# COMPACT MODULI OF ENRIQUES SURFACES OF DEGREE 2

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Abstract. Describe the cusp correspondence in the following:

$$\overline{\mathcal{E}_2}^{\mathrm{bb}} \xrightarrow{\text{Finite}} \overline{\mathcal{E}_\emptyset}^{\mathrm{bb}}, \qquad \overline{\mathcal{E}_2}^{\mathrm{bb}} \xrightarrow{\text{Finite-to-one}} \overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}}$$

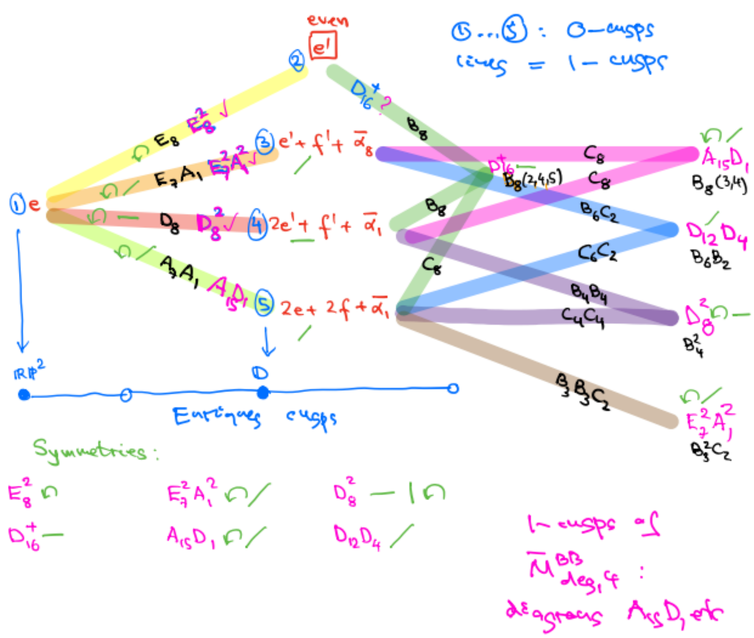


Figure 1. A cusp correspondence

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LIST OF SYMBOLS

| Notation | Description | Page List |
|---|---|---|
| $E_8$ | The $E_8$ root lattice. | 100 |
| $\mathrm{II}_{3,19}$ | The K3 lattice. | 101 |
| $\mathrm{II}_{1,9}$ | The Enriques lattice. | 101 |
| $\mathrm{NS}(S)$ | The Néron-Severi lattice of a surface $S$ | 101 |
| $\mathrm{T}(S)$ | The transcendental group of a surface $S$ | 101 |

TODO LIST

## 1. INTRODUCTION (ATTEMPT)

Enriques surfaces $Y$ are minimal algebraic surfaces of Kodaira dimension zero satisfying $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$ and $K_Y \neq 0$ but $2K_Y = 0$. A fundamental property of an Enriques surfaces $Y$ is that its universal cover $X$ is isomorphic to a K3 surface[1]. For both K3 and Enriques surfaces, the theory of compactifications is very rich: once a polarization $L$ is fixed, there is a Hodge-theoretic period domain parametrizing isomorphism classes of polarized K3 or Enriques surfaces. These are bounded Hermitian symmetric domains, and thus for appropriate choices of arithmetic subgroups, the resulting arithmetic quotients admit Baily–Borel [?], toroidal [?], and Looijenga semitoroidal compactifications [?].

It is then natural to ask about geometric compactifications such as stable pair compactifications and how they relate to these Hodge theoretic compactifications. In a series of recent papers, Alexeev–Engel–Thompson have made breakthroughs for K3 surfaces (see [?, ?, ?, ?, ?]). In particular, there are explicit and effective answers to this question for K3 surfaces equipped with a non-symplectic involution whose fixed locus is a curve.

Stable pair compactifications of the moduli space of Enriques surfaces are less well-studied. In [?] the second author studied the stable pair compactification

[1]A K3 surface is a smooth projective surface $X$ with $K_X = 0$ and $h^1(\mathcal{O}_X) = 0$

of the moduli space of Enriques surfaces with a degree 6 polarization (Enriques' original construction) and give a full description for a 4−dimensional subfamily of the moduli space. One of the obstructions to extending similar results to the entire 10−dimensional family is the high degree $d$ of the polarization needed. It is thus natural to consider instead the lowest value possible, which is $d = 2$. In this situation, $X$ is naturally equipped with a non-symplectic involution, the *Enriques involution*, whose fixed locus is a curve, and thus the theory developed in [**?**] is applicable.

However, the theory in [**?**] does not immediately apply in this situation – one must account for the fact that there are certain natural geometric automorphisms. Horikawa gives a construction of a K3 surface $X$ as a degree 2 cover $\rho : X \to (\mathbb{P}^1)^2$ branched over a divisor $D \in -2K_{(\mathbb{P}^1)^2}$ of bidegree $(4,4)$. It can be shown that $\rho$ is symplectic and that $\rho$ commutes with the Enriques involution, and thus the theory in [**?**] must be modified to keep track of this extra symmetry.

## 2. Preliminaries

We follow closely the exposition in [**?**, § 2].

2A. **Lattices.** By a *lattice* we mean a finitely generated free abelian group $L$ of finite rank equipped with a nondegenerate symmetric bilinear form $b\colon L \times L \to \mathbb{Z}$. In particular, two lattices are *isometric* there exists an isomorphism of the underlying abelian groups which preserves the bilinear forms. Given a set of generators $e_1, \dots, e_r$ of $L$, we can associate a *Gram* matrix given by $(b(e_i, e_j))_{i,j}$. The lattice $L$ is called *unimodular* provided the determinant of a Gram matrix is $\pm 1$. The lattice $L$ is called *even* provided $b(v,v) \in 2\mathbb{Z}$ for all $v \in L$. Given a lattice $L$ we denote by $L^*$ its dual $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. As the bilinear form is nondegenerate, we have an inclusion $L \hookrightarrow L^*$ and the quotient $A_L = L^*/L$ is a finite abelian group called the *discriminant group* of $L$. The discriminant group $A_L$ comes equipped with a quadratic form $q_L\colon A_L \to \mathbb{Q}/\mathbb{Z}$ by sending $v + L \mapsto b(v,v) \bmod \mathbb{Z}$. The lattices for which $A_L \cong \mathbb{Z}_2^a$ for some positive integer $a$ are called 2−*elementary.*

2B. **K3 surfaces, and nonsymplectic involutions.** A lot of the geometry and moduli theory of K3 surfaces is regulated by lattice theory. For a K3 surface $X$ it is well-known that $H^2(X, \mathbb{Z})$, endowed with the cup product, is an even, unimodular lattice of signature $(3, 19)$. It follows that $H^2(X, \mathbb{Z})$ is isometric to the so-called *K3 lattice* $\mathrm{II}_{3,19} := U^{\oplus 3} \oplus E_8^{\oplus 2}$, where $\mathrm{II}_{1,1}$ is the hyperbolic plane $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $E_8$ is the negative definite root lattice associated to the corresponding Dynkin diagram. In particular, symmetries of the surface $X$ translate into symmetries of the K3 lattice $\mathrm{II}_{3,19}$.

A particularly rich setting is provided by *nonsymplectic involutions*, i.e. order 2 automorphisms $\iota\colon X \to X$ such that the induced map $\iota^*\colon H^{2,0}(X) \to H^{2,0}(X)$ satisfies $\iota^*\omega_X = -\omega_X$. Then we can look at the action of $\iota^*$ on $H^2(X, \mathbb{Z})$ and we denote by $S$ its $(+1)$−eigenspace. It turns out that $S$ is a hyperbolic 2−elementary lattice, and all the possibilities for $S$ up to isometries were classified by Nikuln. More precisely, there are 75 cases which correspond bijectively to the triples of invariants $(r, a, \delta)$, where $r$ is the rank of $S$, $A_S \cong \mathbb{Z}_2^a$, and $\delta$ is the so-called *coparity* of $L$: $\delta = 0$ provided $q_L(v) \equiv 0 \bmod \mathbb{Z}$, and $\delta = 1$ otherwise.

## 3. Moduli via period domains

### 3A. A general construction.

**Remark 3.1.** We describe here a construction common to the construction of many Hodge-theoretic moduli spaces. Let $\Lambda$ be an ambient lattice, $S \leq \Lambda$ a primitive sublattice, and $T \coloneqq S^{\perp\Lambda}$ its orthogonal complement in $\Lambda$. Define the *period domain associated to SS* to be

$$\Omega_S^{\pm} \coloneqq \left\{ [v] \in \mathbb{P}(S \otimes_{\mathbb{Z}} \mathbb{C}) \mid v^2 = 0 \text{ and } v\overline{v} = 0 \right\}.$$

As a matter of notation, we also set

$$\Omega^S \coloneqq \Omega_{S^\perp} \coloneqq \Omega_T.$$

In cases of interest, we have a decomposition $\Omega_S^{\pm} = \Omega_S^{+} \amalg \Omega_S^{-}$ into irreducible components, both of which are type IV bounded Hermitian symmetric domains which are permuted by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We fix a choice of component $\Omega_S^{+}$, and let $\mathrm{O}(S)^{+} \leq \mathrm{O}(S)$ be the subgroup fixing this component. We then form a locally symmetric space and a corresponding Baily-Borel compactification

$$F(S) \coloneqq {}_{\mathrm{O}^+(S)}\backslash^{\Omega_S^+} \qquad \overline{F(S)}^{\mathrm{bb}} \coloneqq \overline{{}_{\mathrm{O}^+(S)}\backslash^{\Omega_S^+}}^{\mathrm{bb}}.$$

More generally, one can let $\Gamma$ be any neat arithmetic group that acts properly discontinuously on $S\Omega_S^{+}$. One can then similarly form

$$F(S,\Gamma) \coloneqq {}_{\Gamma}\backslash^{\Omega_S^+}, \qquad \overline{F(S,\Gamma)}^{\mathrm{bb}} \coloneqq \overline{{}_{\Gamma}\backslash^{\Omega_S^+}}^{\mathrm{bb}}.$$

Specific choices of $S$ are used throughout our work to construct various coarse moduli spaces. In some instances, we must remove a hyperplane arrangement to form the correct moduli space. Let

$$\mathcal{H}_{-2} \coloneqq \left( \bigcup_{\delta \in \Phi^2(N)} \delta^{\perp} \right) \cap \Omega_N^{+} = \bigcup_{\substack{\delta \in N, \\ \delta^2 = -2}} \left\{ [v] \in \Omega_N^{+} \mid v \cdot \delta = 0 \right\}.$$

and define

$$F(S,\Gamma,\mathcal{H}_{-2}) \coloneqq {}_{\Gamma}\backslash^{\left(\Omega_S^+ \setminus \mathcal{H}_{-2}\right)}, \qquad \overline{F(S,\Gamma,\mathcal{H}_{-2})}^{\mathrm{bb}} \coloneqq \overline{{}_{\Gamma}\backslash^{\left(\Omega_S^+ \setminus \mathcal{H}_{-2}\right)}}^{\mathrm{bb}}$$

### 3B. Generally finding cusps.

**Remark 3.2.** We now discuss how $\partial F(S,\Gamma,\mathcal{H}_{-2})$ can be described lattice-theoretically. Let $\mathrm{Gr}^{\mathrm{iso}}(S)$ be the isotropic Grassmannian of the lattice $S$, and write $\partial F(S,\Gamma,\mathcal{H}_{-2}) = \bigcup_{i \geq 0} \partial F(S,\Gamma,\mathcal{H}_{-2})_i$ for a stratification of the boundary by $i-$dimensional components. One can show that there are bijections

$$\mathrm{Gr}_1^{\mathrm{iso}}(L)/\Gamma \cong \partial F(S,\Gamma,\mathcal{H}_{-2})_0, \qquad \mathrm{Gr}_2^{\mathrm{iso}}(L)/\Gamma \cong \partial F(S,\Gamma,\mathcal{H}_{-2})_1,$$

and so 0-cusps correspond to $\Gamma-$orbits of primitive isotropic lines and 1-cusps to orbits of isotropic planes.

3C. **Moduli of K3 surfaces with nonsymplectic involution.**

**Remark 3.3** (Constructing moduli of quasi-polarized K3 surfaces lattice-theoretically)**.** The coarse moduli space $F_{2d}$ of polarized K3 surfaces $(X, L)$ can be realized using the construction described in subsection 3A. Recall that $H^2(X;\mathbb{Z}) \cong \mathrm{II}_{3,19}$. Fix a marking $\varphi : H^2(X;\mathbb{Z}) \to \mathrm{II}_{3,19}$ and a polarization $L$ of degree $2d$, and let $h := \varphi([L]) \in \mathrm{II}_{3,19}$. One can then show that $h^\perp \cong \mathrm{II}_{3,19}\langle 2d\rangle$. Let

$$\mathrm{Stab}_{\mathrm{O}(\mathrm{II}_{3,19})}(h) := \{\gamma \in \mathrm{O}(\mathrm{II}_{3,19}) \mid \gamma(h) = h\}$$

be the stabilizer of $h$ in $\mathrm{II}_{3,19}$ and define

$$\Gamma_h := \mathrm{Stab}_{\mathrm{O}(\mathrm{II}_{3,19})}(h)^+$$

to be the finite index subgroup fixing $\Omega^+_{\Lambda_{2d}}$. Letting $\mathcal{F}^{\mathrm{qp}}_{2d}$ be the moduli stack of quasi-polarized K3 surfaces of degree $2d$, there is an analytic isomorphism at the level of coarse spaces

$$F^{\mathrm{qp}}_{2d} \cong {}_{\Gamma_h}\backslash^{\Omega^+_{\Lambda_{2d}}}.$$

However, $\mathcal{F}^{\mathrm{qp}}_{2d}$ is generally not a separated stack. We can instead use the stack $\mathcal{F}^{\mathrm{ADE}}_{2d}$ of polarized K3s with ADE singularities, since there is an isomorphism $F^{\mathrm{ADE}}_{2d} \cong F^{\mathrm{qp}}_{2d}$ at the level of coarse spaces.

**Definition 3.4** (Constructing moduli of marked K3s)**.** The theory of moduli of pairs $(X, \iota)$ with $X$ a K3 surface and $\iota$ a nonsymplectic involution can be approached using the construction in subsection 3A as well. Let $S \subseteq \mathrm{II}_{3,19}$ be a primitive hyperbolic $2-$elementary sublattice which is the $(+1)-$eigenspace of an involution $\rho$ of $\mathrm{II}_{3,19}$. A *$\rho-$marking* of $(X, \iota)$ is an isometry $\varphi\colon H^2(X,\mathbb{Z}) \to \mathrm{II}_{3,19}$ such that $\iota^* = \varphi^{-1} \circ \rho \circ \varphi$. Fix such a marking $\rho$. We have a *period domain* $\Omega^+_S$ associated to $S$, and we define the change-of-marking group associated to $\rho$ to be

$$\Gamma_\rho = \{\gamma \in \mathrm{O}(\mathrm{II}_{3,19}) \mid \gamma \circ \rho = \rho \circ \gamma\}.$$

One can then show that the coarse moduli space of $\rho-$markable K3 surfaces is analytically isomorphic to the locally symmetric space

$$F_S := F(S^\perp, \Gamma_\rho, \mathcal{H}_{-2}) := {}_{\Gamma_\rho}\backslash^{(\Omega_{S^\perp} \backslash \mathcal{H}_{-2})}.$$

In particular, the point corresponding to $(X, \iota)$ is $[\varphi(\mathbb{C}\omega_X)]$.

3D. **Hodge theoretic compactifications.** Hodge theory provides different ways to compactify $\Omega_{S^\perp}/\Gamma$ for any finite index subgroup $\Gamma \subseteq \mathrm{O}(S^\perp)$. A standard way that involves no choices is provided by the *Baily–Borel compactification* $\overline{\Omega_{S^\perp}/\Gamma}^{\mathrm{bb}}$. This is a projective normal compactification whose boundary is stratified into $0-$cusps and $1-$cusps which correspond to $\Gamma-$orbits of isotropic vectors $I \subseteq T$ and isotropic planes $J \subseteq T$. *Toroidal compactifications* $\overline{\Omega_{S^\perp}/\Gamma}^{\mathfrak{F}}$ are blow-ups of $\overline{\Omega_{S^\perp}/\Gamma}^{\mathrm{bb}}$ which depend on the choice of a compatible system of admissible fans $\mathfrak{F} = \{\mathfrak{F}_K\}$ for each isotropic vector $I$ or plane $J$. The fan $\mathfrak{F}_K$ is a rational polyhedral decomposition of the rational closure $C_{K,\mathbb{Q}}$ of the positive cone $C_K \subseteq K^\perp/K \otimes \mathbb{R}$. It is required to satisfy the usual fan axioms, and additionally be $\Gamma-$invariant with only finitely many orbits of cones. As this datum is trivial for isotropic planes, it is sufficient to provide the fan only for the isotropic vectors $I$, hence $\mathfrak{F} = \{\mathfrak{F}_I\}$. Lastly, *Semitoroidal compactifications* are due to Looijenga and simultaneously generalize the Baily–Borel and toroidal compactifications by allowing the fans $\mathfrak{F}_I$ to be not necessarily finitely generated.

## 4. Hyperelliptic K3s

**Definition 4.1** (Hyperelliptic K3 surfaces)**.** Let $X$ be a K3 surface and let $L \in \mathrm{Pic}(X)$ be a line bundle with $L^2 > 0$ where the linear system $|L|$ has no fixed components. We say that $|L|$ is a **hyperelliptic linear system on** $X$ and $X$ is a **hyperelliptic** K3 **surface** if $|L|$ contains a hyperelliptic curve.

**Remark 4.2.** The induced morphism $\varphi_{|L|} : X \to \mathbb{P}^g$ where $L^2 = 2g-2$ in this case is a generally 2-to-1 morphism onto a surface $F$ of degree $g-1$ in $\mathbb{P}^g$. By the classification of surfaces, either $F \cong \mathbb{P}^2$ or $\mathbb{F}_n$[2] with $n \in \{0,1,2,3,4\}$ ramified over a curve $C \in |-2K_F|$.

**Remark 4.3.** The open locus of $\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}}$ can be realized using the standard construction of $L-$polarized K3 surfaces, taking $L = {\mathrm{II}_{1,1}}^{\oplus^2} \oplus D_{16}$. More generally, degree $n$ hyperelliptic K3 surfaces can be constructed by taking $L = {\mathrm{II}_{1,1}}^{\oplus^2} \oplus D_{n-2}$.

### 4A. Hyperelliptic quartic K3s.

**Remark 4.4.** We now focus back on our main case of interest: hyperelliptic quartic K3s, i.e. hyperelliptic K3 surfaces of degree 4. In this case, the hyperbolic $2-$elementary even lattice $S$ is given by $\mathrm{II}_{1,1}(2)$, which corresponds to the invariants $(r,a,\delta) = (2,2,0)$.

The Baily–Borel compactification $\overline{\Omega_{S^\perp}/\Gamma}^{\mathrm{bb}}$ ... for which $\Gamma$? Was studied by Laza–O'Grady.

Now relate $\overline{\mathbf{K}}_{\mathrm{h}}$ with $\overline{\Omega_{S^\perp}/\Gamma}$ and an appropriate Looijenga semitoroidal. Where is this in Valery and Phil's work? Give appropriate references.

**Remark 4.5.** Following [**?**], consider the period domain construction described in subsection 3A using the lattice $\Lambda_N \coloneqq {\mathrm{II}_{1,1}}^{\oplus^2} \oplus D_{N-2}$ and $\Gamma = \mathrm{O}(\Lambda_N)^+$.[3] We then obtain a sequence of locally symmetric spaces

$$\mathcal{F}(N) \coloneqq F(\Lambda_N, \mathrm{O}(\Lambda_N)^+) \coloneqq {}_{\mathrm{O}(\Lambda_N)^+}\backslash^{\Omega^+_{\Lambda_N}}.$$

In particular, taking $N = 19$ yields the $F_4$, the coarse moduli space of standard polarized K3 surfaces of degree 4, and taking $N = 18$ yields a coarse moduli space $F_{4,\mathrm{h.e.}}$ of quartic (i.e. degree 4) hyperelliptic K3 surfaces. The lattice embedding $\Lambda_{18} \hookrightarrow \Lambda_{19}$ induced by $D_{16} \hookrightarrow D_{17}$ produces an inclusion $F_{4,\mathrm{h.e.}} \subsetneq F_4$ realizing $F_{4,\mathrm{h.e.}}$ as a normal Heegner divisor in $F_4$. This in turn induces a morphism $\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}} \to \overline{F_4}^{\mathrm{bb}}$. The Baily-Borel compactification $\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}}$ was studied in LO16 and [**?**], where in the latter they show

$$\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}} \cong \mathrm{Chow}_{2,4} /\!\!/ \mathrm{SL}_4,$$

a GIT quotient of the Chow variety of $(2,4)$ curves in $\mathbb{P}^3$.

**Theorem 4.6** ([**?**, Theorem 2.3])**.** *The Baily–Borel compactification*

$$\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}} \cong \overline{{}_{\mathrm{O}(\Lambda_{18})^+}\backslash^{\Omega^+_{\Lambda_{18}}}}^{\mathrm{bb}}$$

[2] A Hirzebruch surface $\mathbb{F}_n \coloneqq \mathrm{Proj}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1})$.

[3] For moduli-theoretic purposes, if $N \equiv 6 \mod 8$, one then instead passes to a finite index subgroup as detailed in [**?**].

*has two* $0-$*cusps (type III boundary components) and eight* $1-$*cusps (type II boundary components). The incidences between* $0-$*cusps and* $1-$*cusps are represented in Figure 2.*

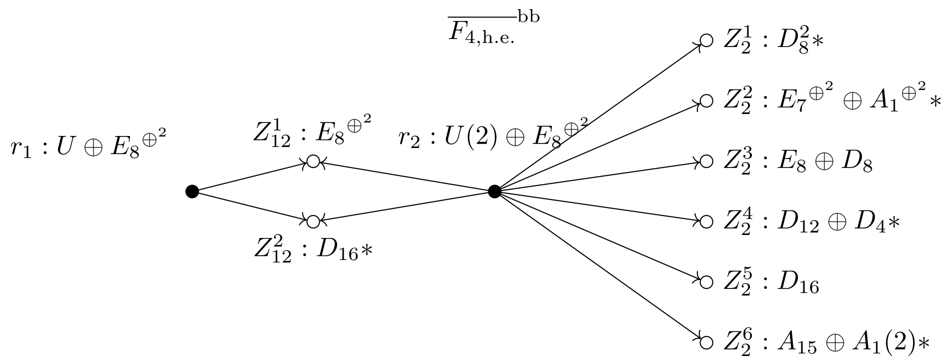


FIGURE 2. Cusp diagram for degree 4 hyperelliptic K3 surfaces $\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}}$.

**Remark 4.7.** If $C \subseteq (\mathbb{P}^1)^2$ is a smooth curve of bidegree $(4,4)$ and $\pi : X_C :\to (\mathbb{P}^1)^2$ is the double cover branched along $C$, then $X_C$ is a smooth hyperelliptic polarized K3 surface of degree 4 and thus $X_C \in \overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}}$. Letting $M := \left|\mathcal{O}_{(\mathbb{P}^1)^2}(4,4)\right| /\!\!/ \operatorname{Aut}((\mathbb{P}^1)^2)$ be the GIT quotient, LO21 describes a birational period map $M \dashrightarrow \overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}}$.

The K3 surfaces parameterized by $\overline{\mathbf{K}}$ are double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along curves of class $(4,4)$ in the monomials listed in (1). More in general, the double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along general curves of class $(4,4)$ give rise to K3 surfaces known as *hyperelliptic* K3 surfaces. Let us construct their family and the KSBA compactification.

Let $\mathbb{P}^{24}$ be the space of coefficients, up to scaling, for a bidegree $(4,4)$ polynomial in $\mathbb{P}^1 \times \mathbb{P}^1$. In this case, a monomial $X_0^i X_1^j Y_0^k Y_1^\ell$ is indexed by

$$M_{\mathrm{h}} := \{(i,j,k,\ell) \in \mathbb{Z}^4_{\geq 0} \mid i+j = k+\ell = 4\}.$$

Let $\mathbf{U}_{\mathrm{h}} \subseteq \mathbb{P}^{24}$ be the dense open subset of coefficients $[\ldots : c_{ijk\ell} : \ldots]$ such that the corresponding $(4,4)$ curve is smooth. We can define a KSBA-stable family

$$\left(\mathcal{X}_{\mathrm{h}} := \mathbf{U}_{\mathrm{h}} \times (\mathbb{P}^1 \times \mathbb{P}^1), \frac{1+\epsilon}{2}\mathcal{B}_{\mathrm{hyp}}\right) \to \mathbf{U}_{\mathrm{hyp}},$$

where $\mathcal{B}_{\mathrm{h}}$ is the relative divisor given by

$$\sum_{(i,j,k,\ell)\in M_{\mathrm{h}}} c_{ijk\ell} X_0^i X_1^j Y_0^k Y_1^\ell = 0.$$

We can consider the fiberwise double cover $(\mathcal{T}_{\mathrm{h}}, \epsilon\mathcal{R}_{\mathrm{h}}) \to \left(\mathcal{X}_{\mathrm{h}}, \frac{1+\epsilon}{2}\mathcal{B}_{\mathrm{h}}\right)$, which gives rise to the family of hyperelliptic K3 surfaces. The automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$ acts on $\mathbf{U}_{\mathrm{h}}$ identifying isomorphic fibers. In particular, $\mathbf{U}_{\mathrm{h}}/\operatorname{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is the moduli space of smooth hyperelliptic K3 surfaces. To compactify it, we can consider the stack $\overline{\mathcal{P}}'_{\mathrm{h}}$ given by the closure of the image of the morphism $\mathbf{U}_{\mathrm{h}} \to \mathcal{SP}\left(\frac{1+\epsilon}{2}, 2, 8\epsilon^2\right)$. Let $\overline{\mathbf{P}}'_{\mathrm{h}}$ be the corresponding coarse moduli space and denote by $\overline{\mathbf{P}}_{\mathrm{h}}$ its normalization, which gives rise to a compactification of the 18−dimensional moduli space $\mathbf{U}_{\mathrm{h}}/\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$. Alternatively, by using the family $(\mathcal{T}_{\mathrm{h}}, \epsilon\mathcal{R}_{\mathrm{h}}) \to \mathbf{U}_{\mathrm{h}}$ and the moduli functor $\mathcal{SP}\left(\epsilon, 2, 16\epsilon^2\right)$ instead, we obtain the compactifications $\overline{\mathbf{K}}_{\mathrm{h}}$, which instead parameterize generically the hyperelliptic K3 surfaces. We have that $\overline{\mathbf{K}}_{\mathrm{h}} \cong \overline{\mathbf{U}}_{\mathrm{h}}$.

