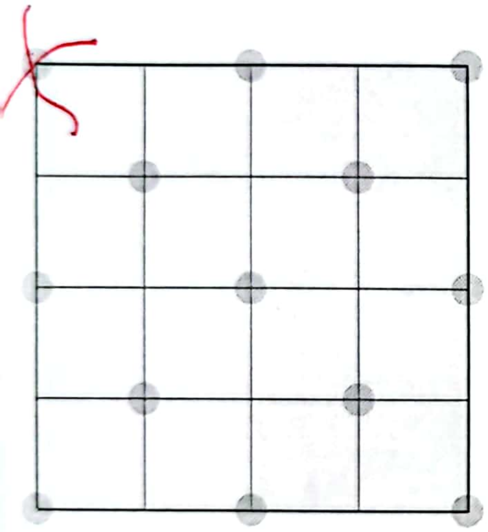
2.2. The main diagram, special case. The previous section describes the general case, when the K3 cover X is non-unigonal. The special case corresponds to a Heegner divisor in $F_{\mathrm{En},2}$ for which $(Y,L) = (\mathbb{P}(1,1,2), \mathcal{O}(4))$ is a singular quadric. The toric surfaces Y and W correspond to the same polytope Q shown in the right

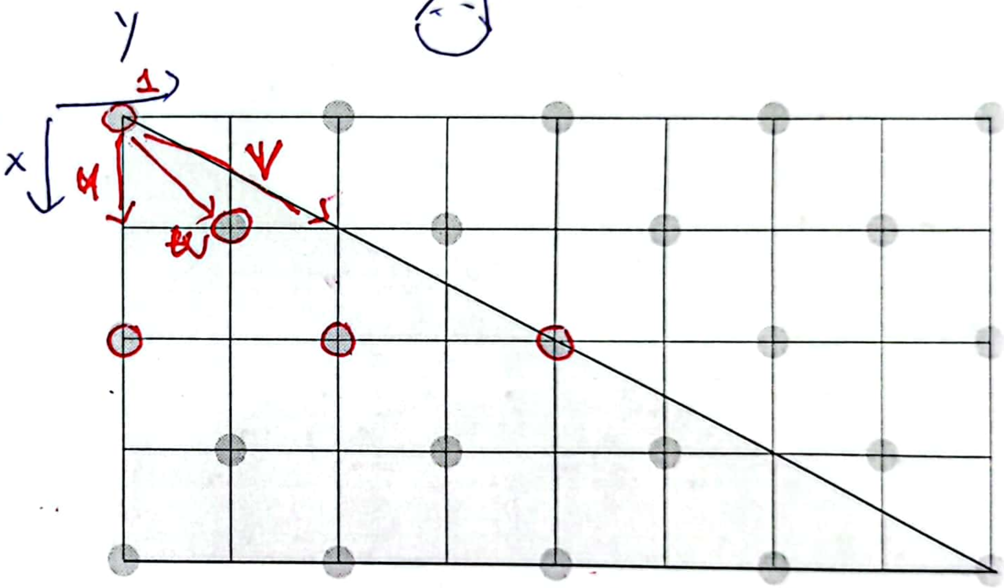
$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\text{ramified}"'] & Z \\ X & \end{tikzcd}$$

$\boxed{uv = w^2}$ = sings of X

$f = au^2 + bv^2 + cw$







$Y = \mathbb{P}(1,1,2) = \mathbb{F}_2^0$

= quad. cone

$Y: \quad z^2 + f = 0$

$$\supset \mathbb{A}^4_{u,v,w,z} \quad \begin{cases} z^2 + (u^2 + v^2 + w) = 0 \\ uv = w^2 \end{cases}$$

$u = x^2$

$w = xy$

$v = x^2 y^4$

$\iota_{dp}: (u,v,w,z) \to (u,v,w,-z)$

$\iota_{dp}: (x,y,z) \to (x,y,-z)$

$\iota_{En}: ($

$\iota_{En}: (x,y,z) \to (-x,-y,-z)$

$(u,v,w,z) \to (u,v,w,-z)$

$$\frac{(1,1)}{4} \neq \frac{(1,5)}{8} \quad \frac{(1,3)}{8}$$

smoothable

||

T-sings

$$\frac{(1,\ an+1)}{dn^2} \qquad \frac{(1,\ 2n+1)}{8}$$

$d = n = 2$