**Remark 4.8.** The inclusion $\mathbf{U} \hookrightarrow \mathbf{U}_{\mathrm{h}}$ induces an inclusion of the stacks $\overline{\mathcal{P}}' \hookrightarrow \overline{\mathcal{P}}'_{\mathrm{h}}$, and hence an inclusion of the corresponding coarse moduli spaces $\overline{\mathbf{P}}' \hookrightarrow \overline{\mathbf{P}}'_{\mathrm{h}}$. Therefore, we have an induced morphism $\overline{\mathbf{P}} \to \overline{\mathbf{P}}_{\mathrm{h}}$ which is finite and birational onto its image. Luca: The reason why this morphism exists is nontrivial! The normalization is not functorial, so one has to really prove this. The above is also missing the following. Do we have an embedding of $\mathbf{U}/G$ into $\mathbf{U}_{\mathrm{h}}/\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$? Recall $G = \mathbb{G}_m^2 \rtimes (\mathbb{Z}/2\mathbb{Z})$.

The compactification $\overline{\mathbf{P}}_{\mathrm{h}}$ should be fully understood from the work in [?]. Moreover, the GIT and Baily–Borel should be understood by [?].

## 5. Enriques surfaces

5A. **The unpolarized case.** If $Y$ is an Enriques surface, it is well known that the universal cover $\pi\colon X \to Y$ is a $\mu_2$ Galois cover where $X$ is a K3 surface and $Y \cong X/\iota$ for $\iota$ the basepoint-free involution swapping the sheets of the cover. We write $V_{+1}(\iota^*), V_{-1}(\iota^*) \subseteq H^2(X;\mathbb{Z})$ for the $(+1)$ and $(-1)$−eigenspaces respectively of the induced involution in cohomology $\iota^*\colon H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$. It is well-known that $V_{+1}(\iota)^{\perp H^2(X;\mathbb{Z})} = V_{-1}(\iota)$. The covering map $\pi$ induces an embedding of lattices

$$\pi^*\colon H^2(Y;\mathbb{Z}) \hookrightarrow H^2(X;\mathbb{Z}),$$

whose image is $V_{+1}(\iota^*)$. It is well known that

(1) $H^2(X;\mathbb{Z}) \cong \mathrm{II}_{3,19} \cong U^{\oplus 3} \oplus E_8^{\oplus 2}$ is the K3 *lattice*;
(2) $M := H^2(Y,\mathbb{Z})/\mathrm{tors} \cong \mathrm{II}_{1,9} \cong U \oplus E_8$ is the *Enriques lattice*;
(3) $V_{+1}(\iota^*) \cong \mathrm{II}_{1,1}(2) \oplus E_8(2) \cong \mathrm{II}_{1,9}(2)$;
(4) $V_{-1}(\iota^*) \cong \mathrm{II}_{1,1} \oplus \mathrm{II}_{1,1}(2) \oplus E_8(2) = U \oplus \mathrm{II}_{1,9}(2)$.

We will use the decomposition of the K3 lattice into summands involving the Enriques lattice

$$\mathrm{II}_{3,19} = \mathrm{II}_{1,9} \oplus \mathrm{II}_{1,9} \oplus \mathrm{II}_{1,1},$$

and describe a vector in the K3 lattice $\mathrm{II}_{3,19}$ with three coordinates $(x, y, z)$ accordingly. Let $\mathrm{II}_{1,1} = \mathbb{Z}e \oplus \mathbb{Z}f$ with $\{e, f\}$ the standard hyperbolic basis satisfying $e^2 = f^2 = e \cdot f - 1 = 0$.

**Remark 5.1** (Period domain for unpolarized Enriques surfaces)**.** Again following the period domain construction described subsection 3A, now with the lattice

$$N := U \oplus \mathrm{II}_{1,9}(2).$$

The period domain for unpolarized Enriques surfaces is $\Omega_N^+$, and the correct associated locally symmetric space is

$$\mathcal{E}_\emptyset := F(N, \mathrm{O}(N)^+, \mathcal{H}_{-2}) := {}_{\mathrm{O}(N)^+}\backslash^{\left(\Omega_N^+ \backslash \mathcal{H}_{-2}\right)}.$$

**Lemma 5.2** (Torelli for Enriques surfaces, Horikawa)**.** *Points in $\mathcal{E}_\emptyset$ correspond to isomorphism classes of unpolarized Enriques surfaces.*

**Theorem 5.3** ([**?**, Propositions 4.5 and 4.6])**.** *The Baily–Borel compactification*

$$\overline{\mathcal{E}_\emptyset}^{\mathrm{bb}} \coloneqq \overline{{}_{\mathrm{O}^+(N)}\backslash^{(\Omega_N^+\backslash\mathcal{H}_{-2})}}^{\mathrm{bb}}$$

*has two* $0-$*cusps and two* $1-$*cusps. The incidences between* $0-$*cusps and* $1-$*cusps are represented in Figure 3.*

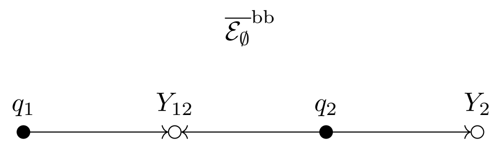


FIGURE 3. Cusp diagram for the moduli space of unpolarized Enriques surfaces $\overline{\mathcal{E}_\emptyset}^{\mathrm{bb}}$.

5B. **Degree** 2 **polarized Enriques surfaces.** For degree 2 polarized Enriques surfaces, we consider the same period domain, but we change the arithmetic group acting on it.

**Definition 5.4.** A *polarization* on an Enriques surface $Y$ is a pseudo-ample (i.e. big and nef) line bundle $L$ on $Y$; we call this an *ample polarization* if $L$ is ample. A *numerical* (resp. ample numerical) *polarization* on $Y$ is a choice $L$ of a numerical equivalence class of pseudo-ample (resp. ample) line bundle $L$. A numerically polarized Enriques surface is a pair $(Y, L)$ where $L$ is a numerical polarization on $Y$.[4]

**Remark 5.5.** Let $L \coloneqq U \oplus {\mathrm{II}_{1,9}}^{\oplus^2}$, noting that $N \leq L$, and define the following involution:

$$I : L \to L,$$
$$(x, y, z) \mapsto (y, x, -z).$$

A result of Horikawa shows that there is an isometry $\mu : H^2(X;\mathbb{Z}) \to L$ such that $I \circ \mu = \mu \circ I^*$ and produces an embedding

$$M \to L$$
$$m \mapsto (m, m, 0).$$

Define

$$\Gamma' \coloneqq \{g \in \mathrm{O}(L) \mid g \circ I = I \circ g \text{ and } g(e+f, e+f, 0) = (e+f, e+f, 0)\},$$

automorphisms in the centralizer of $I$ in $\mathrm{O}(L)$ fixing the point $(e+f, e+f, 0)$. If $g \in \Gamma'$ then $g|_N \in \mathrm{O}(N)$. So define

$$\Gamma \coloneqq \{\, g|_N \mid g \in \Gamma'\} \leq O(N),$$

[4]Why introduce *numerical* polarizations? Recall that $A$ is a polarized abelian variety if it is equipped with an isogeny $\lambda : A \to A^\vee$. If $L$ is a numerical polarization on $A$, it induces a unique isogeny $\lambda_L$, and every such isogeny comes from such an $L$, so numerical polarization strictly generalizes this notion to other varieties.

which is the image of $\Gamma'$ in $\mathrm{O}(L_-)$. Again using the construction in subsection 3A, the moduli space for Enriques surfaces with a polarization of degree 2 is given by the locally symmetric space

$$\mathcal{E}_2 \cong F(N,\Gamma) = {}_{\Gamma}\backslash^{\Omega_N^+}.$$

**Theorem 5.6** ([?, § 4.3])**.** *The Baily–Borel compactification*

$$\overline{\mathcal{E}_2}^{\mathrm{bb}} := \overline{{}_{\Gamma}\backslash^{\Omega_N^+}}^{\mathrm{bb}}$$

*has five 0−cusps and nine 1−cusps. The incidences between 0−cusps and 1−cusps are represented in Figure 4.*

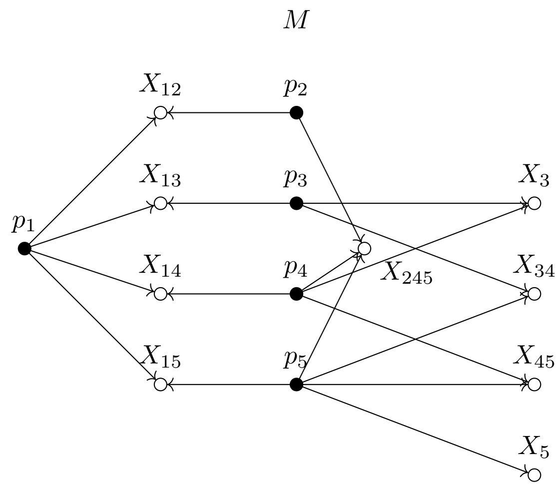


FIGURE 4. Cusp diagram for degree 2 polarized Enriques surfaces $\overline{\mathcal{E}_2}^{\mathrm{bb}}$.

5C. **Numerically polarized.** [?]

5D. **The family of degree 2 polarized Enriques surfaces.** We review the construction of degree 2 polarized Enriques surfaces following [?, Chapter V, § 23]. Let us consider the involution on $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\iota\colon ([X_0 : X_1], [Y_0 : Y_1]) \mapsto ([X_0 : -X_1], [Y_0 : -Y_1]).$$

We have that $\iota$ has precisely four isolated fixed points, namely

$$([0:1],[0:1]),\ ([0:1],[1:0]),\ ([1:0],[0:1]),\ ([1:0],[1:0]).$$

Let $B \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a general $\iota$−invariant curve of class $(4,4)$ not passing through the fixed points of $\iota$. Then, the bi-homogeneous polynomial giving $B$ consists of the following monomials:

$$\begin{gathered} X_0^4Y_0^4,\ X_0^4Y_0^2Y_1^2,\ X_0^4Y_1^4,\ X_0^3X_1Y_0^3Y_1,\ X_0^3X_1Y_0Y_1^3,\ X_0^2X_1^2Y_0^4, \\ X_0^2X_1^2Y_0^2Y_1^2,\ X_0^2X_1^2Y_0^4,\ X_0X_1^3Y_0^3Y_1,\ X_0X_1^3Y_0Y_1^3,\ X_1^4Y_0^4,\ X_1^4Y_0^2Y_1^2,\ X_1^4Y_1^4. \end{gathered} \tag{1}$$

The coefficients of $X_0^4Y_0^4, X_0^4Y_1^4, X_1^4Y_0^4, X_1^4Y_1^4$ must be nonzero to guarantee that $B$ does not pass through the torus fixed points of $\iota$.

The double cover $\pi\colon T \to \mathbb{P}^1 \times \mathbb{P}^1$ branched along $B$ is a well known to be a K3 surface: $T$ is smooth and minimal, $K_T \sim \pi^*\left(K_{\mathbb{P}^1\times\mathbb{P}^1} + \frac{1}{2}B\right) \sim 0$, and $\pi_*\mathcal{O}_T = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\left(-\frac{1}{2}B\right)$, which gives $h^1(\mathcal{O}_T) = 0$.

Let $\mathcal{L}^{\otimes 2} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4)$ and let $p\colon L \to \mathbb{P}^1 \times \mathbb{P}^1$ be the total space of the line bundle $\mathcal{L}$. Then the double cover $T$ of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along $B$ can be viewed inside $L$ as the vanishing locus of $t^2 - p^*s = 0$, where $B = V(s)$ and $t \in \Gamma(L, p^*\mathcal{L})$ is the tautological section. We have that then $\iota$ lifts to an involution $\widetilde{\iota}$ of $T$ with exactly eight fixed points: two over each fixed point of $\iota$. If $\tau$ denotes the deck transformation of the cover, i.e. $t \mapsto -t$, then we have that $\widetilde{\iota}$ commutes with $\tau$ and the composition $\sigma = \tau \circ \widetilde{\iota}$ is a fixed-point free involution of $T$. The quotient $q\colon T \to T/\sigma = S$ is then an Enriques surface called *Horikawa model*, and comes equipped with a degree 2 polarization induced by $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$.

Let $R \subseteq T$ be the ramification locus, so that $2R = \pi^*B$, define $\overline{R} = q(R)$, and let $0 < \epsilon \ll 1$ rational. Then we have the two following covering equalities:

$$K_T + \epsilon R \sim_{\mathbb{Q}} \pi^*\left(K_{\mathbb{P}^1\times\mathbb{P}^1} + \frac{1+\epsilon}{2}B\right),$$

$$K_T + \epsilon R \sim_{\mathbb{Q}} q^*\left(K_S + \frac{\epsilon}{2}\overline{R}\right).$$

The next lemma is straightforward.

**Lemma 5.7.** *With the notation introduced above, we have the following self-intersection numbers:*

(1) $\left(K_{\mathbb{P}^1\times\mathbb{P}^1} + \frac{1+\epsilon}{2}B\right)^2 = 8\epsilon^2$;
(2) $(K_T + \epsilon R)^2 = 16\epsilon^2$;
(3) $\left(K_S + \frac{\epsilon}{2}\overline{R}\right)^2 = 8\epsilon^2$.

We now relativize the above construction. Let $\mathbb{P}^{12}$ be the space of coefficients, up to scaling, for a bidegree $(4,4)$ polynomial in the monomials in (1). So, if $c_{ijk\ell}$ denotes the coefficient of $X_0^iX_1^jY_0^kY_1^\ell$, then $[\ldots : c_{ijk\ell} : \ldots] \in \mathbb{P}^{12}$ with $(i,j,k,\ell)$ within the following set:

$$M \coloneqq \{(i,j,k,\ell) \in \mathbb{Z}^4_{\geq 0} \mid i+j = k+\ell = 4,\ i+k \equiv j+\ell \equiv 0 \bmod 2\}.$$

Let $\mathbf{U} \subseteq \mathbb{P}^{12}$ be the dense open subset of coefficients such that the corresponding $\iota-$invariant $(4,4)$ curve $B \subseteq \mathbb{P}^1\times\mathbb{P}^1$ is smooth and does not pass through the torus fixed points of $\mathbb{P}^1\times\mathbb{P}^1$. Define $\mathcal{X} \coloneqq \mathbf{U}\times(\mathbb{P}^1\times\mathbb{P}^1)$ and let $\mathcal{X} \to \mathbf{U}$ be the projection. Let

$$\mathcal{B} \coloneqq V\left(\sum_{(i,j,k,\ell)\in M} c_{ijk\ell}X_0^iX_1^jY_0^kY_1^\ell\right) \subseteq \mathcal{X}.$$

Then $\left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right) \to \mathbf{U}$ is a family of stable pairs with fibers given by $\left(\mathbb{P}^1\times\mathbb{P}^1, \frac{1+\epsilon}{2}B\right)$ as described above. Additionally, we observe that $\left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right) \to \mathbf{U}$ is a KSBA-stable as defined in [**?**, 8.7].

The family $\left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right) \to \mathbf{U}$ has isomorphic fibers. To eliminate this redundancy, we consider the action of $\mathrm{Aut}(\mathbb{P}^1\times\mathbb{P}^1) \cong (\mathrm{PGL}_2\times\mathrm{PGL}_2)\rtimes\mathbb{Z}/2\mathbb{Z}$ (see [**?**]) on $H^0(\mathcal{O}(4,4))$. More precisely, we want to look at the subgroup $G$ which preserves $\iota-$invariant $(4,4)-$curves not passing through the torus fixed points. Note that the $\mathbb{Z}_2-$action preserves the set of monomials $M$ as $(i,j,k,\ell) \in M$ if and only if $(k,\ell,i,j) \in M$. Now consider a generic $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PGL}_2$ acting on $[X_0 : X_1]$. One can check directly that the action of this matrix preserves the monomials in $M$ if and only if $b = c = 0$, and the same holds if we consider the action of the second copy of $\mathrm{PGL}_2$ which acts on $[Y_0 : Y_1]$. In particular, we have that $G \cong \mathbb{G}_m^2 \rtimes (\mathbb{Z}/2\mathbb{Z})$. Therefore, we have an action $G \curvearrowright \mathbf{U}$ which identifies the isomorphic fibers of $\left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right) \to \mathbf{U}$.

Over $\mathbf{U}$, we can also consider the cover $(\mathcal{T}, \epsilon\mathcal{R}) \to \left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right)$ which gives the family of isomorphism classes of pairs $(T, \epsilon R)$ and the fiberwise quotient by the Enriques involution $(\mathcal{T}, \epsilon\mathcal{R}) \to \left(\mathcal{S}, \frac{\epsilon}{2}\overline{\mathcal{R}}\right)$ which gives the family of isomorphism classes of Enriques surfaces $\left(S, \frac{\epsilon}{2}\overline{R}\right)$. Summarizing, we have the following commutative diagram:

$$\begin{array}{ccc} (\mathcal{T}, \epsilon\mathcal{R}) & \longrightarrow & \left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right) \\ \downarrow & & \downarrow \\ \left(\mathcal{S}, \frac{\epsilon}{2}\overline{\mathcal{R}}\right) & \longrightarrow & \mathbf{U} \end{array}$$

**Definition 5.8.** Following the notation in [**?**, Theorem 8.1], consider the moduli functors $\mathcal{SP}(\mathbf{a}, d, \nu)$ for

$$(\mathbf{a}, d, \nu) = \left(\frac{1+\epsilon}{2}, 2, 8\epsilon^2\right), \ \left(\epsilon, 2, 16\epsilon^2\right), \ \left(\frac{\epsilon}{2}, 2, 8\epsilon^2\right).$$

and the corresponding coarse moduli spaces $\mathrm{SP}(\mathbf{a}, d, \nu)$. We now define the following stacks:

Consider the KSBA-stable family $\left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right) \to \mathbf{U}$. Therefore there is an induced morphism $\mathbf{U} \to \mathcal{SP}\left(\frac{1+\epsilon}{2}, 2, 8\epsilon^2\right)$ and denote by $\mathcal{P}'$ the closure of its image. Let $\overline{\mathbf{P}}'$ be the coarse moduli space corresponding to $\overline{\mathcal{P}}'$, and denote by $\overline{\mathbf{P}}$ its normalization. We have that $\overline{\mathbf{P}}$ provides a projective compactification of $\mathbf{U}/G$. By using the families $(\mathcal{T}, \epsilon\mathcal{R}) \to \mathbf{U}$ and $\left(\mathcal{S}, \frac{\epsilon}{2}\overline{\mathcal{R}}\right) \to \mathbf{U}$ instead, we obtain the compactifications $\overline{\mathbf{K}}$ and $\overline{\mathbf{E}}$ of $\mathbf{U}/G$ respectively, which instead parameterize generically the K3 and Enriques surfaces.

It is a standard observation that the compactifications $\overline{\mathbf{P}}, \overline{\mathbf{K}}, \overline{\mathbf{E}}$ are isomorphic to each other (see [**?**, § 3] for an analogous situation). We will mostly focus on $\overline{\mathbf{P}}$ as it parameterized the simplest objects.

## 6. Morphisms of moduli and cusps

6A. **Mapping the boundaries: matching cusp diagrams.** Let $\mathcal{F}_{2d}$ be the moduli space of polarized K3 surfaces of degree $2d$. How do we match the cusp diagram for the Baily–Borel compactification for the moduli of degree 2 Enriques surfaces with the Baily–Borel compactification for degree 4 hyperelliptic K3 surfaces? This is actually quite subtle, and it works as follows:

- [**?**, § 5] matches cusps for degree 2 Enriques surfaces and degree 4 K3 surfaces,
$$\overline{\mathcal{E}_2}^{\mathrm{bb}} \rightleftharpoons \overline{\mathcal{F}_4}^{\mathrm{bb}}.$$
- Using Scattone's method in [**?**, § 6], we can match the cusps for degree 4 hyperelliptic K3 surfaces and degree 4 K3 surfaces,
$$\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}} \rightleftharpoons \overline{\mathcal{F}_4}^{\mathrm{bb}}.$$

- The above two points imply the matching we need.

What is Scattone's method uses the following observations:

- $1-$cusps of $\overline{\mathcal{F}_4}^{\mathrm{bb}}$ are in one-to-one correspondence with the orthogonal complements of $D_8$ in the Niemeier lattices, and
- The $1-$cusps of $\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}}$ are in one-to-one correspondence with the orthogonal complements of $D_7$ in the Niemeier lattices.

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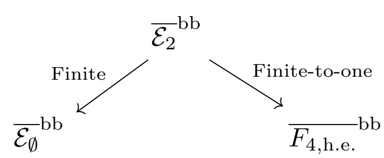
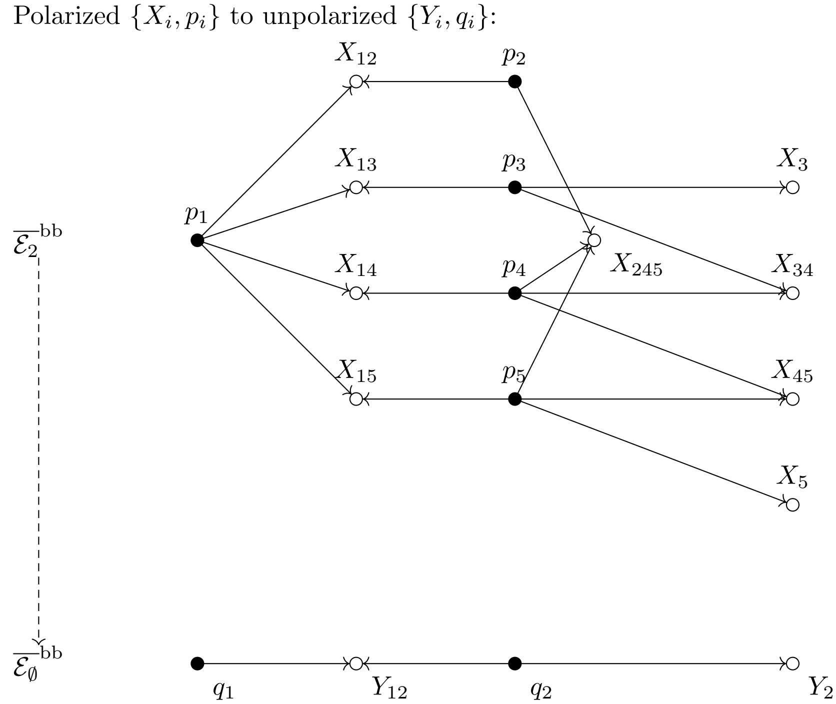
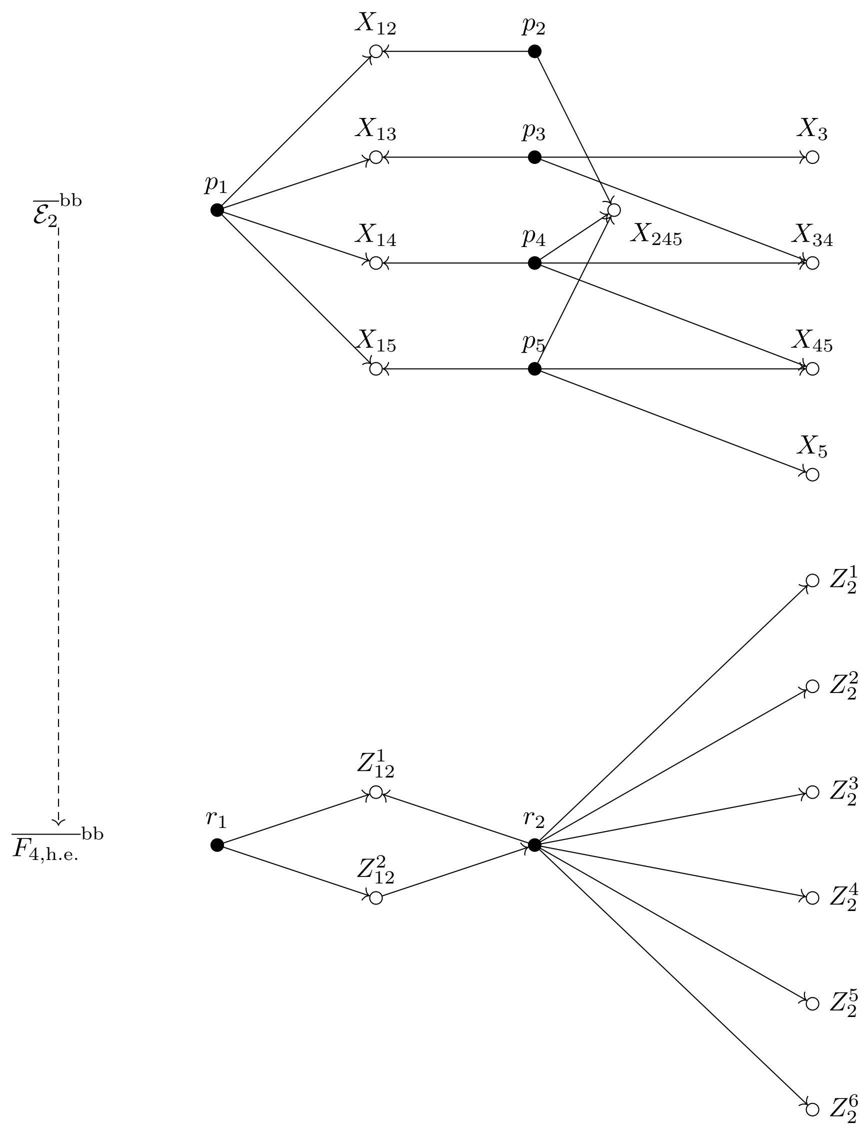



As a result, there are two cusp incidence diagrams to match:

(1) Polarized $\{X_i, p_i\}$ to unpolarized $\{Y_i, q_i\}$:



(2) Polarized $\{X_i, p_i\}$ to hyperelliptic $\{Z_i, r_i\}$:




6B. **Type III KPP model at the (18,2,0) odd 0-cusp.** At the (18, 2, 0) odd 0-cusp the Type III stable models are of pumpkin type

Stable models vs KPP models, write down the definition and the differences.

Let $X$ be a K3 surface with a nonsymplectic involution $\iota$ with induced involution $i^*$ on $H^2(X;\mathbb{Z})$. We define $S$ to be the $(+1)-$eigenspace $\iota^*$; it is a hyperbolic lattice 2-elementary lattice, and all the possibilities for such lattices were classified by Nikulin. We denote by $T$ the orthogonal complement of $S$ in $H^2(X;\mathbb{Z})$.

**Definition 6.1.** What makes an **odd** $0-$cusp different from an **even** $0-$cusp?

**Definition 6.2.** We define two types of stable models $\overline{X}_0 = \cup \overline{V}_i$:

(1) *Pumpkin.* Each surface $\overline{V}_i$ has two sides $\overline{D}_i = \overline{D}_{i,\mathrm{left}} + \overline{D}_{i,\mathrm{right}}$, they are glued in a circle, all of $D_i$ meeting at the north and south poles.

(2) *Smashed pumpkins.* Starting with a surface of the pumpkin type, one short side is contracted to a point, so that the north and south poles are identified.

If the surface $V_i$, say to the left, is $(\mathbb{F}_1, D_1 + D_2)$, where $D_1 \sim f$ is the short side being contracted, $D_2 \sim 2s + 2f$ is the other side, and $C_g \sim f$ on $V$ contract $V_i$ by the $\mathbb{P}^1-$fibration $V_i \to \mathbb{P}^1$. Then on the next surface $V_{i-1}$ to the left the long side will fold $2:1$ to itself, creating a non-normal singularity along that side.

If on $V_i$ the divisor $C_g$ has degree $C_g^2 \geq 2$, then only the short side is contracted and the resulting surface $\overline{V}_i$ is normal in codimension 1, with only two points in the normalization glued together (the poles).

**Theorem 6.3** ([**?**, Theorem 9.9])**.** *Let* $(\overline{X}_0, \cup\overline{V}_i, \epsilon\overline{C}_g)$ *be the stable model of a pair* $(X_0 = \cup V_i, \epsilon C_g)$*, where* $X_0$ *is the KPP model of a Type III Kulikov surface and* $C_g$ *is the component of genus* $g \geq 2$ *in the ramification divisor* $R$*. Then the normalization of each* $\overline{V}_i$ *is an ADE surface with an involution from* [**?**, Table 2]*. Moreover,*

(1) *If* $\overline{T}$ *is an odd* $0-$*cusp of* $F_S$*, then* $\overline{X}_0$ *is of pumpkin type.*
(2) *If* $\overline{T}$ *is an even* $0-$*cusp of* $F_S$*, then* $\overline{X}_0$ *is of smashed pumpkin type. The surfaces* $V_i$ *of the last type in definition Definition 9.8, on which* $V_i \to \overline{V}_i$ *contracts one side are surfaces of* [**?**, Table 2] *for which one of the sides has length* 0*, i.e. those with a double prime or a "+".*

## 6C. ADE surfaces.

**Definition 6.4** (ADE surfaces)**.** An **ADE surface** is a pair $(Y, C)$, where $Y$ is a normal surface. $(Y, C)$ has log canonical singularities and the divisor $-2(K_Y + C)$ is Cartier and ample. $L \coloneqq -2(K_Y + C)$ is referred to as the **polarization** of the ADE surface $(Y, C)$.

Let $B \in |L|$ effective divisor such that $(Y, C + \frac{1+\epsilon}{2}B)$ is log canonical for $0 < \epsilon \ll 1$, then $(Y, C + \frac{1+\epsilon}{2})$ is called an ADE pair. We can take the double cover $X \to Y$ branched along $B$ and I guess possibly along $C$. It can happen that $Y$ is toric and $C$ is part of the toric boundary.

ADE surfaces admit a combinatorial classification. The classes of ADE surfaces are called shapes. A shape can be *pure* or *primed*. Surfaces of pure shape are fundamental. Surfaces of primed shape are secondary and can be obtained from surfaces of pure shape using an operation called *priming*.

The ADE surfaces of pure shape are all toric. To construct these we start from a polarized toric surface $(Y, L)$, where $L = -2(K_Y + C)$. This corresponds to a lattice polytope $P$ in $M \otimes \mathbb{Z}$ Given a surface $(Y, C)$ of pure shape, the irreducible components of $C$ are called **sides**. There are two sides with a point in common called left or right. They decompose $C = C_1 + C_2$. A side can be **long** or **short** depending on whether a side $C'$ satisfies $C' \cdot L = 2, 4$ or $C' \cdot L = 1, 3$ respectively. The ADE surfaces of pure shape are listed in [**?**, Table 1] (see Figures 1, 2, 3 therein). Here are some basic examples

- The ADE surface $(Y, C)$ corresponding to $D_4$ is $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and $C$ is the sum of two incident torus fixed curves.
- The ADE surface $(Y, C)$ corresponding to $A_1$ is $Y = \mathbb{P}^2$ and $C$ is the sum of two torus fixed curves. The polarization is $\mathcal{O}(2)$.

The superscripts minus signs on the left or right denote the location of the short side. Note both sides can be long or short. Do they correspond to the visible length? Not at all! The ADE surface $A_3$ has two long sides, but one edge is shorter than the other. By the way, in this case, $Y = \mathbb{F}_2^0$. $A_2^-$ has a long side on the left and a short side on the right.

Primed shapes. Priming is an operation that produces a new del Pezzo surface $(\overline{Y}', \overline{C}')$ from an old one $(Y, C)$. The priming operation is basically a weighted blow-up given by the composition of two ordinary blow-ups and the contraction of a $(-2)-$curve making an $A_1$ singularity. Weighted blow-ups of this form are the basis of the priming operation. Weighted blow-up with respect to the idea $(y, x^2)$. Priming has the meaning of disconnecting a curve from another. Given an ADE pair $(Y, C + \frac{1+\epsilon}{2})$, then the priming operation is performed on the points of intersection between $C$ and $B$, which intersect transversely by [**?**, Remark 3.3]. Priming may not exist, and there are some necessary and sufficient conditions for priming to exist.

For an ADE shape, we add a prime symbol when priming on a long side. When priming a short side, we change the minus into a plus. All the ADE surfaces, pure or primed are in [**?**, Table 2].

## 7. Our new results

### 7A. **Enriques strategy.**

**Remark 7.1.** This story suggests the following approach to Enriques surfaces:

(1) Fully understand the cusps of the Enriques moduli space, possibly in terms of what has been done for K3s already.
(2) For each cusp, find the Coxeter diagram.
(3) For each Coxeter diagram, cook up the right IAS pair of a manifold and a divisor $R_{\mathrm{IAS}}$. For us, instead of an $\mathrm{IAS}^2$ it may be an $\mathrm{IA}\mathbb{RP}^2$, and may come from some fusion of known $\mathrm{IAS}^2$s for K3s, maybe as simple as quotienting the $\mathrm{IAS}^2$ by the antipodal map.
(4) Reverse-engineer the $\mathrm{IA}\mathbb{RP}^2$ so that it carries two commuting involutions, and probably take $R_{\mathrm{IAS}}$ to be the intersection of the two ramification divisors on the $\mathrm{IA}\mathbb{RP}^2$.
(5) Describe all of the ways the $\mathrm{IA}\mathbb{RP}^2$ can degenerate, a la Valery's pumpkin-type models.

### 7B. **Lemmas/theorems.**

**Lemma 7.2.** *If* $\operatorname{sgn}(L) = (p, q)$ *and* $e \in L$ *is isotropic, then* $\operatorname{sgn}(\mathbb{Z}e) = (?, ?)$ *and* $\operatorname{sgn}(\mathbb{Z}e^\perp) = (?, ?)$.

**Lemma 7.3.** *Let $L$ be a lattice of signature $(p, q)$ and let $e \in L$ be an isotropic vector. Then*
$$\operatorname{sgn}(e^\perp / e) = (p - 1, q - 1).$$

*Proof.* □

**Proposition 7.4.** *Let $\mathrm{II}_{3,19}$ be the K3 lattice and let $h$ be an ample class of degree $d$. Then*
$$h^{\perp \mathrm{II}_{3,19}} \cong \mathrm{II}_{3,19}\langle d \rangle := {\mathrm{II}_{1,1}}^{\oplus^2} \oplus {E_8}^{\oplus^2} \oplus \langle -2d \rangle$$
*and* $\operatorname{sgn} h^{\perp \mathrm{II}_{3,19}\langle d \rangle} = (2, 19)$. *Thus $F_{2d}$ arises as the Hodge-theoretic moduli space associated with the period domain $D_L$ for the lattice $L := \mathrm{II}_{3,19}\langle d \rangle$.*

*Proof.* todo □

The theorem below needs the following notations and conventions (these will all be introduced before as needed).

$L_{\mathrm{K3}} = U^{\oplus 3} \oplus E_8^{\oplus 2} = (U \oplus E_8)^{\oplus 2} \oplus U$ and

$$I(m, m', h) = (m, m', -h).$$

$$L_- = U \oplus U(2) \oplus E_8(2)$$

$$\Omega_- = \{[v] \in \mathbb{P}(L_- \otimes \mathbb{C}) \mid v^2 = 0,\ v \cdot \overline{v} > 0\}$$

**Definition 7.5.** Consider the following subgroup of $O(L_{\mathrm{K3}})$:

$$\Gamma' = \{g \in O(L_{\mathrm{K3}}) \mid g \circ I = I \circ g,\ g(e+f, e+f, 0) = (e+f, e+f, 0)\}$$

Note that we have a natural group homomorphism $\Gamma' \to O(L_-)$ given by $g \mapsto g|_{L_-}$. To prove that $g|_{L_-} \in O(L_-)$ it is enough to observe that $g(L_-) = L_-$. Let $x \in L_-$. We have that $g(x) \in L_-$ if $I(g(x)) = -g(x)$. This holds because

$$I(g(x)) = g(I(x)) = g(-x) = -g(x).$$

We denote by $\Gamma$ the image of $\Gamma' \to O(L_-)$.

$$E_2 = \Omega_-/\Gamma$$

$$\Omega_{4,\mathrm{h}} = \{[v] \in \mathbb{P}(\Lambda_{18} \otimes \mathbb{C}) \mid v^2 = 0,\ v \cdot \overline{v} > 0\}$$

$\Lambda_{18} = U^{\oplus 2} \oplus D_{16}$, $\Gamma_{4,\mathrm{h}} = O(\Lambda_{18})$.

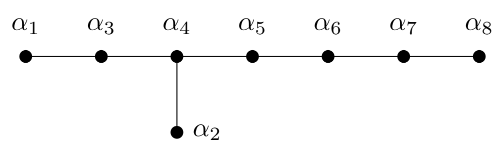



FIGURE 5. The $E_8$ lattice (Sterk's convention).

**Theorem 7.6.** *There exists an injective morphism*

$$E_2 \to F_{4,\mathrm{h}}$$

*which extends to a morphism of the Baily–Borel compactifications*

$$\overline{E}_2^{\mathrm{bb}} \to \overline{F}_{4,\mathrm{h}}^{\mathrm{bb}}.$$

*Proof.* Consider the inclusion of $U(2)$ into $U(2) \oplus E_8(2)$ as direct summand. By considering the orthogonal complements in $L_{\mathrm{K3}}$ we obtain that

$$L_- \subseteq \Lambda_{18}.$$

From this follows from the definitions of $\Omega_-$ and $\Omega_{4,\mathrm{h}}$ that we have an inclusion

$$\Omega_- \hookrightarrow \Omega_{4,\mathrm{h}}.$$

Let us show that this descends to a morphism

$$\Omega_-/\Gamma_2 \to \Omega_{4,\mathrm{h}}/\Gamma_{4,\mathrm{h}}.$$

Let $[v],[w]\in\Omega_2$ and assume there exists $g\in\Gamma$ such that $g([v])=[w]$. We show that there exists $h\in\Gamma_{4,\mathrm{h}}$ such that $h([v])=[w]$. By the definition of $\Gamma$, $g=\widetilde{g}|_{L_-}$, there exists $\widetilde{g}\in O(L_{\mathrm{K3}})$ such that $\widetilde{g}\circ I=I\circ\widetilde{g}$ and $f(e+f,e+f,0)=(e+f,e+f,0)$. Then, by the proof of [**?**, Proposition 2.7], we have that $\widetilde{g}$ preserves $e+f$ and $e-f$ in $L_+=U(2)\oplus E_8(2)$. In particular, $\widetilde{g}$ preserves the summand $U(2)\subseteq U(2)\oplus E_8$. This implies that $\widetilde{g}$ preserves $U(2)^\perp=\Lambda_{18}$. In particular, by setting $h=\widetilde{g}|_{\Lambda_{18}}$ we obtain what we needed.

We now prove that the morphism $\varphi\colon\Omega_-/\Gamma\to\Omega_{4,\mathrm{h}}/\Gamma_{4,\mathrm{h}}$ is injective. Let $x_1,x_2\in\Omega_-/\Gamma$ and assume that $\varphi(x_1)=\varphi(x_2)$. Let $S_i$ be the Enriques surface corresponding to $x_i$. Then $S_i$ is the quotient of a K3 surface $T_i$ which is the double cover $\pi_i\colon T_i\to\mathbb{P}^1\times\mathbb{P}^1$ branched along a $(4,4)$ curve $B_i$ which is invariant with respect to the involution $\iota\colon(x,y)\mapsto(-x,-y)$. Because of the assumption that $\varphi(x_1)=\varphi(x_2)$, we must have that

$$\begin{array}{ccc} T_1 & \xrightarrow{\cong} & T_2 \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ \mathbb{P}^1\times\mathbb{P}^1 & \xrightarrow{\cong} & \mathbb{P}^1\times\mathbb{P}^1 \end{array}$$

where the bottom isomorphism commutes with $\iota$ and the top map commutes with $\widetilde{\iota}$. Let $\tau_i$ be the deck transformation of the cover $\pi_i$, so that we have the two Enriques involutions $\sigma_i=\tau_i\circ\widetilde{\iota}$. Then we have an isomorphism between $S_1=T_1/\sigma_1\cong T_2/\sigma_2$, which implies that the period points $x_1,x_2$ are equal.

The morphism $\Omega_-/\Gamma\to\Omega_{4,\mathrm{h}}/\Gamma_{4,\mathrm{h}}$ extends to a morphism of the Baily–Borel compactifications by [**?**, Theorem 2], and sends boundary components to boundary components.

Next, we describe the cusp correspondence. Recall, $\overline{E}_2^{\mathrm{bb}}$ has five 0-cusps $p_1,\ldots,p_5$ corresponding to the following isotropic vectors in

$$L_-=U\oplus U(2)\oplus E_8(2)=\langle e,f\rangle\oplus\langle e',f'\rangle\oplus\langle\alpha_1,\ldots,\alpha_8\rangle.$$

(1) $\delta_1=e$;
(2) $\delta_2=e'$;
(3) $\delta_3=e'+f'+\overline{\alpha}_8$;
(4) $\delta_4=2e'+f'+\overline{\alpha}_1$;
(5) $\delta_5=2e+2f+\overline{\alpha}_1$.

Note that $e'\cdot f'=2$ and $\overline{\alpha}_i\cdot\alpha_j=\delta_{ij}$. We have that $\delta_1^\perp/\delta_1\cong U(2)\oplus E_8(2)$ and $\delta_i^\perp/\delta_i\cong U\oplus E_8(2)$ for $i=2,\ldots,5$.

On the other hand, $\overline{F}_{4,\mathrm{h}}^{\mathrm{bb}}$ has two 0-cusps $q_1,q_2$ for which the corresponding isotropic vectors $\eta_1,\eta_2\in\Lambda_{18}$ satisfy $\eta_1^\perp/\eta_1\cong U\oplus E_8^{\oplus2}$ and $\eta_2^\perp/\eta_2\cong U(2)\oplus E_8^{\oplus2}$.

To understand whether $p_1\mapsto q_1$ or $p_1\mapsto q_2$, it is enough to compute

$$\delta_1^{\perp_{\Lambda_{18}}}/\delta_1.$$

But this is clear after realizing that $\Lambda_{18}\cong U\oplus U(2)\oplus E_8^{\oplus2}$, and there is the explicit embedding

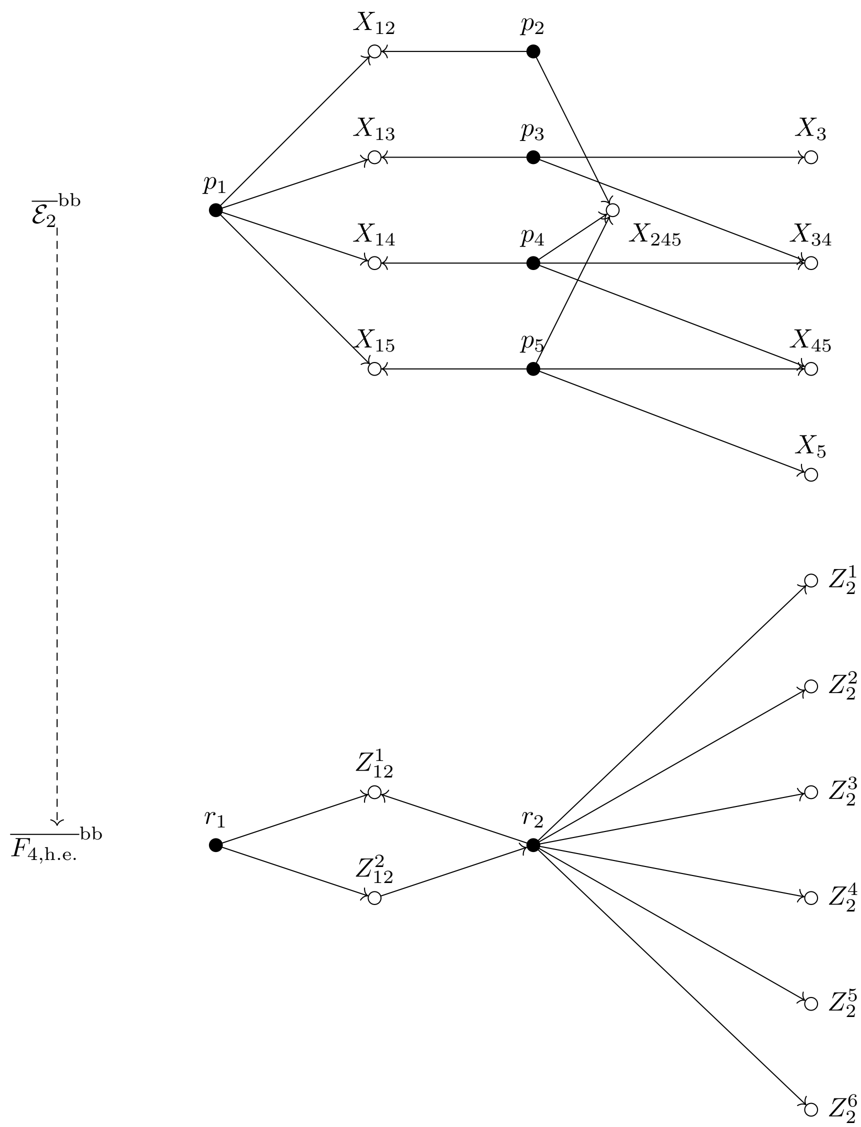
$$\begin{aligned} L_-=U\oplus U(2)\oplus E_8(2)&\subseteq U\oplus U(2)\oplus E_8^{\oplus2}\\ (u,v,w)&\mapsto(u,v,w,w). \end{aligned}$$

So that it is clear that

$$\delta_1^{\perp\Lambda_{18}}/\delta_1 = U(2) \oplus E_8(2).$$

□

**Lemma 7.7** (Cusp correspondence 1)**.** *We have a cusp correspondence from polarized $\{X_i, p_i\}$ in $\partial\overline{\mathcal{E}_2}^{\mathrm{bb}}$ to hyperelliptic $\{Z_i, r_i\}$ in $\partial\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}}$:*




*Proof.* □

## 8. Period Domains and Baily-Borel Compactifications

**Remark 8.1.** The Baily-Borel and toroidal compactifications are defined for quotients of Hermitian symmetric spaces by actions of arithmetic subgroups of their automorphism groups, i.e. those that can be written as $\Gamma\backslash\Omega$. BB compactifications are generally small, e.g. $\dim F_2 = 19$ but $\operatorname{codim} \partial\overline{F_2}^{\mathrm{bb}} = 18$, and this often precludes having a satisfactory modular interpretation of its boundary points. In particular, given an arc in this compactification with endpoint in the boundary, one can not generally construct a birationally unique limit. Toroidal compactifications $\overline{\Gamma\backslash\Omega}^{\mathrm{tor}}$ are obtained as certain blowups of $\overline{\Gamma\backslash\Omega}^{\mathrm{bb}}$, and e.g. for $F_2$ some boundary components become divisors (codimension 1). However these are highly non-unique and depend on choices of fans. One might hope there are canonical such choices. The semitoric compactifications of Looijenga interpolate between $\overline{\Gamma\backslash\Omega}^{\mathrm{bb}}$ and $\overline{\Gamma\backslash\Omega}^{\mathrm{tor}}$.

### 8A. **The Baily-Borel compactification.**

**Definition 8.2.** The group $G$ acts transitively on the set of boundary components $F \subseteq \partial\mathcal{D}_L \coloneqq \widetilde{\mathcal{D}}_L \setminus \mathcal{D}_L$, and $\operatorname{Stab}_G(F) \leq G$ is a maximal parabolic subgroup. Taking stabilizers establishes a bijection

$$\begin{aligned}\{\text{Boundary components } F \subseteq \partial\mathcal{D}_L\} &\to \{\text{Maximal parabolic subgroups } P \leq G\}\\ F &\mapsto P \coloneqq \operatorname{Stab}_G(F)\end{aligned}$$

For $G \coloneqq \mathrm{SO}(V)$, parabolic subgroups $P$ are stabilizers of flags of isotropic subspaces in $V$, and since $\operatorname{sgn}(V) = (2, n)$, a flag has length at most 3 and a maximal flag is of the form $p \subseteq I \subseteq J$ where $p$ is a point, $I$ is an isotropic line, and $J$ is an isotropic plane. The only flags that define maximal parabolic subgroups of $\mathrm{SO}(V)$ are of length 1, consisting of either a single line or a single plane. Thus we have bijections

$$\begin{aligned}\{\text{Rational boundary components of } (\mathrm{SO}_{2,n}(\mathbb{R}), \mathrm{SO}_2(\mathbb{R}) \times \mathrm{SO}_n(\mathbb{R}))\}\\ \Downarrow\!\Uparrow\\ \{\text{Maximal parabolic subgroups of}\, \mathrm{SO}(V)\}\\ \Downarrow\!\Uparrow\\ \{\text{Isotropic lines } I \subset V\} \cup \{\text{Isotropic planes } J \subset V\}\end{aligned}$$

where a boundary component $F$ is rational if $\operatorname{Stab}_G(F)$ is defined over $\mathbb{Q}$. For an arithmetic subgroup $\Gamma \leq G$, letting $\partial(\mathcal{D}_L)_{\mathbb{Q}}$ be the set of all rational boundary components of $\mathcal{D}_L \subset \widetilde{\mathcal{D}}_L$, we produce a compactification

$$\overline{\Gamma\backslash\mathcal{D}_L} = \Gamma\backslash\mathcal{D}_L \bigcup_{F \in \partial\mathcal{D}L,\mathbb{Q}} (G_F(\mathbb{Q}) \cap \Gamma)\backslash F.$$

**Definition 8.3.** Let $L$ be a lattice of signature $(2, n)$ for $n \geq 1$, let $\Omega_L$ be the associated period domain, let $\mathrm{O}^+(L) \leq \mathrm{O}(L)$ be the subgroup preserving $\Omega_L$ and let $\widetilde{\Omega}_L$ be the affine cone over $\Omega_L$. Let $n \geq 3$, let $k \in \mathbb{Z}$, and let $\Gamma \leq \mathrm{O}^+(L)$ be a finite-index subgroup with $\chi : \Gamma \to \mathbb{C}^*$ a character. A holomorphic functional $f : \Omega_L \to \mathbb{C}$ is called a **modular form of weight $k$ and character $\chi$ for $\Gamma$** if

(1) Factor of automorphy: $f(\lambda z) = \lambda^{-k} f(z)$ for any $\lambda \in \mathbb{C}^*$.
(2) Equivariance: $f(\gamma z) = \chi(\gamma) f(z)$ for all $\gamma \in \Gamma$.

**Definition 8.4.** Let $\Omega_L$ as above and let $M_k(\Gamma, \chi)$ be the $\mathbb{C}-$vector space of such modular forms of weight $k$ for $\Gamma$ with character $\chi$. The **Baily-Borel compactification** can be defined as

$$\overline{\Gamma\backslash\Omega_L}^{\mathrm{bb}} := \operatorname{Proj} \bigoplus_{k \geq 1} M_k(\Gamma, \chi_{\mathrm{triv}})$$

where $\chi_{\mathrm{triv}}$ is the trivial character.

**Remark 8.5.** $\partial\overline{\Gamma\backslash\Omega_L}^{\mathrm{bb}}$ decomposes into points $p_i$ and curves $C_j$, which are in bijection with $\Gamma-$orbits of isotropic lines $i$ and isotropic planes $j$ in $L_{\mathbb{Q}}$. Moreover $p_i \in \overline{C_j} \iff$ one can choose representatives lines $i$ and planes $j$ such that $i \subseteq j$.

**Remark 8.6.** A theorem of Baily-Borel gives the existence of an ample automorphic line bundle $\mathcal{L}$ on $\overline{\mathcal{D}_L}$ giving it the structure of a normal projective variety isomorphic to a canonical model $\operatorname{Proj} \bigoplus_{k \geq 0} H^0(L^k)$??. We denote this compactification $\overline{\Gamma\backslash\mathcal{D}_L}^{\mathrm{bb}}$.

I don't quite remember what this graded ring is.

8B. **Toroidal and semitoroidal compactifications.**

**Remark 8.7.** A **toroidal compactification** $\overline{\Gamma\backslash\mathcal{D}_L}^{\mathrm{tor}}$ is a certain blowup of $\overline{\Gamma\backslash\mathcal{D}_L}^{\mathrm{bb}}$, so there is a birational map $\overline{\Gamma\backslash\mathcal{D}_L}^{\mathrm{tor}} \dashrightarrow \overline{\Gamma\backslash\mathcal{D}_L}^{\mathrm{bb}}$. It is defined by a collection of admissible fans $\{F_i\}_{i \in I}$ where $I$ ranges over an index set for all cusps.

A **semitoroidal compactification** is a generalization due to Looijenga for which the cones of $F_i$ are not required to be finitely generated, and [**?**] shows that semitoroidal compactifications are characterized as exactly the normal compactifications $\overline{M}^{\mathrm{semitor}}$ fitting into a tower

$$\begin{array}{c} \overline{M}^{\mathrm{tor}} \\ \uparrow \\ \overline{M}^{\mathrm{semitor}} \\ \uparrow \\ \overline{M}^{\mathrm{bb}} \end{array}$$

where $\overline{M}^{\mathrm{tor}}$ is some toroidal compactification of $M$.

**Remark 8.8.** On the toroidal compactification associated with $\Gamma\backslash\Omega_L$: for a cusp $C_i$ of the BB compactification, let $F \subset L_{\mathbb{Q}}$ be the corresponding $\Gamma-$orbit of an isotropic line or plane. Consider its stabilizer $S(F) := \mathrm{Stab}_{\mathrm{O}^+(L_{\mathbb{R}})}$, and its unipotent radical $U(F)$. Then $U(F)$ is a vector space containing a lattice $U(F) \cap \Gamma$ and an open convex cone $C(F)$. Let $\overline{C(F)}^{\mathrm{rc}}$ be the rational closure of the cone, so the union of $C(F)$ and rational rays in its closure. We then choose a fan $\Sigma(F)$ with $\mathrm{supp}(\Sigma(F)) = \overline{C(F)}^{\mathrm{rc}}$ which is invariant under $\mathrm{Stab}_{\mathrm{O}^+(L_{\mathbb{R}})}(F) \cap \Gamma$ and produce an associated toric variety $X_{\Sigma(F)}$. If one does this for every $F$ to produce a $\Gamma-$admissible collection of polyhedra $\Sigma$, their quotients by $\Gamma$ glue to give a toroidal compactification $\overline{\Gamma\backslash\Omega_L}^{\mathrm{tor}}$, which has the structure of a (complex) algebraic space. There is a surjection $\overline{\Gamma\backslash\Omega_L}^{\mathrm{tor}} \twoheadrightarrow \overline{\Gamma\backslash\Omega_L}^{\mathrm{bb}}$. Why $e^\perp/e$ shows up: if $e$ is an isotropic line in $L$ corresponding to a cusp $F$, there is an isomorphism of lattices $U(F) \cap \widetilde{\mathrm{O}}^+(L) \cong e^\perp/e$ where $\widetilde{\mathrm{O}}^+ := \ker\left(\mathrm{O}^+(L) \to \mathrm{O}(A_L)\right)$.

## 8C. **Misc.**

**Definition 8.9** (Log CY pairs)**.** A **log Calabi-Yau (CY) pair** is a pair $(X, D)$ with $X$ a proper variety and $D$ an effective $\mathbb{Q}-$Cartier divisor such that the pair is log canonical and $K_X + D \sim_{\mathbb{Q}} 0$.

A degeneration $\pi : \mathcal{X} \to \Delta$ is a CY degeneration if $\pi$ is proper, $K_{\mathcal{X}} \sim_{\mathbb{Q}} 0$, and $(\mathcal{X}, \mathcal{X}_0)$ is dlt. This implies that $\mathcal{X}_t$ is a Calabi-Yau variety for $t \neq 0$ and $\mathcal{X}_0$ is a union of log CY pairs $(V_i, D_i)$. If $\mathcal{X}_t$ is a strict CY of dimension $n$, so $\pi_1\mathcal{X}_t = 0$ and $h^i(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$ for $1 \le i \le n-1$, and $\dim\Gamma(\mathcal{X}_0) = n$, we say $\mathcal{X}$ is a large complex structure limit or equivalently a maximal unipotent or MUM degeneration.

If $n = 2$, Kulikov shows that $\Gamma(\mathcal{X}_0)$ is always isomorphic to a 2-sphere $S^2$. Whether $\Gamma(\mathcal{X}_0) \cong S^n$ or a quotient thereof for $n \ge 3$ more generally is an open question, posed by Kontsevich-Soibelman. It has recently been shown by Kollár-Xu that in the case of degenerations of Calabi-Yau or hyperkähler manifolds, the dual complex is always a rational homology sphere.

**Remark 8.10.** Why this is useful to us: one formulation of mirror symmetry is the formulation due to Strominger-Yau-Zaslow, aptly called SYZ mirror symmetry. Conjecturally, the general fiber $\mathcal{X}_t$ of a punctured family of CYs $\mathcal{X}^\circ \to \Delta^\circ$ can be given the structure of a special Lagrangian torus fibration $\mathcal{X}_t \to B$, one can "dualize" the fibration to obtain a mirror CY $\widehat{\mathcal{X}_t} \to B$ over the same base. The common base $B$ of these two fibrations is conjecturally of the form $\Gamma(\mathcal{X}_0)$, the dual complex of a degeneration $\mathcal{X} \to \Delta$ extending $\mathcal{X}^\circ$.

This might have something to do with the discrete Legendre transform Phil mentions.

## 8D. **Baily-Borel cusps and incidence diagrams.**

## 8E. **Other compactifications.**

**Remark 8.11.** Write $F_{2d} := \Gamma_{2d}\backslash D_{2d}$. A cusp $p_i$ of $\overline{F_{2d}}^{\mathrm{bb}}$ determines a cone $C_i$. Toroidal and semitoroidal compactifications are then determined by a collection of $\Gamma_{2d}-$invariant fans supported on $C_i$ for $i$ ranging over an index set for all cusps. If $d = 1$ (or more generally if $2d$ is squarefree), there is a single 0-cusp $p_{2d}$ whose cone $C_{2d}$ has a description as the positive light cone in the rational closure of $C_{2d}$ with respect to a certain lattice $M_{2d}$. This can be written $C_{2d}^{\mathrm{rc}} := \mathrm{Conv}(\overline{C}_{2d} \cap M_{2d} \otimes_{\mathbb{Z}} \mathbb{R})$. A semitoroidal compactification of $F_{2d}$ is then determined by a semitoric fan in $M_{2d,\mathbb{R}}$ supported on $C_{2d}^{\mathrm{rc}}$ which is invariant for a particular subgroup $\Gamma_{2d}^+ \le \mathrm{O}(M_{2d})$. In this case, one can make a canonical choice for such a semitoric fan: the Coxeter fan $\Sigma_{2d}^{\mathrm{Cox}}$ whose cones are precisely the fundamental domains for a Weyl group action on $C_{2d}^{\mathrm{rc}}$, see AET19.

Todo: can spell out what $M_{2d}, \Gamma, \Gamma^+$ are.

## 9. Vinberg-Coxeter boundary theory

Goal for this section: describe how Coxeter-Vinberg diagrams are used to get models at BB cusps.

9A. **The case of abelian varieties.** Let us consider the setup first for moduli of principally polarized abelian varieties. We have the following:

**Theorem 9.1** (Alexeev). *There is an isomorphism*

$$\eta : \overline{\mathcal{A}_g}^{\mathrm{tor}(F)} \xrightarrow{\sim} (\overline{\mathcal{A}_g}^{\mathrm{KSBA}})^{\nu}$$

*for $F$ the second Voronoi fan.*

Is there always a natural morphism from toric compactifications to KSBA? Not in general! We can talk about it when we meet.

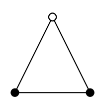
**Corollary 9.2.** *As a result, any punctured 1-parameter family $\mathcal{X}^{\circ} \to \Delta^{\circ}$ has a unique limit $\mathcal{X}_0$ which can combinatorially be described as a tropically polarized abelian variety $(X_{\mathrm{trop}}, \Theta_{\mathrm{trop}})$ with a tropical $\Theta$ divisor $\Theta_{\mathrm{trop}}$.*

*More is true: the fan $F$ is itself a moduli space for such tropical abelian varieties.*

**Remark 9.3** (Motivation from abelian varieties)**.** To see how this works, consider a 1-parameter family of abelian varieties. These are tori of the form

$$\mathcal{X}_t := \mathrm{coker}(\phi_t : \mathbb{Z}^g \hookrightarrow (\mathbb{C}^*)^g),$$

where $\phi_t$ are embeddings that vary in the family. Write this embedding as a matrix $M$; this is a matrix of periods. Then for $t \approx 0$ one exponentiates $M_{ij}$ to get a symmetric positive-definite $g \times g$ matrix $B$. There is a cone $C \subsetneq \{B = B^t > 0\}$ in $\mathrm{GL}_g(\mathbb{Z})$ and a Coxeter fan $F$ supported on its rational closure $\overline{C}^{\mathrm{rc}}$ which corresponds to affine Dynkin diagram $\widetilde{A}_2$:

$\widetilde{A}_2$ : 

This corresponds to a triangular fundamental domain for a reflection group that acts on $C$. For a cartoon picture, think of a hyperbolic disc $\mathbb{D}$ and let the fundamental domain be a triangle with ideal vertices:

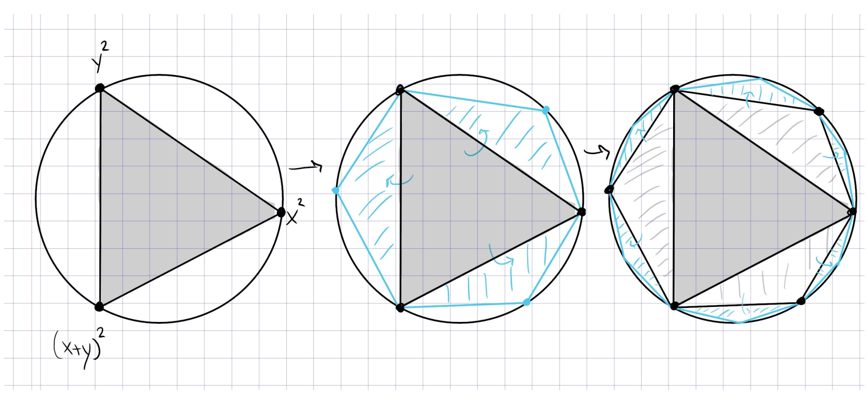



Figure 6. Tessellating a hyperbolic disc by triangles

Note that the straight lines forming the edges should "really" be curved hyperbolic geodesics. One continues reflecting in order to tessellate the hyperbolic disc, then puts this disc in $\mathbb{R}^3$ at height one and cones it to the origin to get an infinite-type fan $F$:

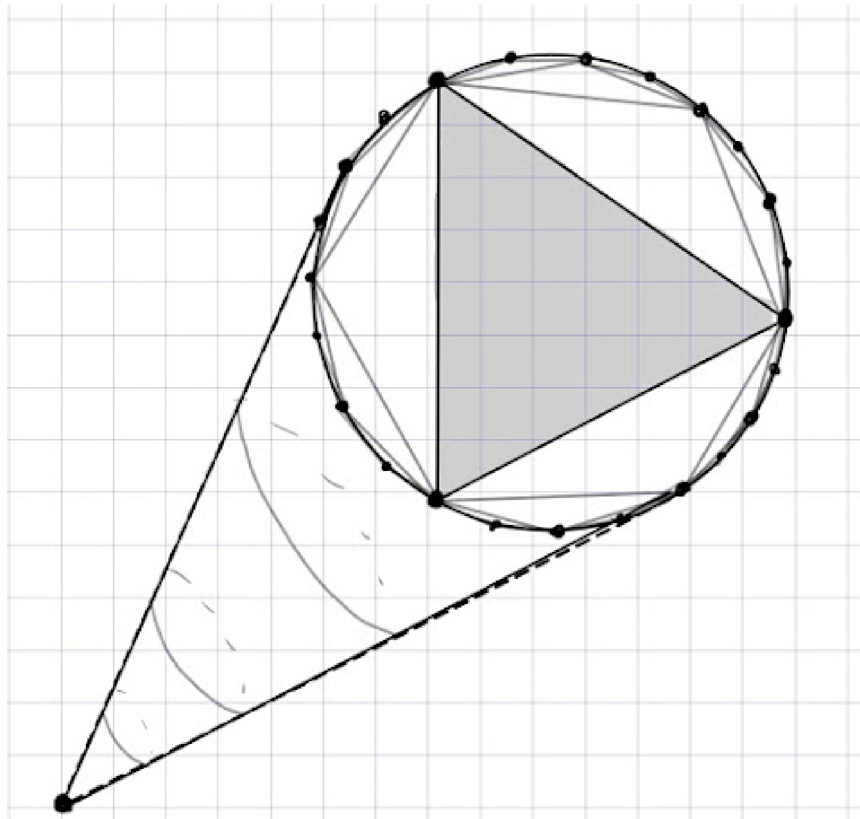


FIGURE 7. Coning off the hyperbolic disc to form a fan.

Note that the entire fan $F$ admits an $\mathrm{SL}_2(\mathbb{Z})$ action. Now if one rewrites $B$ as a form $B(x,y) = ax^2 + by^2 + c(x+y)^2$ with $a, b, c \in \mathbb{Z}_{\geq 0}$, the coordinate vector $\vec{v} := (a, b, c)$ defines a point in some chamber of $\overline{C}^{\mathrm{rc}}$. In turn, $\vec{v}$ defines a 1-parameter degeneration of abelian varieties, and thus a pair $(X_{\mathrm{trop}}, \Theta_{\mathrm{trop}})$. When $B$ is integral it defines an embedding $B : \Lambda \hookrightarrow \Lambda^{\vee}$ and thus one can construct a torus $T := \Lambda^{\vee}_{\mathbb{R}}/\Lambda \cong \mathbb{R}^2/\mathbb{Z}^2$ which has finitely many integral points defined by $\Lambda$.

What is $\Lambda$?

Recall that for any lattice $L$ there is an associated Voronoi tessellation by polytopes $P_i$, one such $P_i$ centered around each lattice point $\ell_i$. Let $\mathrm{Vor}_B$ be the Voronoi tessellation of $\Lambda$; this can be identified with a hexagonal honeycomb tessellation of $\mathbb{R}^2$:

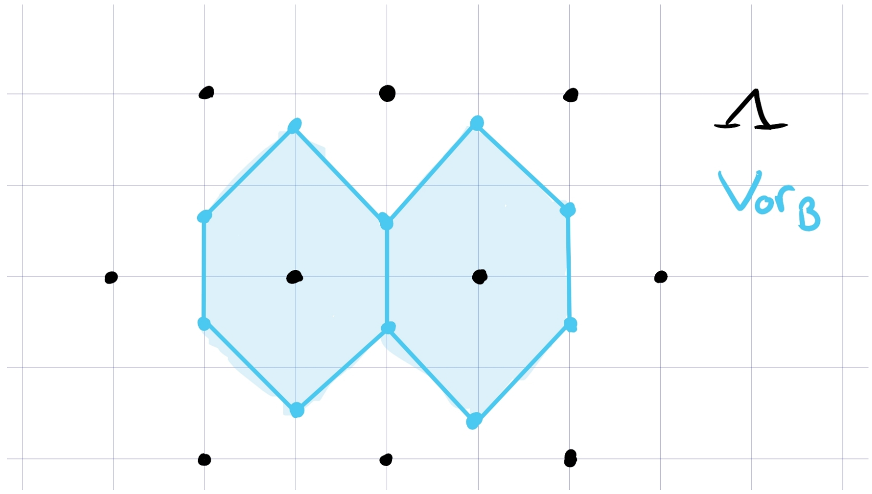



FIGURE 8. The Voronoi tesselation associated to $\widetilde{A}_2$, the triangular lattice.

One then defines $\Theta_{\text{trop}} \coloneqq B(\text{Vor}_B)/\Lambda$, a quotient of the image of the hexagonal tessellation. Although the blue vertices in $\text{Vor}_B$ generally have vertices with fractional coordinates, the vertices in the image have integral coordinate vertices with respect to $\Lambda^\vee$. The image of a regular hexagon is now a hexagon with side lengths $a, b, c$, and since we've quotiented by $\Lambda$, $\Theta_{\text{trop}}$ is determined by two hexagons with side lengths determined by $\vec{v} = (a, b, c)$ which are glued together:

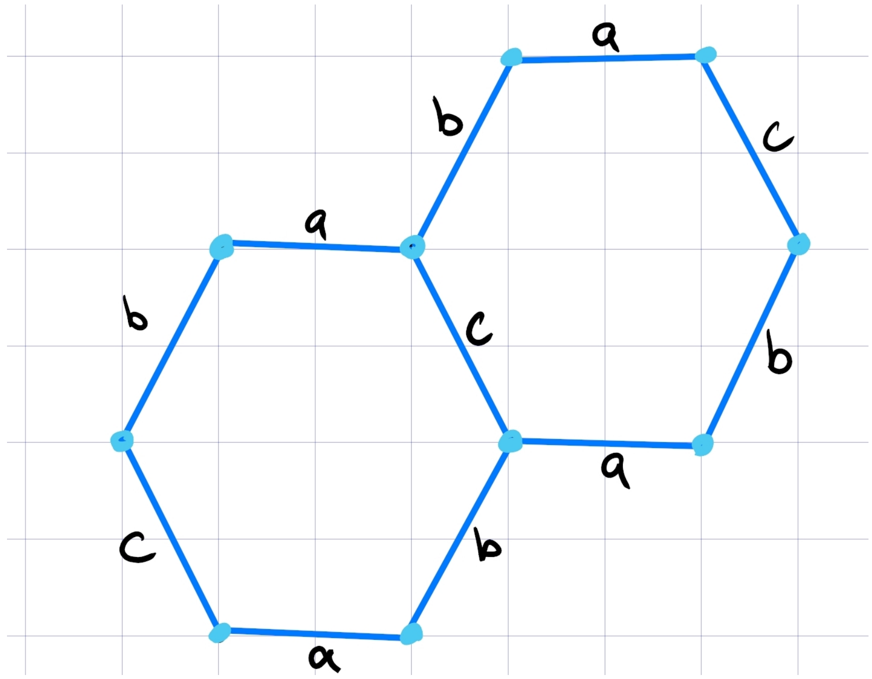



FIGURE 9. A picture of two relevant polytopes in the image of the Voronoi tessellation, which tessellates the entire dual lattice. After quotienting, these will be the only two relevant polygons.

The claim is that this picture describes an entire degeneration $\mathcal{X}$ of abelian varieties. To see the central fiber: every vertex $w_i$ in this new tessellation defines an honest fan via $\mathrm{Star}(w_i)$; here there are 2 vertices of valence 3 and 3 edges in the quotient, so the central fiber $\mathcal{X}_0$ is two copies of $\mathbb{P}^2$ corresponding to $w_1, w_2$ glued together along three curves corresponding to $a, b, c$. To see the entire family: put this entire picture at height 1, cone to the origin to get a fan, and quotient that fan by a $\mathbb{Z}^2$ action to get $\mathcal{X}$.

Note that in the K3 case, things are harder because the combinatorics only describes $\mathcal{X}_0$ and not the entire family $\mathcal{X}$, so one has to appeal to abstract smoothing results to obtain the existence of a family $\mathcal{X}$ extending $\mathcal{X}_0$.

Moreover, the original fan $F$ is a moduli of these polyhedral pictures. One can degenerate $\Theta_{\mathrm{trop}}$ by sending some coordinates $a, b, c$ to zero. This degenerates the honeycomb 6-gons into 4-gons if just one side goes to zero. For example, if $a \to 0$, this corresponds to being on a wall in Figure 7. If two coordinates degenerate, say $a, b \to 0$, this corresponds to being on a ray. This can be read off by recalling $B(x,y) = ax^2 + by^2 + c(x+y)^2$ and labeling the ideal vertices with monomials $x^2, y^2, (x+y)^2$ as in Figure 6. Thus varying $\vec{v} = (a, b, c)$ corresponds to varying the side lengths of hexagons and correspondingly moving through $\overline{C}^{\mathrm{rc}}$. Staying in the fundamental chamber doesn't change the overall combinatorial type of Figure 9, but passing through a wall will flip the hexagonal tiling in various ways.

9B. **To the K3 case.** The claim is that a similar story more or less goes through for K3s: the Coxeter diagram is much more complicated, and the relevant combinatorial device is an $\mathrm{IAS}^2$ with 24 singularities instead of a tropical variety. We have the following:

**Theorem 9.4.** *There is a morphism*

$$\eta : \overline{F_2}^{\mathrm{tor}(F)} \to (\overline{F_2}^{\mathrm{KSBA}})^{\nu}$$

*where $F$ is fan of a Coxeter diagram associated to a cusp of $F_2$, and the Stein factorization of $\eta$ is through a semitoroidal compactification.*

**Corollary 9.5.** *Any punctured 1-parameter family $\mathcal{X}^\circ \to \Delta^\circ$ has a unique limit $\mathcal{X}_0$ which can be combinatorially described as a singular integral-affine sphere with an integral-affine divisor* $(\mathrm{IAS}^2, R_{\mathrm{IAS}})$.

**Remark 9.6.** This is a much harder theorem than the $\mathcal{A}_g$ case: periods of K3 surfaces are highly transcendental, and the period map is not well-understood. Also note that the relevant Coxeter diagram for $\mathcal{A}_g$ was relatively simple, while the diagram for $F_2$ is the following:

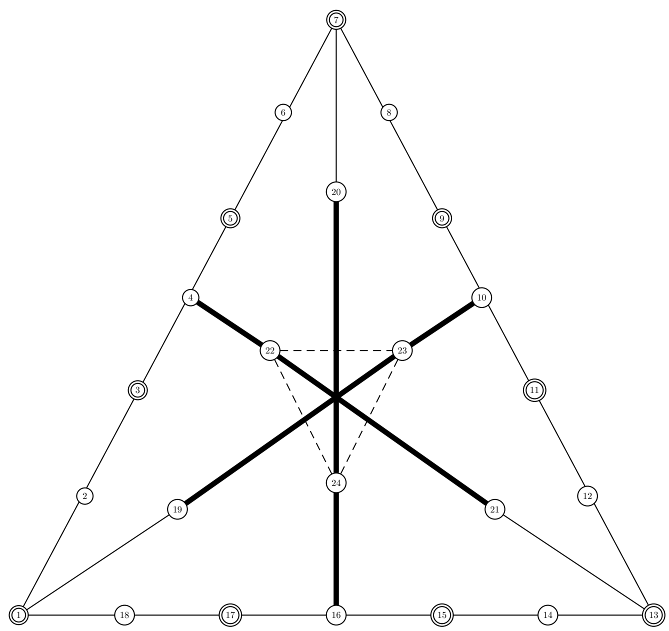



FIGURE 10. The Coxeter diagram for type $(19, 1, 1)$.

Nodes in Figure 10 correspond to roots spanning a hyperbolic lattice

$$N := U \oplus E_8(-1)^{\oplus^2} \oplus A_1(-1), \qquad \mathrm{sgn}(N) = (1, 18)$$

which is the Picard lattice of the Dolgachev-Nikulin mirror K3. Decorated nodes $v_i$ record self-intersection numbers $v_i^2$, and edges between $v_i$ and $v_j$ record the intersection numbers $v_i.v_j$. Note that the Coxeter diagram also captures the data of all $(-2)$ curves on the mirror K3 surface and their intersections.

This diagram again describes the fundamental chamber of a reflection group, and the cone in this case $C = \{v^2 > 0\}$. Toroidal compactifications of $F_2$ correspond to fans whose support is $\overline{C}^{\mathrm{rc}}$ (i.e. the interior, plus rational rays on the boundary). There is a natural fundamental chamber defined by $\{v \mid v.r_i \geq 0\}$ where $\{r_i\}$ are roots, the difference is that now some vertices of the fundamental chamber may be ideal vertices:

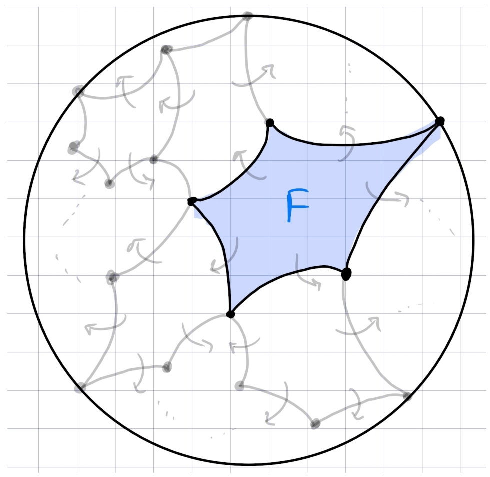



FIGURE 11. A fundamental chamber $F$ for a reflection group. Reflecting over walls of $F$ successively generates a tiling of the hyperbolic disc by copies of $F$. Note that one vertex is an ideal vertex, i.e. it is in $\partial\overline{\mathbb{H}^n}$.

Proceeding similarly to take the cone over this picture and allow rational boundary points yields the cone $\overline{C}^{\mathrm{rc}}$ and a corresponding infinite-type fan $\mathcal{F}$ – this is a fan since the faces are rationally generated, $F$ is a fundamental chamber for the reflection group $W(N)$, the fan is $W-$invariant by construction and moreover invariant under $\mathrm{O}(N)$. Since there is a short exact sequence

$$0 \to W(N) \to \mathrm{O}(N) \to S_3 \to 0$$

the index of $W(N)$ is finite and thus $F$ is finite volume.

Points in this fan can naturally be interpreted as period points, so a choice of a point in the fan yields a degenerating family of K3 surfaces by the Torelli theorem. Let $v \in F$ be a point in the fundamental chamber, we will next consider how this corresponds to a combinatorial object, the same way $\vec{v} = (a, b, c)$ did in the case of $\mathcal{A}_g$.

First consider a fan with 18 rays, corresponding to a toric surface $\Sigma$ with 18 curves. Note that the rays alternative between long and short vectors:

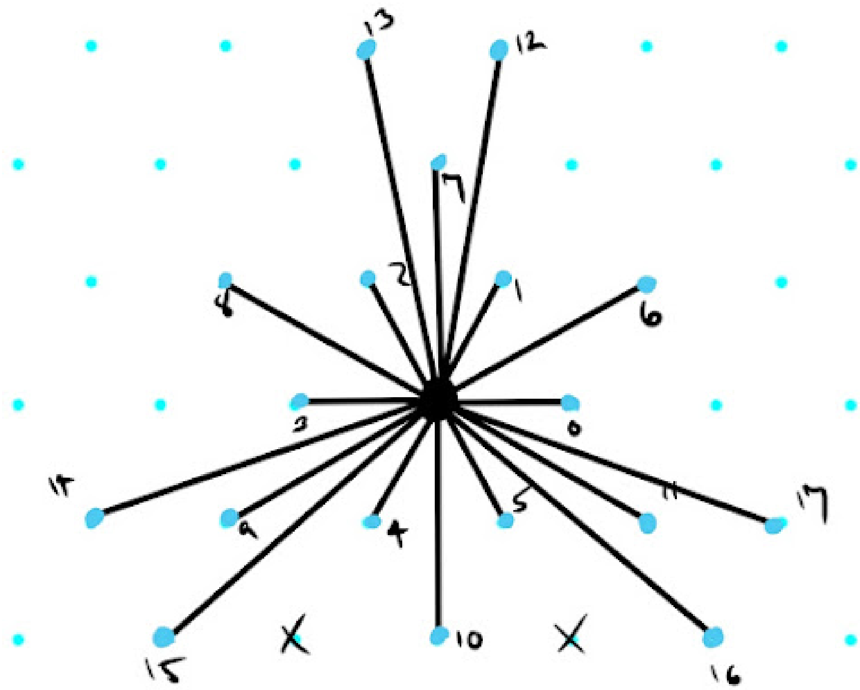



Figure 12. The starting point: a toric surface with 18 rays.

This corresponds to a polytope $P_\Sigma$ which is an 18-gon (not necessarily regular) which is the moment polytope for $X_\Sigma$ where $\Sigma$ is the fan in Figure 12 and has edge lengths $\ell_0, \cdots, \ell_{17} \in \mathbb{R}$, which determines a polarization $L$ for $X_\Sigma$. Although not shown in the picture here, we can call each edge "long" if it was dual to a long vector, and similarly "short" if dual to a short vector.

Note also that each edge can be written as $\ell_i v_i$ for $v_i$ some unit vectors, and it is a nontrivial condition on $\vec{\ell}$ that this polygon closes. In particular, one needs $\sum_{i=0}^{17} \ell_i v_i = 0$.

Now cut triangles out of sides $0, 6, 12$ and call the resulting polygon non-convex polygon $P$. Each triangle cut corresponds to a non-toric blowup of $X_\Sigma$, i.e. a blowup at a point $p$ which is not $T-$invariant. This introduces three new length parameters $\ell_{18}, \ell_{20}, \ell_{20}$ corresponding to the heights of these three triangles. Each will introduce an $\mathrm{I}_1$ singularity to the moment polytope.

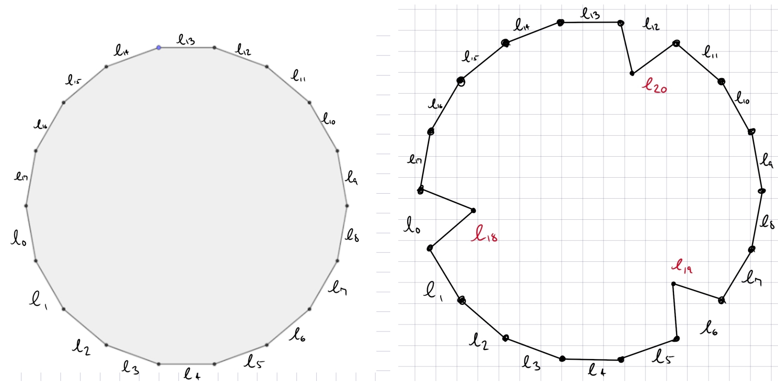


FIGURE 13. The Symington polytope: an 18-gon, before and after a nontoric blowup corresponding to cutting out triangles.

Regarding such polytopes as the Symington polytopes, which are bases of Lagrangian torus fibrations, these are in particular elliptic fibrations and these singularities precisely correspond to introducing singular type $I_1$ fibers in Kodaira's classification.

Take two copies of $P$, say $P$ and $P^{\mathrm{op}}$, and glue them together along the outer edges and call the result $B$. This is topolologically the gluing of two discs, and thus $B$ is homeomorphic to $S^2$. Each gluing along the outer edges introduces a new $I_1$ singularity, yielding $3+3=6$ singularities in the hemispheres and 18 singularities along the equator for a total of 24 singularities of type $I_1$ and thus an $\mathrm{IAS}^2$ with charge 24.

Note that there are now 24 length parameters: $\ell_0, \cdots, \ell_{17}$ along the equator, $\ell_{18}, \ell_{19}, \ell_{20}$ in the northern hemisphere, and $\ell_{21}, \ell_{22}, \ell_{23}$ in the southern hemisphere. The tuple $\vec{\ell} = (\ell_1, \cdots, \ell_{23})$ turns out to correspond to 24 vectors in a 19 dimensional space, and there are enough conditions to ensure the polygons actually close.

This produces the tropical sphere $\mathrm{IAS}^2$, so one also needs to describe its tropical divisor $R_{\mathrm{IAS}}$. The above construction works for any K3 with a nonsymplectic involution, e.g. an elliptic K3, and the $\mathrm{IAS}^2$ is naturally equipped with an involution $\iota$ that swaps $P$ and $P^{\mathrm{op}}$. The ramification divisor of $\iota$ is the equator, highlighted in blue in the following cartoon picture of $B$, and one takes $R_{\mathrm{IAS}}$ to be the sum of the blue edges with coefficient 2 for even (short?) sides and coefficient 1 for odd (long?) sides:

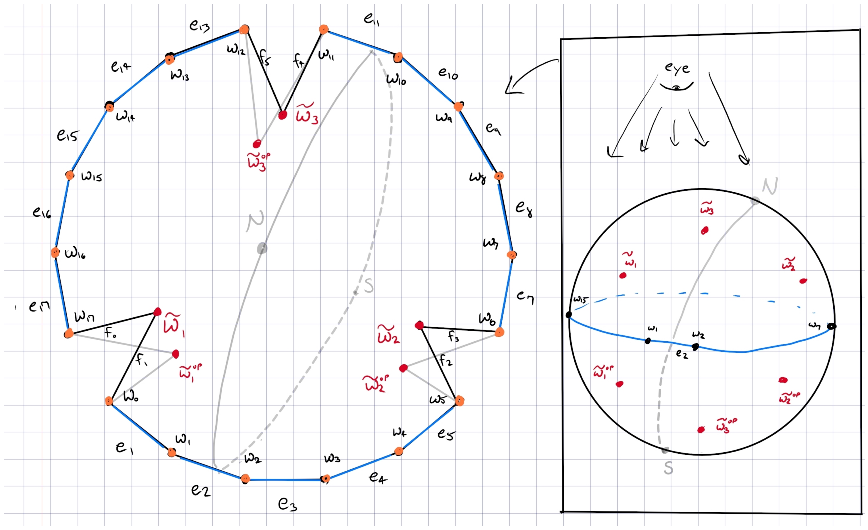



FIGURE 14. Caption

We now describe how one obtains a degeneration $\mathcal{X}$ of K3 surfaces from this combinatorial picture. One must first extend this $\mathrm{IAS}^2$ to a complete triangulation by basis triangles. This triangulation should be done on $P$ first, before the doubling construction, so that the vertices and edges in the northern hemisphere are perfectly matched with those in the southern. Here is a cartoon of what this might look like on one copy of $P$, before gluing:

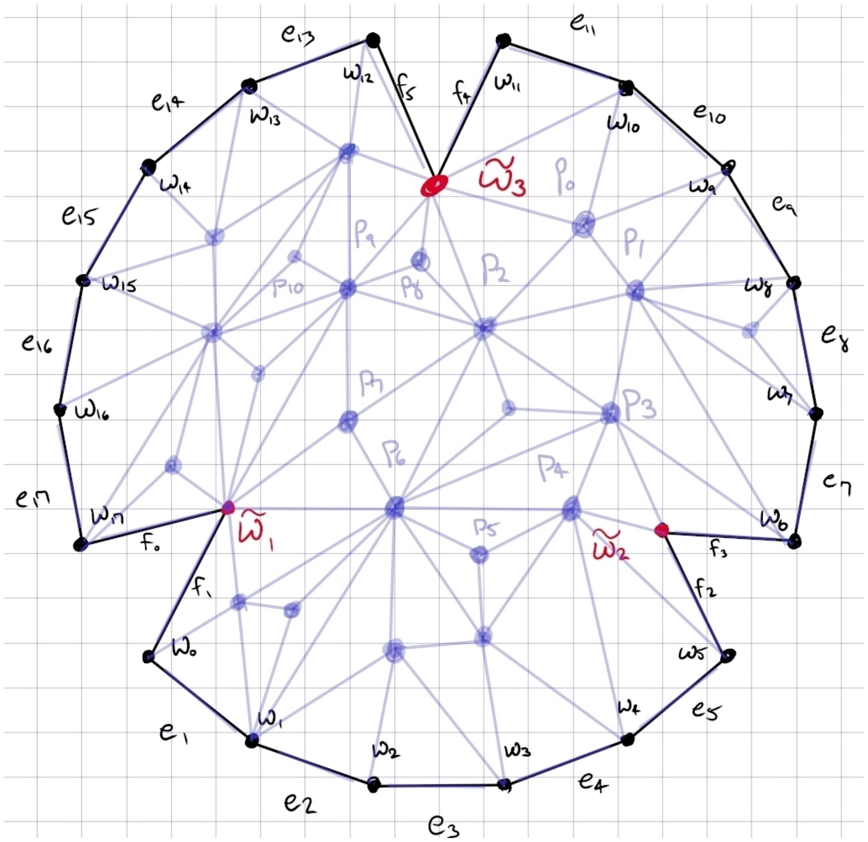



FIGURE 15. The IAS on $P$ extended to a complete triangulation by basis triangles.

This is again a cartoon picture, meant to show how vertices and triangles in the hemispheres should match in pairs exchanged by the involution $\iota$. Here e.g. the blue triangles are meant to match, as well as $\mathrm{Star}(\widetilde{w}_1)$ and $\mathrm{Star}(\widetilde{w}_1^{\mathrm{op}})$:

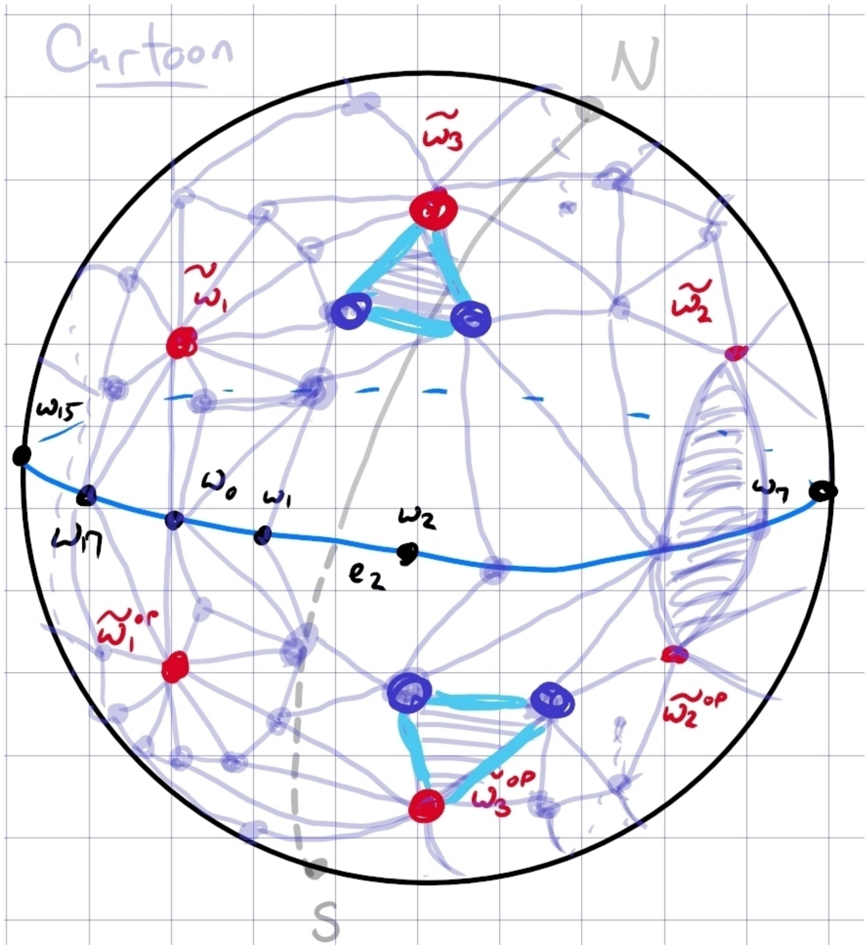



FIGURE 16. A completely triangulated $\mathrm{IAS}^2$ defined by $B := P \cup P^{\mathrm{op}}$.

This final picture describes the central fiber $\mathcal{X}_0$ of a Kulikov degeneration of K3 surfaces in the following way: there are many non-singular vertices $p_i$, and exactly 24 singular vertices $w_i$ and $\widetilde{w}_i, \widetilde{w}_i^{\mathrm{op}}$. For the $p_i$, there is a fan defined by $\mathrm{Star}(p_i)$ which defines a toric surface $V_i$. For the 24 singular vertices, there is a modified recipe to cook up a semi-toric surface – since the singularity is type $\mathrm{I}_1$, this will be a charge 1 surface, and thus realizable as a toric surface with a single non-toric blowup, a *semitoric* surface. How to make this blowup is uniquely determined by an additional omitted decoration called the *monodromy ray* at the singular vertex. Roughly, this is a preferred ray cooked up from the Picard-Lefschetz monodromy operator around the singular vertex. One can think of this as a "singular fan".

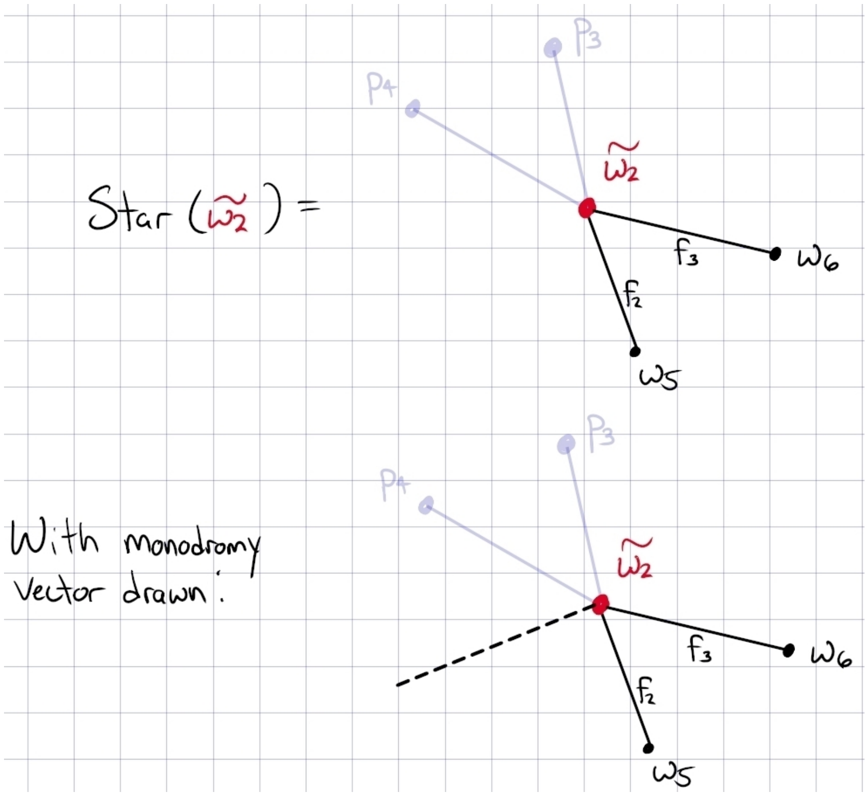



FIGURE 17. Star($\widetilde{w}_2$) in Figure 15 with the extra data of a monodromy vector.

So

$$\mathcal{X}_0 = \bigcup_i V_i \cup \bigcup_{j=1}^{24} W_j$$

where the $V_i$ are all toric surfaces and the $W_j$ are all semitoric surfaces of charge 1, and the triangulation determines how they are all glued together. To see how this gluing is done, consider the following local picture in the triangulation:

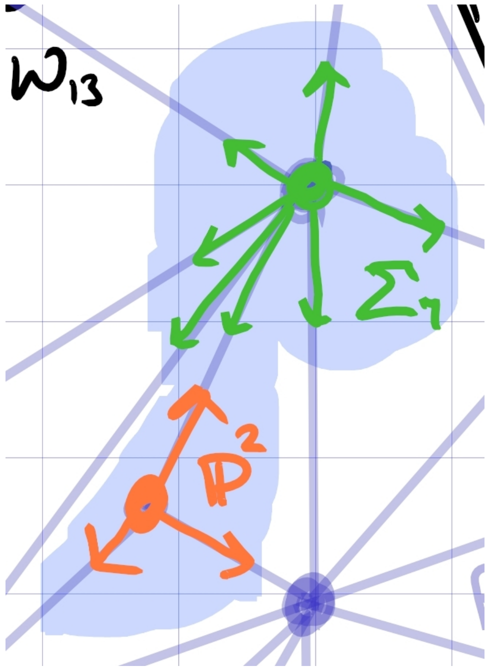



FIGURE 18. Local gluing in the $\text{IAS}^2$ of two toric surfaces $\Sigma_7$ and $\mathbb{P}^2$

At the orange vertex, taking the star we see three rays and thus a copy of $\mathbb{P}^2$. At the green vertex, we see 7 rays, and thus some toric surface $\Sigma_7$ which is probably something like a Hirzebruch surface $\mathbb{F}_n$ with 3 toric blowups. Since the orange and green vertices are adjacent by exactly one edge, this means we glue $\mathbb{P}^2$ to $\Sigma_7$ along the the curves determined by rays pointing along that edge. Moreover, whenever there is a triangle, this corresponds to three surfaces glued together along a triple point.

The general case is that the $\text{IAS}^2$ has 24 copies of $\text{I}_1$ singularities; these singularities can collide to produce semitoric surfaces with multiple nontoric blowups.

**Remark 9.7.** Note that this *only* describes $\mathcal{X}_0$ and not an entire family $\mathcal{X}$. Friedman solved this problem: there is a technical condition called $d-$**semistability**, and if this is satisfied then $\mathcal{X}_0$ is smoothable. Moreover the smoothing will have the correct period and/or monodromy vector $\lambda$.

Haven't discussed $\lambda$ here yet!

To obtain all degenerations, one considers all of the ways this combinatorial object can degenerate. Sending some $\ell_i \to 0$ causes the 18-gon to collapse into a small polygon, or causes some hemispherical singularities to descend into the equator. This corresponds to moving an interior point of original fundamental chamber $F$ onto a wall, and wall-crossing mutates the $\text{IAS}^2$ in some other ways.

**Remark 9.8.** Some miscellaneous remarks:

- The Kulikov models are highly non-unique, differing by flops. Adding the divisor $R_{\text{IAS}}$ fixes this and pins down $\mathcal{X}$ uniquely.
- It seems one can read off the stable model from the $\text{IAS}^2$. For the honeycombs in the $\mathcal{A}_g$ case, everything was contracted down to two $\mathbb{P}^2$s glued

along their 3 boundary curves in a $\Theta-$graph. In the $F_2$ case, one contracts everything in the IAS$^2$ except for the equator, i.e. the interiors of the hemispheres are contracted. The most general degeneration is 18 copies of $\mathbb{P}^2$ glued in a cycle; one can then send some $\ell_i \to 0$ to collide the vertices and get fewer than 18 surfaces.

- It seems one can also read off Type II degenerations from the IAS$^2$. Here there are 4 Type II cusps, 3 correspond to collapsing the 18-gon in the IAS$^2$ in the equatorial plane to an interval. The 4th involves collapsing the 18-gon to a point with bits sticking out. Type I degenerations correspond to collapsing everything to a point.
- Why everything works simply here: there is only one relevant cusp in $F_2$, and the involution propagates to everything including the IAS$^2$. The Coxeter diagram is also highly symmetric, hinting at how to make the right toric and IAS construction.

9C. **Notes from Phil's talk.**

**Remark 9.9.** For $\Sigma_g$ a compact complex curve of genus $g$, choose a symplectic basis $\{\alpha_i, \beta_i\}_{i\leq g}$ of $H_1(\Sigma_G;\mathbb{Z})$, then there is a unique basis $(\omega_1, \cdots, \omega_g)$ of $H^0(\Omega_{\Sigma_g})$ such that $\int_{\alpha_i}\omega_j = \delta_{ij}$. In this basis, form the **period matrix** $\tau = (\int_{\beta_i}\omega_j)_{i,j=1}^g$. This satisfies $\tau^t = \tau$ and $\Im(\tau) > 0$ is positive-definite, and is thus an element in The **Siegel upper half-space**

$$\mathcal{H}_g := \left\{\tau \in \mathrm{Sym}_{g\times g}(\mathbb{C}) \mid \Im(\tau) > 0\right\}.$$

The **Jacobian** of $\Sigma$ is defined as $\mathrm{Jac}(\Sigma) := \mathbb{C}^g/(\mathbb{Z}^g \oplus \tau\mathbb{Z}^g)$. Note that we made a choice of "marking" by choosing the symplectic basis $\{\alpha_i, \beta_i\}$, and any two such choices are related by $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$, the isometry group of $\mathbb{Z}^{2g}$ with the standard symplectic form, where the action is $\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} I_g \\ \tau \end{bmatrix} = \begin{bmatrix} A+B\tau \\ C+D\tau \end{bmatrix}$, the analogue of a linear fractional transformation. To renormalize the 1-forms, we change basis to get a similar matrix $\begin{bmatrix} I_g \\ (A+B\tau)^{-1}(C+D\tau) \end{bmatrix}$. Thus to get an invariant, we consider

$$[\tau] \in \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g := \mathcal{A}_g,$$

the moduli space of PPAVs. Here one can realize the polarization on $A$ as a symplectic form on $H_1(A;\mathbb{Z})$ which is represented by a holomorphic line bundle $L \in \mathrm{Pic}(A)$, i.e. identifying a symplectic form on $H_1(A;\mathbb{Z})$ as an element of $H^2(A;\mathbb{Z})$ which we want to be a $(1,1)$ form.

**Remark 9.10.** Now $\mathcal{A}_g$ is not compact, so we consider degenerations over $\mathcal{X}^\circ \to \Delta^\circ$ and let $\Delta^\circ \to \mathcal{A}_g$ be the associated period mapping – how does the period map degenerate as $t \to 0$? The answer is that a certain isotropic subspace $I \leq (\mathbb{Z}^{2g}, \omega_{\mathrm{std}})$ becomes distinguished by the fact that periods against $I^\perp$ remain finite.

**Example 9.11.** Let $y^2 = x^3 + x^2 + t$ be a family of elliptic curves over $\mathring{\mathbb{A}}^1 \setminus 0$. At $t = 0$ this degenerates to a nodal cubic. There is a vanishing cycle $\alpha$, and the distinguished isotropic subspace is precisely $I := \mathbb{Z}\alpha$. One shows $I^\perp = \mathbb{Z}\alpha$ as well. In this case, to be in $I^\perp$ means to be a curve that does not pass through the thinning neck of the torus that degenerates; any curve that does pass through should intuitively have a period that blows up. We normalize by picking a $c_t$ such that $\int_\alpha c_t \frac{dx}{y} = 1$, then $\int_\beta \omega_t = \tau_t \in \mathbb{C}$. However, this isn't well-defined: one can parallel-transport $\beta$ around $t = 0$ and the monodromy action will be a Dehn twist, so integrals against $\beta$ are only well-defined up to $\mathbb{Z}p$ where $p$ are periods against $\alpha$, here $p$ is normalized to 1. So $\int_\beta \omega_t \in \tau_h + \mathbb{Z} \in \mathbb{C}/\mathbb{Z}$. As $t \to 0$, one was $\tau_t \to +i\infty$ if $\alpha, \beta$ are oriented properly. We can fix this ambiguity by exponentiation, getting a well-defined invariant $\exp(2\pi i \int_\beta \omega_t) \in \mathbb{C}^*$.

**Remark 9.12.** How this works for $g \geq 1$: assume $I$ is Lagrangian, so $I^\perp = I$, corresponding to a maximally unipotent degeneration. If this were a genus $g$ curve, we could pinch $\leq g$ disjoint cycles simultaneously, and a maximal degeneration will pinch exactly $g$. Since $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts transitively on Lagrangian subspaces in $\mathrm{LGr}(V)$, we can assume $I = \bigoplus_i \mathbb{Z}\alpha_i$ is generated by the $\alpha$ curves. Generalizing the $\mathbb{C}^*$ embedding in the previous case, we obtain a torus embedding

$$\begin{aligned} E : \mathcal{H}_g &\hookrightarrow (\mathbb{C}^*)^{\binom{g}{2}} \\ \tau &\mapsto \begin{bmatrix} \exp(2\pi i \tau_{11}) & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \exp(2\pi i \tau_{gg}) \end{bmatrix} \end{aligned}$$

Since $\tau$ is symmetric, the image $E(\tau)$ is again symmetric. Note that the $\beta_i$ cycles are well-defined up to translation in $I$, but because the 1-form was normalized so that integrals of $\alpha_j$ along $\beta_i$ were 1 or 0, the entries in this matrix are well-defined up to integers. Thus we can exponentiate every entry in the period matrix to get a well-defined symmetric matrix. The unipotent orbit theorem of Schmid gives an asymptotic estimate

$$E(\tau_t) \sim_{t \to 0} \begin{bmatrix} c_{11} t^{n_{11}} & c_{12} t^{n_{12}} & \cdots & \\ \vdots & \ddots & & \vdots \\ & \cdots & & c_{gg} t^{n_{gg}} \end{bmatrix} \in \mathrm{Mat}_{n \times n}(\mathbb{C}^*)$$

which is a cocharacter of $(\mathbb{C}^*)^{\binom{g}{2}}$, i.e. an inclusion $\mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^{\binom{g}{2}}$ which is a composition of a group morphism and a translation. Here the $c_{ij}$ are the translation parts, and if $c_{ij} = 1$ for all $i, j$ this yields an honest group morphism. Such a cocharacter is called a unipotent orbit. This asymptotic estimate is quantified, so there is a precise speed at which the period matrix approaches the cocharacter.

Setting $N \coloneqq (n_{ij})$, we have $N \in \mathrm{Sym}_{g \times g}(\mathbb{Z})$ and $N > 0$. These entries capture the relative speeds at which the various cycles are collapsing. Since the $c_{ij}$ are ultimately just translations, we'll omit them from here onward.

**Remark 9.13.** Define a cone

$$P_g \coloneqq \left\{ N \in \mathrm{Sym}_{g \times g}(\mathbb{Z}) \mid N > 0 \right\} \subseteq \mathbb{Z}^{\binom{g}{2}}$$

and consider the family

$$\frac{(\mathbb{C}^*)^g}{\langle (t^{n_{11}}, t^{n_{12}}, \cdots, t^{n_{1g}}), (t^{n_{21}}, t^{n_{22}}, \cdots, t^{n_{2g}}), \cdots, (t^{n_{g1}}, t^{n_{g2}}, \cdots, t^{n_{gg}}) \rangle}$$

over $\Delta^\circ$. How does one extend this family over $t = 0$? If $N$ has full rank, this entire expression is isomorphic to $(\mathbb{C}^g/\mathbb{Z}^g)/\tau\mathbb{Z}^g$. There are two answers, one by fans and one by polytopes.

**Remark 9.14.** The following is the fan construction due to Mumford, which most easily generalizes to K3 surfaces.

Consider the example

$$N = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \rightsquigarrow \frac{(\mathbb{C}^*)^2}{\langle (t^2, t), (t, t^3) \rangle}$$

Note $N > 0$, since $\det N > 0, \mathrm{Trace} N > 0$, and $N(\mathbb{Z}^g)$ is generated by the vector $(2,1)$ and $(1,3)$. First quotient $\mathbb{R}^2$ by this lattice to get a flat real 2-torus, then take a polyhedral tiling whose vertices are integer points. Here we take a tiling of the fundamental domain and translate it everywhere. This gives a tiling $\mathcal{F}_0$ on the universal cover $\mathbb{R}^g$. Now put this picture at height 1 in $\mathbb{R}^{g+1}$ to get a tiling $\widetilde{\mathcal{F}}_0$ of $\mathbb{R}^g \times \{1\} \subseteq \mathbb{R}^{g+1}$, and let $\widetilde{\mathcal{F}} \coloneqq \mathrm{Cone}(\widetilde{\mathcal{F}}_0) \subset \mathbb{R}^{g+1}$ be its cone. Taking the toric variety $X(\widetilde{\mathcal{F}})$, and define $X(\mathcal{F}) \coloneqq X(\widetilde{\mathcal{F}})/N(\mathbb{Z}^g)$, where the quotient makes sense precisely because $N(\mathbb{Z}^g)$ acts on $\mathbb{R}^g \times \{1\}$ by translation, and this extends to a linear action on $\mathbb{R}^g$, which moreover preserves $\widetilde{\mathcal{F}}$ and thus acts on the toric variety. There is a morphism $\phi : X(\mathcal{F}) \to \mathring{A}^1$ induced by the morphism of fans given by the height function: projection in $\mathbb{R}^{g+1}$ onto the last coordinate, whose image is in $\mathbb{R}_{\geq 0}$. This map descends to the quotient since the linear action preserves the height function.

This produces a degenerating fan of abelian varieties. A fiber $\phi^{-1}(t)$ of $X(\widetilde{F})$ for $t \neq 0$ yields $(\mathbb{C}^*)^g$, and the action $N(\mathbb{Z}^g)$ acts by translations of the form $(t^{n_{i1}}, t^{n_{i2}}, \cdots, t^{n_{ig}})$ in the original family. Thus we recover the original family as an infinite quotient of a toric variety. But the toric variety has a toric boundary, encoded in the tiling. The fiber $\phi^{-1}(0)$ has dual complex $\Gamma(X_0(\mathcal{F})) = \mathcal{F}_0$ equal to the original tiling, and $X_0(\mathcal{F})$ is a union of toric varieties.

In the original lattice, in the quotient there are precisely 3 0-cells, and we interpret the star of each 0-cell as the fan of a toric surface. They are glued according to the tiling.

**Remark 9.15.** The polytope construction, which builds the projective coordinate ring instead. One defines $Q$ to be the hull of certain points, constructs a theta function, and takes Proj of a certain graded algebra generated by such functions with an explicit multiplication rule and structure constants. These define a certain PL function with a "bending locus" which gives a polyhedral decomposition of $\mathbb{R}^g/\mathbb{Z}^g$. For any $N \in P_g$ one can define the Delaunay decomposition $\mathrm{Del}(N)$, and the central fiber $\mathcal{X}_0$ of the family will have **intersection** complex $\mathrm{Del}(N)$ – the loci where the PL function is linear will be polytopes which are the cells of the Delaunay decomposition. The second Voronoi fan $F^{\mathrm{Vor}}$ is a decomposition of $P_g$ into loci where $\mathrm{Del}(N)$ is constant. One then takes $\mathrm{Sym}_{g\times g}(\mathbb{Z})\backslash \mathcal{H}_g \hookrightarrow (\mathbb{C}^*)^{\binom{g}{2}} \to X(F^{\mathrm{Vor}})$. One the quotients by conjugation in $\mathrm{GL}_g(\mathbb{Z})$ to get $X(F^{\mathrm{Vor}})/\operatorname{GL}_g(\mathbb{Z}) \hookrightarrow \overline{\mathcal{A}_g}^{\mathrm{Vor}}$. Correspondingly, for any $\mathcal{X}^\circ$ in $\mathcal{A}_g$, tracing through this construction gives a proper family $\mathcal{X}$ in $\overline{\mathcal{A}_g}^{\mathrm{Vor}}$ – note that we've only described what toric compactification to take for the maximally unipotent degenerations, but one can carry out similar constructions for the other cusps of $\overline{\mathcal{A}_g}^{\mathrm{bb}}$.

**Remark 9.16.** One should ask if $\overline{\mathcal{A}_g}^{\mathrm{Vor}}$ actually solves a moduli problem, and the answer is yes (up to normalization) by a theorem of Alexeev. The moduli problem is the moduli of semi-abelic pairs. Define $\overline{\mathcal{A}_g}^{\Theta}$ to be the closure of pairs $(X, \varepsilon R)$

where $R$ is their theta divisors, then Alexeev shows

$$\overline{\mathcal{A}_g}^{\mathrm{Vor}} = (\overline{\mathcal{A}_g}^{\Theta})^{\nu}$$

**Remark 9.17.** How do we do something similar for K3 surfaces? Fix $v \in \mathrm{II}_{3,19}$ primitive with $v^2 = 2d$ and define

$$\begin{aligned}
\Omega_L &\coloneqq \left\{ \mathbb{C}x \in \mathrm{Gr}_1(\mathrm{II}_{3,19_{\mathbb{C}}}) \mid (x,x) = 0, (x,\bar{x}) > 0 \right\} \\
\Omega_{2d} &\coloneqq v^{\perp_{\Omega_{\mathrm{II}_{3,19}}}} = \left\{ x \in \mathrm{II}_{3,19_{\mathbb{C}}} \mid (x,v) = 0 \right\} \\
\Gamma_{2d} &\coloneqq \mathrm{Stab}_{\mathrm{O}(\mathrm{II}_{3,19})}(v) = \{\gamma \in \mathrm{O}(\mathrm{II}_{3,19}) \mid \gamma(v) = v\} \\
F_{2d} &\coloneqq \Gamma_{2d} \backslash \Omega_{2d}
\end{aligned}$$

Here $\Omega_{2d}$ plays the role of $\mathcal{H}^g$ in the abelian variety case, and is a Hermitian symmetric domain of type IV or $\mathrm{SO}_{2,n}$, and $F_{2d}$ is an arithmetic quotient. Fixing a marking $\phi : H^2(X;\mathbb{Z}) \to \mathrm{II}_{3,19}$, the period map for a family $\mathcal{X}^\circ \to \Delta^\circ$ is given by taking $H^{2,0}(X) = \mathbb{C}\omega$ and looking at $[\omega] \coloneqq \phi(\omega) \in F_{2d}$, since $[\omega] \in \Omega_{2d}$ but is ambiguous up to change of marking (elements of $\Gamma$). This is a map $\Delta^\circ \to \Gamma_{2d}\backslash\Omega_{2d}$.

Given a degenerating family, there is a distinguished isotropic lattice $I \leq v^\perp$ where $\operatorname{sgn} v^\perp = (2,19)$. Note $I$ can only have rank 1 or 2. The rank 1 case (Type III degenerations) is a maximally unipotent degeneration; the central fiber is as singular as possible, and $\mathcal{X}_0$ will always have 0-strata. In contrast, in the rank 2 case (Type II degenerations) there are models of the degeneration with no 0-strata.

In the rank 1/Type III case, there is a vanishing cycle $\delta$ associated to a 0-stratum in $\mathcal{X}_0$ which is topologically a 2-torus. It turns out that $\delta$ is an isotropic vector that spans the isotropic lattice, so we can write $I = \mathbb{Z}\delta \subseteq v^\perp$. In the degeneration, the 2-torus collapses to a point.

In the rank 2/Type II case, there are two linearly independent isotropic vectors $\delta$ and $\lambda$ in $v^\perp$ corresponding to 2-tori collapsing simultaneously not to isolated points as in the previous case, but rather to circles in $\mathcal{X}_0$. They are in the singular locus of $\mathcal{X}_0$, which is an elliptic curve.

**Remark 9.18.** We henceforth assume $\operatorname{rank}_{\mathbb{Z}} I = 1$ and write $I = \mathbb{Z}\delta$ for $\delta$ the isotropic vanishing cycle. Normalize $\omega_t$ so that $\int_\delta \omega_t = 1$ for $t \neq 0$. Let $\{\gamma_i\}_{i=1}^{19}$ be a basis of $\delta^\perp/\delta$. Since $\delta \in v^\perp$ was isotropic of signature $(2,19)$, we have $\operatorname{sgn}(\delta^\perp/\delta) = (1,18)$ and this gives us a hyperbolic lattice of rank 19. Consider the integral $\int_{\gamma_i} \omega_t \in \mathbb{C}$. For this to make sense, one needs to lift the $\delta_i$ from $\delta^\perp/\delta$ to $\delta^\perp$, and the choice of lift is ambiguous up to a multiple of $\delta$. By the normalization of the integral, we get a well-defined period

$$\int_{\gamma_i} \omega_t \in \mathbb{C}/\mathbb{Z}$$

As in the PPAV case, we use the exponential to get rid of the quotient by $\mathbb{Z}$. Letting $U_\delta \leq \Gamma_{2d}$ be the unipotent subgroup stabilizing $\delta$, we get the following torus embedding

$$\begin{aligned}
U_\delta \backslash \Omega_{2d} &\xrightarrow{\psi} (\mathbb{C}^*)^{19} \\
&\quad \mathbb{C}[\omega_t] \mapsto \left( \exp\left(2\pi i \int_{\gamma_1} \omega_t\right), \cdots, \exp\left(2\pi i \int_{\gamma_{19}} \omega_t\right) \right)
\end{aligned}$$

and a nilpotent orbit theory yielding an asymptotic estimate

$$\psi_t \sim (c_1 t^{\lambda_1}, \cdots, c_{19} t^{\lambda_{19}})$$

with $c_i \in \mathbb{C}^*$, so the periods are approximated by a cocharacter where the $\lambda_i$ measure how fast the periods degenerate.

**Remark 9.19.** Degenerations: a theorem of KPP shows that after a finite base change and birational modifications, any degeneration of K3s has a model where

- $X$ is smooth
- $X_0$ is RNC
- $K_X = \mathcal{O}_X$

The most famous degeneration of K3s is the Fermat degeneration is a non-example, since smoothness fails:

$$V(x_0x_1x_2x_3 = t(x_0^4 + x_1^4 + x_2^4 + x_3^4))$$

This threefold has precisely 24 conifold singular points. The central fiber at $t = 0$ is a tetrahedron, 4 planes $\mathbb{P}^2$ in $\mathbb{P}^3$, and the singular points come from intersecting each edge of the tetrahedron with the residual quartic. One can get a smooth threefold by taking a small resolution of the singular points. There are choices for the resolutions, differing by flops, so here is a heuristic of a symmetric choice where along each edge there are two resolutions extending into each component:

image

The result has four components $V_i$ which are isomorphic to $\mathrm{Bl}_6\,\mathbb{P}^2$, 2 points on each of 3 lines in $\mathbb{P}^2$.

An observation originally due to GHK: there is an IAS on $\Gamma(X_0)$, i.e. there are charts to $\mathbb{R}^2$ up to post-composition with $\mathrm{SL}_2(\mathbb{Z}) \rtimes \mathbb{R}^2$.

Here is an example of $\mathcal{X}_0 = \cup V_i$ for a Kulikov degeneration (written as a decomposition into irreducible components). Each unlabeled edge has an implicit label of $-1$:

image

This forms a tiling of the sphere. Each tile corresponds to an irreducible component $V_i$ of $\mathcal{X}_0$. The edges correspond to components $V_i, V_j$ glued along an anti-canonical cycle of rational curves $V_{ij}$. The edge numbers record the self-intersection numbers of the cycles $V_{ij}$ regarded as a cycle in $V_i$ and $V_{ji} = V_{ij}$ regarded as a cycle in $j$.

A general fact about degenerations of CYs: $\mathcal{X}_0$ is generally a union of log CY varieties, i.e. there are meromorphic 2-forms on components and they are glued along their poles so that the residues agree. The red lines in the image denotes the pole locus of these forms. Each triple point is where 3 surfaces are glued. Since the overall variety is a SNC surface, there are only double curves and triple points. Note that this picture is the **intersection complex** of $\mathcal{X}_0$, and not the dual complex $\Gamma(\mathcal{X}_0)$. To obtain the dual complex, take the dual tiling, regard each integral 0-cell in the result as a fan, and glue the fans.

Here are the fans:

image

Here is how this interacts with the original diagram:

image

This works fine at most vertices, but at most 24 components are non-toric. Note that from toric geometry, if $(V, D)$ is a toric pair then $-D_i^2 v_i = v_{i-1} + v_{i+1}$ and so one can enforce this formula on such components. For example, the following pair has all $-1$ curves since $v_2 = v_1 + v_3$:

image

Enforcing this formula locally, non-toric points force some $\mathrm{SL}_2(\mathbb{Z})$ monodromy in the IAS.

**Remark 9.20.** This is the analogue of the Mumford fan construction. Note that in the PPAV case, the lattice didn't specify a Kulikov degeneration since it was not a complete triangulation. But completing this to a complete triangulation of the corresponding real 2-torus does yield a Kulikov model. For K3s, instead of a complete triangulation on $T^2$, we're taking a complete triangulation of an $\mathrm{IAS}^2$. Note that unlike the PPAV case, a triangulated $\mathrm{IAS}^2$ only gives $\mathcal{X}_0$ (glued from ACPs) and not the entire family $\mathcal{X}$. An abstract theorem of Friedman says it smooth to a K3, but one does not get an explicit construction of the smoothing $\mathcal{X}_t$.

There is also no polytope construction here whatsoever, only the fan construction for the central fiber. GHK and Siebert have been working on the polytope side. It's hard: it's not clear what the multiplication rule for theta functions should be. We represent an $\mathrm{IAS}^2$ with the following data:

Missing, see video.

This recovers $\mathcal{X}_0$ by taking fans at vertices.

**Remark 9.21.** Joint work with Valery: a polarizing divisor is a divisor $R$ in the generic K3 surface in $F_{2d}(\mathbb{C})$. This corresponds to a choice of ample divisor on a Zariski open subset of $F_{2d}(\mathbb{C})$. For such a choice, we define $\overline{F_{2d}}^R$ to be the closure of K3 pairs $(X, \varepsilon R)$ in the space of KSBA stable pairs. A generic K3 has Picard rank 1, and it's in the ample class, so any divisor on the generic K3 is automatically ample. Thus $K_X + \varepsilon R > 0$ since $K_X = 0$. The pair also has slc singularities. Note that we've allowed all K3s to have ADE singularities, these are examples of slc, and taking $\varepsilon$ small enough resolves any problems. One needs $R$ not to pass through log canonical centers, and there are no log canonical centers on an ADE K3. Their theorem gives an explicit description of such a moduli space.

**Remark 9.22.** We say $R$ is recognizable if it extends to a unique divisor $R_0$ on any Kulikov surface. Idea: for $\mathcal{X}_0$ there are many different smoothing families $\mathcal{X}_i$ and choices of divisors $R_i$. For any 1-parameter family, taking the Zariski closure of $R_i$ yields a flat limit $R_{i,0}$ on $\mathcal{X}_0$. If $R$ is recognizable, these flat limits do not vary, so the choice of divisor can be made on *any* K3, even a smooth K3. If $R$ is a recognizable polarizing divisor, there is a unique semifan $F_R$ such that

$$\overline{F_{2d}}^{F_R} = (\overline{F_{2d}}^R)^\nu$$

This relates a Hodge-theoretic compactification on the left with a geometric compactification on the right.

**Remark 9.23.** A semitoroidal compactification simultaneously generalizes toroidal and BB compactifications.

Recall that assocaited to a degeneration of K3s we had $\vec{\lambda} \coloneqq (\lambda_1, \cdots, \lambda_{19}) \in \delta^\perp/\delta$, a signature $(1, 18)$ lattice. Friedman-Scattone show that $\vec{\lambda}^2$ is the number of triple points in $\mathcal{X}_0$. The semifan $F_R$ is a locally polyhedral $\Gamma_\delta \coloneqq \mathrm{Stab}_\Gamma(\delta)$ invariant decomposition of the positive cone $C^+ \subset \delta^\perp/\delta$. This is the future light cone in the corresponding hyperbolic space. Roughly $\overline{F_{2d}}^{F_R}$ is $X(F_R)/\Gamma_\delta$.

Why this? We had a torus embedding of the first partial quotient $U_\delta \backslash \mathbb{D} \to (\mathbb{C}^*)^{19}$ and the latter is canonically identified with $\delta^\perp/\delta \otimes \mathbb{C}^*$. The monodromy invariant $\vec{\lambda}$ was approximated by the cocharacter $\lambda \otimes \mathbb{C}^*$. We extend that torus by a toric variety whose fan has support in $\delta^\perp/\delta$.

Note that here semitoroidal corresponds to *locally* polyhedral. A globally polyhedral tiling condition would just yield a usual fan. For instance, the cones here might have infinitely many rational polyhedral walls. On the other hand, the BB compactification corresponds to the trivial compactification of $C^+$ which is just the entirety of $C^+$.

**Remark 9.24.** AE prove that recognizable divisors $\overline{R}^{\mathrm{rc}} := \sum_{C \in |L|, C^\nu \cong \mathbb{P}^1} C$ exist. The rational curve divisor is always recognizable for any degree $2d$, so this exhibits some semitoroidal compactifications with geometric meaning.

AET give some explicit examples for $F_2$. Degree 2 K3s are generically 2-to-1 covers $\pi : X \to \mathbb{P}^1$ branched over a sextic, take $L := \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$. One takes the $R$ to be the ramification divisor $R \in |3L|$; it is a recognizable divisor. They construct a semifan $F_R$ which is a coarsening of the Coxeter fan for the root system in $\delta^\perp/\delta$; one takes a subset of the root mirrors.

The construction of the singular K3 surface: start with the heart $\mathrm{IAS}^2$, triangulate completely, double this construction, replace each vertex with the surface defined by the star. Note the cuts introducing shears along the boundary. The cuts introduce 3 singularities in each hemisphere, and angular defects of the polygonal gluing introduce 18 singularities along the equator. This $\mathrm{IAS}^2$ has an involution, and this $\mathcal{X}_0$ naturally has an involution. The ramification divisor of the $\mathrm{IAS}^2$ is in blue, it's a tropical ramification divisor. It is the dual complex of the limit of ramification divisors.

Why is this recognizable? $\mathcal{X}_0$ admits an involution $\iota_0$. From this we can determine the limit of $\mathrm{Fix}(\iota)$. Note that $\mathcal{X}_0$ alone determines $R_0$, the limit of $R_t$, and $R_0 = \mathrm{Fix}(\iota_0)$. This implies recognizability since the choice of divisor $R$ can be made on any Kulikov surface.

**Remark 9.25.** On joint work with ABE for elliptic K3s. Take $X \to \mathbb{P}^1$ an elliptic fibration with fiber $f$ and section $s$. This is not of the form $F_{2d}$, since here one takes $H = \mathbb{Z}s \oplus \mathbb{Z}f$ for the polarization. Generically the fibration has 24 singular fibers. They show $R := s + m \sum f_i$ for $f_i$ the singular fibers is recognizable for any multiple $m$. The lattice $\mathrm{II}_{1,17}$ is reflective, and $F_R$ here refines the Coxeter chamber into 9 subchambers. This is a fan which is strictly not a semifan. There is a corresponding tropical elliptic K3 given by the following $\mathrm{IAS}^2$.

image

Here one glues the top too the bottom, identifying the segments by a vertical shear. Note it has an $S^1$ fibration which tropicalizes the elliptic fibration. The blue vertical lines are limits of singular fibers, the blue horizontal is the limit of the section.

## 10. Coxeter and Dynkin diagrams

### 10A. Coxeter groups and diagrams.

**Remark 10.1.** Main ideas:

- Elliptic subdiagrams of rank $r$ correspond to codimension $r$ faces of a polytope $P$
- Parabolic subdiagrams (of rank $n-1$) correspond to cusps of $P$

**Remark 10.2** (A summary of hyperbolic Coxeter diagram conventions)**.** Regarding this as a group of reflections in hyperplanes, we have the following interpretations:

| Description | Diagram | Notation | $m_{ij}$ | $\angle(H_i, H_j)$ | $w_{ij}$ |
|---|---|---|---|---|---|
| Labeled simple edge | $H_1$ $m_{ij}$ $H_2$ | $H_i \pitchfork H_j$ | $m_{ij}$ | $\pi/m_{ij}$ | $\cos\left(\frac{\pi}{m_{ij}}\right)$ |
| No Edge | $H_1$ $H_2$ | $H_i \perp H_j$ | 2 | $\pi/2$ | 0 |
| Simple Edge | $H_1$ $H_2$ | $H_i \pitchfork H_j$ | 3 | $\pi/3$ | $\frac{1}{2}$ |
| Double Edge | $H_1$ $H_2$ | $H_i \pitchfork H_j$ | 4 | $\pi/4$ | $\frac{\sqrt{2}}{2}$ |
| Triple Edge | $H_1$ $H_2$ | $H_i \pitchfork H_j$ | 5 | $\pi/5$ | $\frac{1+\sqrt{5}}{4}$ |
| Thick/bold edge | $H_1$ $H_2$ | $H_i \parallel H_j$ | $\infty$ | 0 | 1 |
| Dotted Edge | $H_1$ $w_{ij}$ $H_2$ | $H_i \,)(\, H_j$ | 0 | $\infty$ | $\cosh(\rho(H_i, H_j))$ |
| Simple vertex | ○ | $h_i^2 = -1$ | | | 1 |
| Black vertex | ● | $h_i^2 = -2$ | | | 2 |
| Double-circled vertex | ◎ | $h_i^2 = -4$ | | | 4 |

TABLE 2. A summary of conventions for Coxeter-Vinberg diagrams

**Definition 10.3** (Coxeter groups)**.** A group $W$ is a **Coxeter group** if it has a presentation of the following form:

$$W = \langle r_1, \cdots, r_n \mid (r_i r_j)^{m_{ij}} \ \forall 1 \leq i, j \leq n \rangle \qquad m_{ij} \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$$

where

- $m_{ii} = 1$ for all $i$,
- $m_{ij} \geq 2$ for $i \neq j$, and
- $m_{i,j} = \infty$ means there is no relation imposed.

If $S = \{r_1, \cdots, r_n\}$ is a fixed generating set, we call the pair $(W, S)$ a **Coxeter system**.

**Definition 10.4** (Coxeter diagrams)**.** Given a Coxeter system $(W, S)$, the **the pre-Coxeter diagram** of $(W, S)$ is weighted undirected graph with a single vertex $v_i$ for each $r_i \in S$, and for each pair $i \neq j$, an edge $e_{ij}$ of weight $w_{ij} := m_{ij}$ connecting $v_i$ to $v_j$. Note that this yields a complete[5] graph on $|S|$ vertices. The **Coxeter diagram** $D(W)$ of $(W, S)$ is the partially weighted graph obtained from the pre-Coxeter diagram by the following modifications:

- Edges $e_{ij}$ of weight $w_{ij} = 2$ are deleted.
- Edges $e_{ij}$ of weight $w_{ij} \geq 7$ are labeled with their weights.
- Edges $e_{ij}$ of $w_{ij} = 3, 4, 5, 6$ follow one of two conventions: they are either replaced with an $(w_{ij} - 2)-$fold multi-edge, or are unmodified and retain their label of $w_{ij}$.
- Edges $e_{ij}$ of weight $w_{ij} = \infty$ are replaced by bold/thick edges.

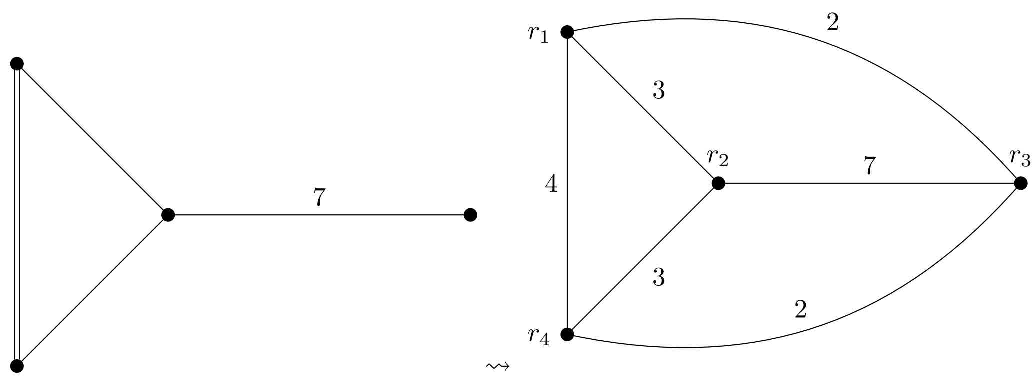
**Remark 10.5** (Facts about Coxeter diagrams)**.** We summarize several facts about the full Coxeter diagram:

[5]Recall that a graph is **complete** if every vertex is adjacent to every other vertex.

- Vertices $v_i$ and $v_j$ are non-adjacent if and only if $w_{ij} = 2$,
- Vertices $v_i$ and $v_j$ are adjacent if and only if $w_{ij} \geq 3$,
- Edge weights are suppressed for small weights $w_{ij} \leq 6$, and explicitly included for every $w_{ij} \geq 7$.

**Remark 10.6** (How to read a group presentation from a Coxeter diagram)**.** One can recover the presentation of a Coxeter group from any Coxeter diagram. Explicitly, given a diagram $D$, one constructs a group $W$ such that $D = D(W)$ in the following way: first one transforms the Coxeter diagram into a pre-Coxeter diagram by adding weight 2 edges between every pair of non-adjacent vertices, forming a complete graph. One then replaces double/triple/quadruple edges with weight 4/5/6 edges respectively. Finally, reads the group presentation off of the weighted adjacency matrix of the resulting graph. Explicitly, the group $W$ will have a generator for every vertex and a relation $(r_i r_j)^{w_{ij}}$ for each edge $e_{ij}$ of weight $w_{ij}$.d

**Example 10.7** (Passing between Coxeter diagrams and Coxeter groups)**.** Every Coxeter diagram is naturally associated with a weighted graph whose edge weights are all integers $m_{ij} \geq 2$, and from this presentation, one can immediately read off the group presentation. For example, consider the following diagram and the associated weighted graph:




Reading generators and relations off of this graph, we obtain a group freely generated by $r_1, r_2, r_3, r_4$ subject to the following relations:

$$W := \left\langle r_1, r_2, r_3, r_4 \;\middle|\; \begin{array}{l} r_1^2 = r_2^2 = r_3^2 = r_4^2 = 1 \\ (r_1 r_2)^3 = (r_2 r_3)^7 = (r_1 r_4)^4 = (r_2 r_4)^3 = 1 \\ (r_1 r_3)^2 = (r_3 r_4)^2 = 1 \end{array} \right\rangle .$$

Letting $A$ be the weighted adjacency matrix of this weighted graph, we can read this group presentation directly off of the following symmetric matrix:

$$A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 1 & 7 & 3 \\ 2 & 7 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

This matrix defines an exact sequence of $\mathbb{Z}-$modules

$$0 \to \mathbb{Z}^4 \xrightarrow{A} \mathbb{Z}^4 \to W \to 0,$$

realizing $W \cong \operatorname{coker} A$ as a presentation of $W$ by generators and relations.

10B. **Coxeter polytopes.**

**Remark 10.8.** Recall the cosine formula for Euclidean inner product spaces: in $\mathbb{E}^n$, the norm is $\|x\| \coloneqq \sqrt{x^2} \coloneqq \sqrt{x.x}$, and we have

$$vw = \|v\|\|w\| \cos(\angle(v,w)) = \sqrt{v^2}\sqrt{w^2}\cos(\angle(v,w)) = \sqrt{v^2w^2}\cos(\angle(v,w))$$

For a general bilinear form, we can define

$$\angle(v,w) \coloneqq \cos^{-1}\left(\frac{vw}{\sqrt{v^2w^2}}\right).$$

We can thus interpret the pairing as measuring angles in the following way:

$$vw = \frac{\cos(\angle(v,w)}{\sqrt{v^2w^2}},$$

which moreover allows one to compute intersections $vw$ from knowledge of $v^2, w^2$, and angles $\angle(v,w)$, which is precisely the data that is encoded in a Coxeter diagram.

**Definition 10.9** (Dihedral angles between hyperplanes)**.** If $H_i, H_j$ are intersecting hyperplanes in $\mathbb{E}^n$, we write $H_i \pitchfork H_j$. We write $h_i \coloneqq H_i^\perp$ and $h_j \coloneqq H_j^\perp$ for unit normal vectors spanning their orthogonal complements, and define the **dihedral angle** between $H_i$ and $H_j$ as

$$\angle(H_i, H_j) \coloneqq \angle(h_i, h_j).$$

If $H_i$ is parallel to $H_j$, we write $H_i \parallel H_j$ and define $\angle(H_i, H_j) = 0$. We similarly write $H_i \perp H_j$ if $\angle(H_i, H_j) = \pi/2$.

**Remark 10.10.** Note that there is a common trick to get rid of the square root in these formulas: one writes

$$(vw)^2 = v^2w^2\cos^2(\angle(v,w))$$

For $\angle(v,w) = \pi/m_{ij}$, this gives a way to recover $m_{ij}$ from the bilinear form.

**Definition 10.11** (Coxeter polytopes)**.** Let $X \coloneqq \mathbb{E}^n, \mathbb{S}^n, \mathbb{H}^n$ be a Euclidean, spherical, or hyperbolic geometry. A polytope $P \subseteq X$ is **Coxeter polytope** if all dihedral angles between pairs of intersecting facets $H_i$ and $H_j$ are of the form $\pi/m_{ij}$ for $m_{ij} \in \mathbb{Z}_{\geq 2}$, and any two non-intersecting facets are parallel.

**Remark 10.12** (Coxeter group $G_P$ of a Coxeter polytope $P$)**.** Every Coxeter polytope $P$ defines a Coxeter group $G_P \leq \mathrm{Isom}(\mathbb{X})$ generated by reflections through the supporting hyperplanes $H_i$ of facets of $P$ and a corresponding Coxeter diagram $D_P$. For $\mathbb{X} = \mathbb{E}^n$, one constructs $G_P$ in the following way:

- A generator $r_i$ for each facet $H_i$ of $P$ with relation $r_i^2 = 1$, representing reflection through the hyperplane $H_i$,
- For any facets $H_i, H_j$ where $H_i \pitchfork H_j$, there is a relation $(r_i r_j)^{m_{ij}} = 1$ where $m_{ij}$ is defined by $\angle(H_i, H_j) = \pi/m_{ij}$.
- For non-intersecting facets $H_i \parallel H_j$, we set $m_{ij} = \infty$ and take a relation $(r_i r_j)^\infty = 1$, i.e. no relation is imposed at all.

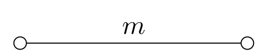
**Remark 10.13.** Note that $P$ is a fundamental domain for the action of $G_P$ on $\mathbb{X}$. Moreover, if $G \leq \mathrm{Isom}(\mathbb{X})$ is any discrete finitely generated reflection group, then its fundamental domain is always a Coxeter polytope. If $\mathbb{X} = \mathbb{S}^n$ or $\mathbb{E}^n$, Coxeterpolytopes are classified and are either simplies or products of simplices respectively, and full lists can be found. For $\mathbb{X} = \mathbb{H}^n$, the general classification is an open problem. Poincaré classified them in $\mathbb{H}^2$. Vinberg showed that no compact Coxeter polytopes exist in $\mathbb{H}^n$ for $n \geq 30$, and no non-compact but finite volume polytopes exist for $n \geq 996$. These bounds are not sharp. Finding explicit examples of high-dimensional compact Coxeter polytopes is interesting because these can be used to explicitly construct high-dimensional hyperbolic manifolds.

**Definition 10.14** (Volumes and covolumes of Coxeter groups/polytopes/diagrams)**.** We define the **covolume** of $G_P$ as the volume of $P \cong \mathbb{X}/G_P$, where the metric on the quotient is induced from the metric defining the geometry on $\mathbb{X}$.

**Remark 10.15.** We collect some facts about the corresponding Coxeter diagram $D(P)$:

- $D(P)$ has vertices $v_i$ corresponding to $H_i$, where $v_i, v_j$ are non-adjacent if and only if $H_i \perp H_j$[6],
- Edges $e_{ij}$ are plain if $m_{ij} < \infty$ and $m_{ij} \neq 0$, so $H_i \pitchfork H_j$,
- Edges $e_{ij}$ are bold if $m_{ij} = \infty$, so $H_i \parallel H_j$ and $\angle(H_i, H_j) = \pi/\infty = 0$.

**Example 10.16** (Euclidean Coxeter polytopes)**.** Consider the following Coxeter diagram:




This corresponds to a non-compact polytope in $\mathbb{E}^2$ bounded by two hyperplanes $H_1, H_2$ through the origin (i.e. lines), one corresponding to each node, intersecting at an angle of $\pi/m$. Without loss of generality, we can take $H_1$ to be the $x-$axis and $H_2$ to be a line of slope $\pi/m$:

[6]Recalling that edges with $m_{ij} = 2$ are deleted by convention.

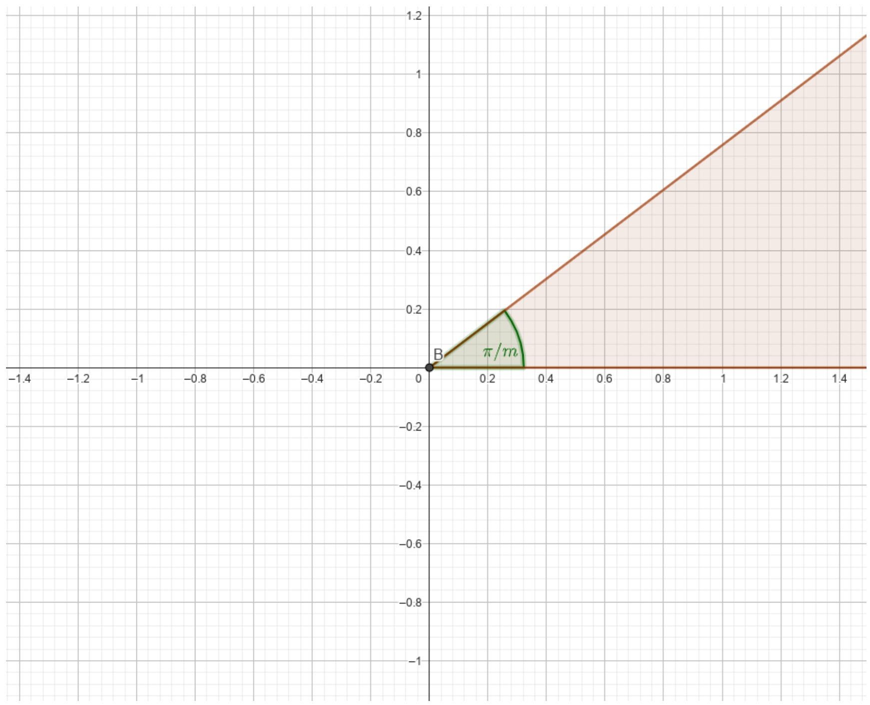
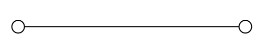



FIGURE 19. Caption

One can note that if $m = 2$, then one deletes the edge by convention to get the Coxeter diagram

This is the Dynkin diagram of $A^1 \times A^1$, which indeed has fundamental chamber the first quadrant. Similarly, if one takes $m = 3$ on recovers the standard Dynkin diagram for $A_2$:



We get a fundamental chamber with two walls at a dihedral angle of $\pi/3$, corresponding to the dual hyperplanes of the two standard short roots$\alpha$ and $\beta$ with $\angle(\alpha, \beta) = 2\pi/3$ in Lie theory:

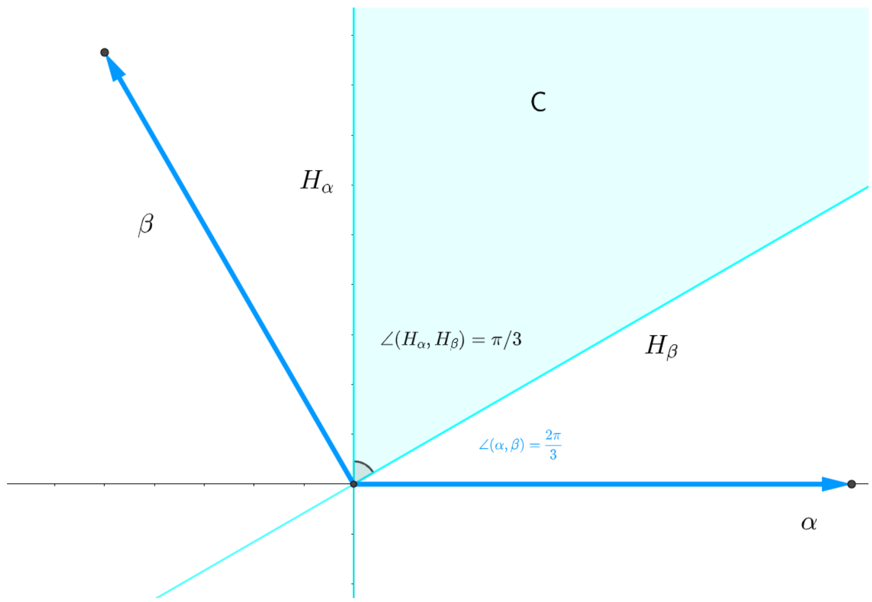



FIGURE 20. Caption

Todo: weighted $\widetilde{A}_2$ as a simplex.

**Example 10.17** (Affine examples)**.**

**Remark 10.18.** Note that taking reflections of the fundamental domain $C$ by the Weyl group generates a **tiling** of the hyperbolic disc in these cases.

**Remark 10.19** (Importance of tilings!)**.** Why this is important: given *any* tiling of $\mathbb{E}^2$ or $\mathbb{H}$ the hyperbolic disc, we can place it at height one and take a cone to get an infinite-type toric variety. Alternatively, given any tiling we can construct a surface that is a union of toric pairs by interpreting every vertex of the tiling as a fan and the edges of tiles as gluing instructions.

Finally, we can interpret an IAS$^2$ has an irregular spherical tiling, i.e. a tiling $\mathbb{S}^2$ which is not necessarily generated by reflections, but one which has finitely many tiles. We then regard the tiling as a union of toric surfaces as described above.

**Definition 10.20** (The Gram matrix of a Euclidean Coxeter polytope)**.** Let $P \subset \mathbb{E}^n$ be a Euclidean Coxeter polytope, not necessarily compact. One defines the **Gram matrix** $G(P)$ of $P$ as

$$G(P)_{ij} = \begin{cases} 1 & i = j \\ -\cos\left(\frac{\pi}{m_{ij}}\right) & H_i \pitchfork H_j, \quad \angle(H_i, H_j) = \pi/m_{ij} \\ -1 & H_i \parallel H_j, \quad \angle(H_i, H_j) = \pi/\infty = 0 \end{cases}.$$

## 10C. Hyperbolic Coxeter polytopes.

**Remark 10.21.** See the section on hyperbolic geometry for a description of $\mathbb{H}^n := \left\{x \in \mathbb{E}^{n,1} \mid x^2 = -1, x_0 > 0\right\}$ and terminology (space/time/light-like vectors). As a convention, $\mathbb{H}^n$ means the interior of $\overline{\mathbb{H}^n} := \mathbb{H}^n \cup \partial\mathbb{H}^n$ where $\partial\mathbb{H}^n$ is the boundary at infinity consisting of ideal points.

Some unsorted notes:

- The distance $\rho$ on $\mathbb{H}^n$ is defined such that $\rho(v,w) := \operatorname{arccosh}(vw)$.
- In $\mathbb{H}^n$ Vinberg defines the dihedral angle as $\angle(f_i, f_j) := \pi - \angle(f_i^{\perp}, f_i^{\perp})$.
- The diagram $E_{10}$ describes a polytope in $\mathbb{H}^9$.

**Remark 10.22** (Hyperplane incidence relations in hyperbolic spaces)**.** In hyperbolic geometry ($\mathbb{H}^2$ to simplify), there are two types of parallelism: asymptotically parallel (converging) lines, or ultraparallel (diverging) lines. Both are characterized by sharing a common orthogonal line, however, asymptotically parallel lines have a common perpendicular in $\partial\overline{\mathbb{H}^2}$ going through their ideal point of intersection, while ultraparallel lines share a common perpendicular at a point in the interior $\mathbb{H}^2$. By the ultraparallel theorem, $H_i, H_j$ are ultraparallel if and only if $H_i \cap H_j = \emptyset$ in $\overline{\mathbb{H}^2}$.

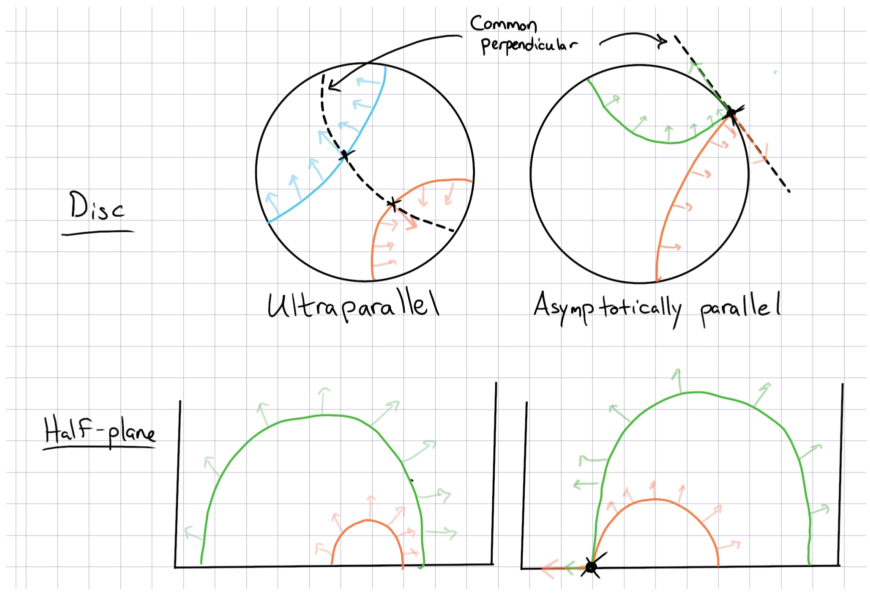



FIGURE 21. The two types of parallelism in hyperbolic space, visualized in the ball model and half-plane model respectively.

Thus given a pair of hyperplanes $H_i$ and $H_j$, there are thus three possibilities for their incidence relations:

(1) $H_i, H_j$ are not parallel and thus intersect in $\mathbb{H}^n$. We write $H_i \pitchfork H_j$ and define $\angle(H_i, H_j)$ as the usual dihedral angle.
(2) $H_i, H_j$ are asymptotically parallel/converging and thus intersect in an ideal point in $\partial\mathbb{H}^n$. We write $H_i \parallel H_j$ and define $\angle(H_i, H_j) = \frac{\pi}{\infty} = 0$.
(3) $H_i, H_j$ are ultraparallel/diverging and do not intersect in $\overline{\mathbb{H}^n}$. We write $H_i )( H_j$.

**Remark 10.23** (Hyperbolic distance between hyperplanes)**.** Note that in the last case above, $\angle(H_i, H_j)$ is undefined but there is a minimal distance $\rho(H_i, H_j)$ between the two hyperplanes. By geometric axioms, if $H_i \cap H_j = \emptyset$ then there is a unique geodesic $L_{ij}$ that is simultaneously orthogonal to both $H_i$ and $H_j$, intersecting them at points $p_i$ and $p_j$. One then defines $\rho(H_i, H_j)$ as the length of a geodesic segment along $L_{ij}$ with endpoints at $p_i$ and $p_j$.

**Remark 10.24** (Extending Coxeter diagrams for hyperbolic polytopes)**.** Following Vinberg, one can extend the notion of a Coxeter diagram to a weighted graph with positive weights $w_{ij} > 0$ where all $w_{ij} \in (0,1)$ can be written in the form $w_{ij} = \cos\left(\frac{\pi}{m_{ij}}\right)$ for some $m_{ij} \in \mathbb{Z}_{\geq 2}$ and $w_{ij} \in [1, \infty]$ can be arbitrary real (possibly infinite) numbers. In this convention,

- $w_{ij} = \cos\left(\frac{\pi}{m_{ij}}\right) \in (0,1)$ get simple edges of labeled weight $m_{ij}$ (or multi-edges) corresponding to $H_i \pitchfork H_j$ and $\angle(H_i, H_j) = \left(\frac{\pi}{m_{ij}}\right)$
- $w_{ij} = 1$ get **bold** unlabeled edges of weight 1 corresponding to $H_i \parallel H_j$ and $\angle(H_i, H_j) = \frac{\pi}{\infty} = 0$.
- $w_{ij} \in (1, \infty)$ get **dotted** labeled edges of weight $w_{ij}$ (or unlabeled) corresponding to $H_i )( H_j$ and $w_{ij}$ corresponds to $\rho(H_i, H_j)$

More generally, given a Coxeter-Vinberg diagram set

$$g_{ij} = \frac{h_i h_j}{\sqrt{h_i^2 h_j^2}},$$

then one interprets

- $g_{ij} < 1 \implies g_{ij} = \cos(\angle(h_i, h_j))$ and $H_i \pitchfork H_j$ with $\angle(h_i, h_j) = \pi/m_{ij}$,
- $g_{ij} = 1 \implies H_i \parallel H_j$ with $\angle(h_i, h_j) = 0$,
- $g_{ij} > 1 \implies H_i )( H_j$.

Todo: $(\infty, \infty, \infty)$.

**Example 10.25** (Hyperbolic examples)**.**

**Remark 10.26.** As in the Euclidean case that taking reflections of the fundamental domain $C$ by the corresponding Weyl group naturally constructs a **tiling** of $\mathbb{E}^2$ in all of these cases:

- $A_1 \times A_1$ tiles $\mathbb{E}^2$ with 4 non-compact quadrants,
- $A_2$ tiles $\mathbb{E}^2$ with 6 non-compact sectors of angle $\pi/3$,
- In general, taking $\circ \to^m \circ$ with $m \in \mathbb{Z}_{\geq 1}$ tiles $\mathbb{E}^2$ with $2m$ non-compact sectors of angle $\pi/m$,
- $\widetilde{A}_2$ tiles $\mathbb{E}^2$ with infinitely many compact equilateral triangles of with internal angles $\pi/3$.

**Definition 10.27** (The Gram matrix of a hyperbolic polytope)**.** Let $P \subseteq \overline{\mathbb{H}^n}$ be a Coxeter polytope, possibly with ideal points. The **Gram matrix** of $P$ is the matrix

$$G(P)_{ij} = \begin{cases} 1 & i = j \\ -\cos\left(\frac{\pi}{m_{ij}}\right) & H_i \pitchfork H_j, \quad \angle(H_i, H_j) = \pi/m_{ij}, \\ -1 & H_i \parallel H_j, \quad \angle(H_i, H_j) = \pi/\infty = 0, \\ -\cosh(\rho(H_i, H_j)) & H_i )( H_j, \quad \angle(H_i, H_j) = \pi/0 = \infty, \end{cases}.$$

**Remark 10.28.** When labeling the Coxeter graph, one often puts $m_{ij}$ or $\cosh(\rho(H_i, H_j))$ as the labels, mixing conventions slightly. Edges of weight 2 are deleted, edges of weight 3 are unlabeled simple edges.

**Remark 10.29.** If $P \subseteq \mathbb{H}^n$ is a compact hyperbolic Coxeter polytope, the quotients $\mathbb{H}^n/G_P$ are hyperbolic orbifolds. The simplest examples of such polytopes are the hyperbolic $n-$gons defined by integers $p_1, \cdots, p_k \geq 2$ satisfying $\sum p_i^{-1} < k - 2$.

**Definition 10.30** (Simple systems)**.** We say $\Delta = \{r_i\}$ is a **simple system** of generators for a polytope $P$ if $r_i r_j \geq 0$ for all $i$ and $j$, and $P$ has a facet presentation by the mirrors $H_{r_i}$. This allows one to write

$$P = \left\{ v \in L_{\mathbb{R}} \mid v^2 = 0,\, r_i v \geq 0 \right\}.$$

We call $P$ a **Weyl chamber**[7]. The closure $\overline{P}$ is a fundamental domain for the action of $W(L)$ and the Weyl group acts simply transitively on the set of chambers.

**Remark 10.31** (Decomposing the future orthogonal group into a Weyl and symmetry group)**.** Let $W(L)$ be the reflections in all negative norm vectors. There is an identification

$$\mathrm{O}^+(L) \cong W(L) \rtimes S(C), \qquad S(C) \coloneqq \mathrm{Stab}_{\mathrm{O}^+(L)}(C)$$

where $C \subset \mathbb{B}^n$ be a fundamental chamber of $W(L)$ with respect to some choice of a simple set of generators.

The semidirect might be in the wrong direction here, which one is normal?

**Definition 10.32** (Reflective lattices)**.** We say $L$ is **reflective** if $W(L) \leq \mathrm{O}^+(L)$ is finite-index. More generally, if we define $\mathrm{O}^+(L)_k$ as the subgroup generated by all $k-$reflections, i.e. reflections in roots $v$ with $v^2 = k$, we say $L$ is $k-$reflective if $W()$ is finite index in $\mathrm{O}^+(L)$.

**Remark 10.33.** If $L$ as above is reflective, it is well-known $C$ is a hyperbolic Coxeter polytope of finite volume.

**Definition 10.34** (Vinberg-Coxeter diagrams)**.** A **Vinberg-Coxeter diagram** is an extension of a Coxeter diagram with adds the following decorations:

- Black edges
- Double-circled edges
- Dotted edges
- Thick edges

It is a weighted graph with positive edge weights $w_{ij} > 0$ where we require that any $w_{ij} \in (0,1)$ is of form $w_{ij} = \cos\left(\frac{\pi}{m_{ij}}\right)$ for some $m_{ij} \in \mathbb{Z}_{\geq 2}$, but we explicitly allow some $w_{ij} \in [1, \infty]$ to be real (possibly infinite) numbers. We additionally specify vertex weights $r_i$ for each vertex $v_i$. In this convention,

- $w_{ij} = \cos\left(\frac{\pi}{m_{ij}}\right) \in (0,1)$ get **simple edges** of labeled weight $m_{ij}$ (or unlabeled multi-edges of multiplicity $m_{ij} - 2$ for $m_{ij} = 3, 4, 5, 6$),
- $w_{ij} = 1$ get **bold unlabeled edges** of weight 1
- $w_{ij} \in (1, \infty)$ get **dotted labeled edges** of weight $w_{ij}$.

10D. **Elliptic and Parabolic subdiagrams.**

**Remark 10.35.** Given these weights, one can construct the weighted adjacency matrix $A$ with $a_{ij} = w_{ij}$ if $v_i, v_j$ are adjacent and zero otherwise.

A matrix $A$ is a **direct sum of matrices** $A_i$ if $A$ is similar via permutations of rows and columns to the block diagonal matrix whose blocks are the $A_i$. If $A$ can not be written as a direct sum of two matrices, we say $A$ is **indecomposable**. Every matrix has a unique representation as a sum of indecomposable components. We say a Coxeter polytope is indecomposable if its Gram matrix $G_P$ is indecomposable.

[7]This is also sometimes notated $C$.

Any matrix $G_P$ arising from an irreducible Coxeter polytope is either positive-definite, positive-semidefinite, or indefinite. We say a diagram $D_P$ is elliptic if $G_P$ is PD, parabolic if every subdiagram is elliptic and it has at least one degenerate irreducible component.

Connected components of the diagram correspond to indecomposable sub-block matrices of $A$. A diagram is elliptic of $A$ is positive-definite, and is parabolic if any indecomposable component of $A$ is degenerate and positive-semidefinite. There are finitely many indecomposable elliptic and parabolic diagrams. If a Coxeter diagram describes a Coxeter polytope $P$, elliptic subdiagrams of codimension 1 correspond to facets of $P$. Moreover, $P$ has finite volume iff every such elliptic subdiagram can be extended in exactly 2 ways to either an elliptic subdiagram of rank $n$ or a parabolic subdiagram of rank $n-1$, corresponding to every facet of the polytope meeting each of its adjacent facets at either an interior point or an ideal point of $\mathbb{H}^n$ respectively.

**Remark 10.36.** Idea: a subdiagram is elliptic if the Gram matrix is negative definite of full rank, and parabolic if negative semidefinite of corank equal to the number of components of the diagram. Elliptic diagrams of rank $r$ biject with codimension $r$ faces of $C$. Parabolic diagrams of corank 1 correspond to ideal points of $C$. Vinberg's algorithm produces a simple system $\Delta$ of generators for $W(L)$ which determines a hyperbolic polytope $C$ via the corresponding Weyl chamber. If the algorithm terminates, $C$ is of finite volume.

**Definition 10.37** (Ranks of subdiagrams)**.** The **rank** of a subdiagram is its number of vertices minus its number of connected components.

**Definition 10.38** (Elliptic and parabolic Coxeter subdiagrams)**.** A Coxeter diagram $G$ is called **elliptic** (resp. **parabolic**) if every connected component of $G$ is a Coxeter diagram underlying classical (resp. affine) Dynkin diagram. This is summarized in the following table; note that the classical diagrams $B_n$ and $C_n$ become identified when the arrow is omitted:

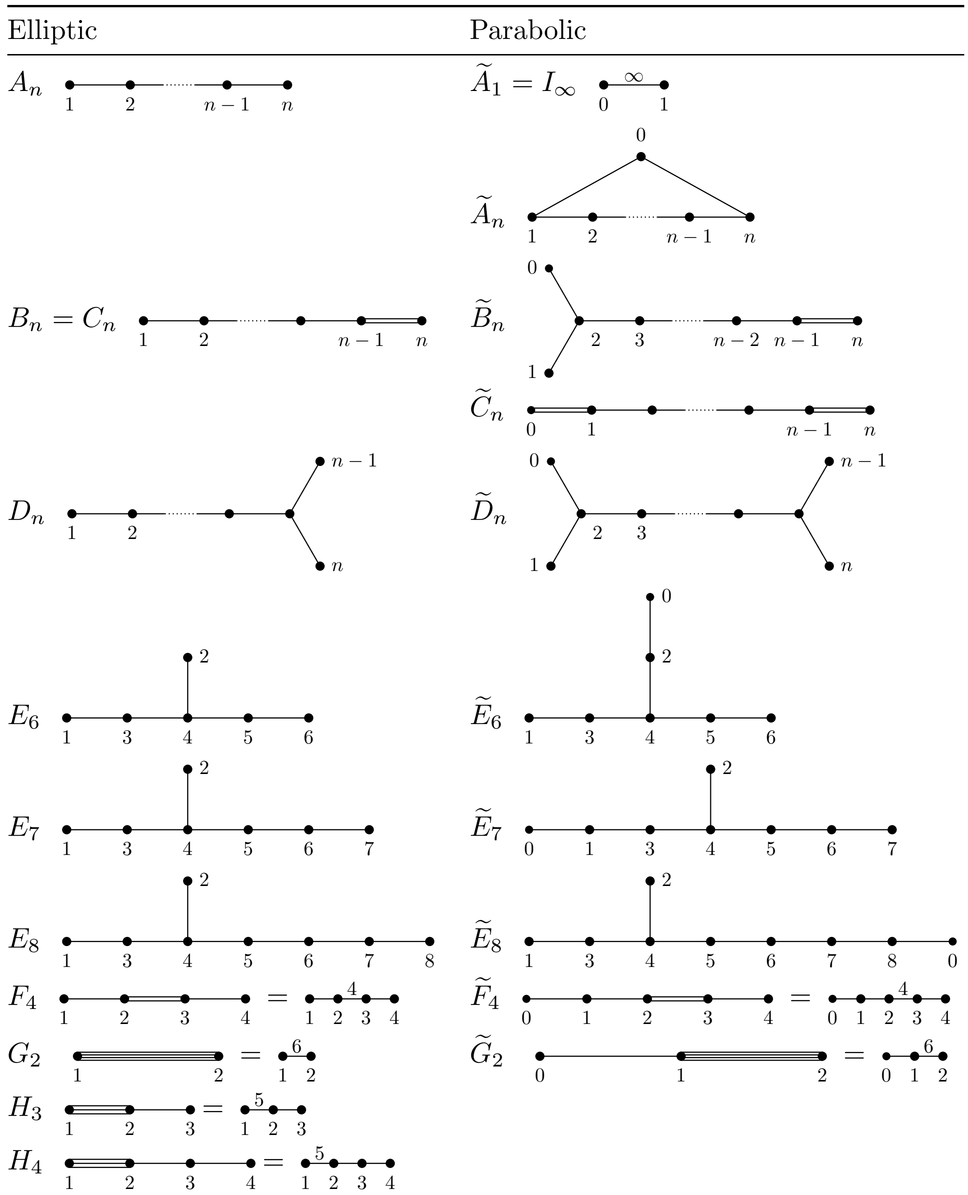


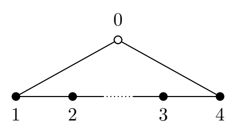

Table 3. Classification of elliptic and parabolic subdiagrams of a Coxeter diagram

10E. **Some discrepancies.** Note the following discrepancies when comparing the classification of diagrams of Coxeter diagrams to the usual notions of Dynkin diagrams:

- These are not Dynkin diagrams: we forget the arrows on double, triple, etc edges.

- **Warning**: in a **Coxeter** diagram, an edge of label $m$ always corresponds to an $(m-2)$−fold edge. **In a Dynkin diagram, a 3-fold edge corresponds to $m = 6$. We do not use this convention in the table above!** Compare $G_2, \widetilde{G}_2$ in the table, which have 4-fold edges corresponding to $m = 6$ to the following classical diagrams for $G_2$ and $\widetilde{G}_2$ which still correspond to $m = 6$:


The reason for this discrepancy: in a **Dynkin** diagram, the edge labels $m$ must satisfy a crystallographic condition and thus $m = 2, 3, 4, 6$. Since $m = 5$ is not possible, this makes the interpretation in that special case unambiguous.
- This discrepancy also occurs for $H_i$; here a triple edge truly corresponds to $m = 5$.
- In the affine case, we do not distinguish the "new" node, usually denoted by a white dot labeled 0. Compare to the usual diagram e.g. for $\widetilde{A}_n$:




**Remark 10.39.** Elliptic subdiagrams are a disjoint union of classical Dynkin diagrams, while parabolic subdiagrams are a disjoint union of *affine* Dynkin diagrams.

Why these matter: we are working with Coxeter polytopes $P$ in a hyperbolic space, i.e. hyperbolic Coxeter polytopes. Vinberg has a general theory which says the Coxeter diagram $D$ records the combinatorics of $P$:

- Facets of $P \rightleftharpoons$ nodes of $D$,
- Dihedral angles between two facets of $P \rightleftharpoons$ edges of $D$,
- $k-$faces of $P \rightleftharpoons$ elliptic subdiagrams of $P$ of co-rank $k$,
- Ideal vertices of $P \rightleftharpoons$ parabolic subdiagrams of rank $k$.

Idea: in $F_2$, Type II strata are classified by maximal parabolic subdiagrams of the single Coxeter diagram, and Type III strata by elliptic subdiagrams. Dimensions of strata correspond to number of vertices in these subdiagrams, and inclusion of diagrams corresponds to degenerations (smaller diagrams correspond to "more degenerate").

10F. **Edge conventions for Coxeter diagrams.** The interpretation of these Coxeter diagrams in terms of root systems:

Needs some notation from [**?**]:

| | |
|---|---|
| $V(M)$ | the light cone $V(M)=\left\{x \in M \otimes \mathbb{R} \mid x^2>0\right\}$ of a hyperbolic lattice $M$ |
| $V^{+}(X)$ | the half containing polarization of the light cone $V\left(S_X\right)$ |
| $\mathcal{L}(S)=V^{+}\left(S_X\right)/\mathbb{R}^{+}$ | the hyperbolic space of a surface $S$ |
| $W^{(2)}(M)$ | the group generated by reflections in all $f \in M$ with $f^2=-2$ |
| $W^{(4)}(M)$ | the group generated by reflections in all $(-4)$ roots of $M$ |
| $W^{(2,4)}(M)$ | the group generated by reflections in all $(-2)$ and $(-4)$ roots of $M$ |
| $\mathcal{M}^{(2)}$ | a fundamental chamber of $W^{(2)}(S)$ in $\mathcal{L}(S)$ |
| $\mathcal{M}^{(2,4)}$ | a fundamental chamber of $W^{(2,4)}(S)$ in $\mathcal{L}(S)$ |
| $P^{(2)}\left(\mathcal{M}^{(2,4)}\right)$ | all $(-2)$-roots orthogonal to $\mathcal{M}^{(2,4)}$ |
| $P^{(4)}\left(\mathcal{M}^{(2,4)}\right)$ | all $(-4)$-roots orthogonal to $\mathcal{M}^{(2,4)}$ |
| $(X,\theta)$ | a K3 with involution $\theta$ |
| $X^{\theta}$ | the fixed locus of an involution |
| $P(X)_{+I}$ | the subset of exceptional classes of $(X,\theta)$ of type I |

| Description | Symbol |
|---|---|
| Black vertices: $f \in P^{(4)}\left(\mathcal{M}^{(2,4)}\right)$, i.e. $f^2=-4$ | $f$ ● |
| White vertices: $f \in P^{(2)}\left(\mathcal{M}^{(2,4)}\right)$, i.e. $f^2=-2$ | $f$ ○ |
| Double-circled vertices: $f \in P(X)_{+I}$, i.e. the class of a rational component of $X^{\theta}$. | $f$ ◎ |
| No edge: $f_1 \neq f_2 \in P\left(\mathcal{M}^{(2,4)}\right)$ with $f_1 \cdot f_2=0$, so $\angle(f_1 f_2)=\pi/2$ | $f_1$ ○ ○ $f_2$ |
| Simple edges of weight $m$, or $m-2$ simple edges when $m$ is small: $\frac{2 f_1 f_2}{\sqrt{f_1^2 f_2^2}}=2\cos\frac{\pi}{m}$, so $\angle(f_1 f_2)=\pi/m$ | $f_1$ ○—$m$—○ $f_2$ |
| Thick edges: $\frac{2 f_1 \cdot f_2}{\sqrt{f_1^2 f_2^2}}=2$ | $f_1$ ○━━○ $f_2$ |
| Broken edges of weight $t$: ? | $f_1$ ○- - -$t$- - -○ $f_2$ |

Table 4. Edge conventions for Coxeter diagrams

Edge conventions for Coxeter polytopes: nodes correspond to facets $f_i, f_j$ of $P$ and edges record relations in $G_P$.

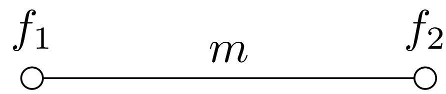
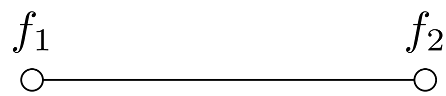
| Description | Diagram |
|---|---|
| $\angle(f_i f_j) = \pi/2$ |   |
| $\angle(f_i f_j) = \pi/m$ |   |
| $\angle(f_i f_j) = \pi/3$ |   |
| $\angle(f_i f_j) = \pi/4$ |   |
| $\angle(f_i f_j) = \pi/5$ |   |
| $f_i, f_j$ do not intersect |   |
| $f_i, f_j$ intersect in $\partial\overline{\mathbb{H}^n}$ |   |

Table 5

10G. **Surfaces associated with Coxeter diagrams.**

**Remark 10.40.** As described in [**?**, Prop. 4.6], for the Halphen case $S \coloneqq (10, 10, 1)$, there exists a K3 surface with $S_X^+ = S$ with $\pi : X \to Y \coloneqq X/\iota$ where $\mathrm{Nef}(Y)$ can be identified with the Coxeter chamber for the full reflection group $W_r$.

Moreover, [**?**, Cor. 4.8] shows that the Coxeter diagram of $S$ can be used to write the dual graph of exceptional curves on $Y$ under the following modifications:

| Description | Symbol | Description |
|---|---|---|
| A single-circled white vertex | $F$ ○ | $F \cong \mathbb{P}^1$ with $F^2 = -1$ |
| A double-circled white vertex | $F$ ◎ | $F \cong \mathbb{P}^1$ with $F^2 = -4$ |
| A black vertex | $F$ ● | $F \cong \mathbb{P}^1$ with $F^2 = -2$. |
| Any single, plain edge | $F_i$ ○—○ $F_j$ | $F_i, F_j \cong \mathbb{P}^1$ with $F_i F_j = 1$ |
| Any bold edge | $F_i$ ○**━**○ $F_j$ | $F_i, F_j \cong \mathbb{P}^1$ with $F_i F_j = 2$ |
| Any double edge | $F_i$ ○═○ $F_j$ | ?? |

Table 6. How to read a surface off of a Coxeter diagram

Unclear what double edges are, need to read further.

### 10H. **Incidence diagrams/dual complexes.**

### 10I. **Dual complexes.**

**Definition 10.41** (Dual complex)**.** Let $D$ be an RSNC divisor. The **dual complex** $\Gamma(D)$ of $D$ is the PL-homeomorphism type of the simplicial complex whose $d-$cells correspond with codimension $d$ strata in $D$, i.e. irreducible components of $d-$fold intersections $V_{i_0} \cap \cdots \cap V_{i_d}$.

**Example 10.42.** Let $\mathcal{X} \to \Delta$ be a semistable degeneration and let $\mathcal{X}_0 = V_1 \cup \cdots \cup V_n$ be the smooth surfaces forming the irreducible components of the central fiber. Writing $C_{ij} \coloneqq V_i \cap V_j$ and $p_{ijk} \coloneqq V_i \cap V_j \cap V_k$ for their intersections along curves and points, we call each irreducible component of $C_{ij}$ a **double curve** and the points $p_{ijk}$ **triple points**.

Semistability ensures that the dual complex has dimension at most 3, i.e. there are at worst triple points. Thus concretely the dual complex has

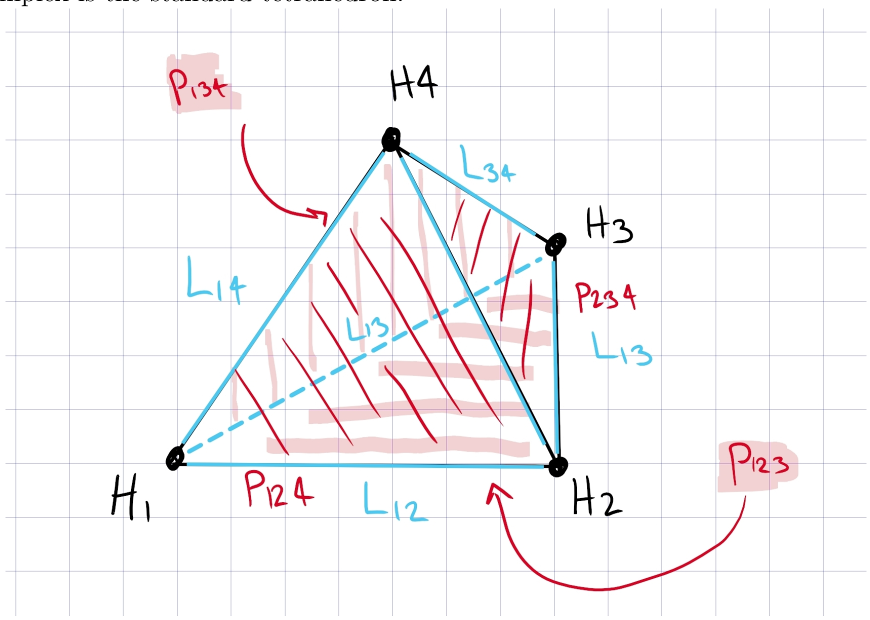
- a vertex for each component $V_i$,
- an edge from $V_i$ to $V_j$ for each double curve $C_{ij}$, and
- a 2-simplex spanning $V_i, V_j, V_k$ for each triple point $p_{ijk}$.

**Remark 10.43.** For a double curve $C = C_{ij} = C_{ji}$ regarded as a curve in $V_i$ and $V_j$ respectively, Persson's triple point formula holds:

$$C_{ij}^2 + C_{ji}^2 = -T_C$$

where $T_C$ is the number of triple points on $C$.

**Example 10.44.** Let $H_i \subset \mathbb{P}^3$ for $0 \leq i \leq 3$ be the four standard coordinate hyperplanes, i.e. $H_i = \{[z_1 : z_2 : z_3 : z_4] \mid z_i = 0\}$ and let $D = \sum H_i$. Any 2 planes intersect in a line and any 3 planes intersect in a point, so there are $\binom{4}{2} = 6$ double curves $C_{ij} := H_i \cap H_j$ and $\binom{4}{3} = 4$ triple points $p_{ijk} := H_i \cap H_j \cap H_k$. The dual complex is the standard tetrahedron:




**Definition 10.45** (Incidence complex)**.** Let $(X, D)$ be a RNC compactification. The **incidence complex** $I(D)$ **of** $D$ is the simplicial complex built in the following way: let $D = \sum_i D_i$ be a decomposition into prime divisors, and take a complex $I(D)$ whose $k-$dimensional cells are in bijection with irreducible components of $k-$fold intersections of the $D_i$. The **colored incidence complex** is $I(D)$ with an integer weight (or a coloring) attached to each 0-cell indicating the dimension of the corresponding stratum.

**Remark 10.46.** In our case of interest, $(X, D)$ will be a Baily-Borel compactification of a moduli space where $D := \partial\bar{X}$ is a union of boundary strata of various dimensions. Because we primarily work with hyperbolic lattices, $D$ will only contain strata of dimensions 0 and 1, i.e. points and curves. Thus $I(D)$ will reduce to a graph whose vertices are in bijection with points and curves in $D$ and whose edges record when a point $p_j$ is contained in the closure of a curve $C_i$. We can thus form the colored incidence complex $I(D)$ with two colors, taking points to be black and curves to be white.

**Definition 10.47** (Cusp incidence diagrams)**.** Let $\Omega_N$ be the period domain associated with a lattice $N$ and let $\Gamma \subseteq \mathrm{O}(N)$ be a finite-index subgroup. The Baily-Borel compactification $\overline{\Omega_S/\Gamma}^{\mathrm{bb}}$ is a projective variety with a boundary stratification

$$\overline{\Omega_S/\Gamma}^{\mathrm{bb}} = (\Omega_S/\Gamma) \cup \mathcal{I} \cup \mathcal{J}, \qquad \partial\overline{\Omega_S/\Gamma}^{\mathrm{bb}} = \mathcal{I} \cup \mathcal{J}$$

where

- $\mathcal{I}$ is a set of points referred to as 0-*cusps*, which are in bijective correspondence with $\Gamma$-orbits of primitive isotropic 2-dimensional sublattices of $N$, and
- $\mathcal{J}$ is a set of modular curves referred to as 1-*cusps*, which are in bijective correspondence with $\Gamma$-orbits of primitive isotropic 1-dimensional sublattices of $N$.

We summarize below what information the colored incidence complex $I(\partial\overline{\Omega_S/\Gamma}^{\text{bb}})$ captures:

| Cusp Type | Type II, $\mathcal{J}$ | Type III, $\mathcal{I}$ |
|---|---|---|
| Boundary Strata | 1-cusps/curves $C_i$ | 0-cusps/points $p_j$ |
| Vertex type | $C_i$ ○ | $p_j$ ● |
| Sublattice Type | Isotropic lines $[\mathbb{Z}e] \in \text{Gr}_1^{\text{iso}}(L)/\Gamma$ | Isotropic planes $[\mathbb{Z}e \oplus \mathbb{Z}f] \in \text{Gr}_2^{\text{iso}}(L)/\Gamma$ |
| Subdiagram Type | Maximal parabolic | Elliptic |

Table 7. Cusp types

Moreover, we draw an edge between a black and white node to denote a point $p_i$ contained in the closure of a curve $C_j$:


**Example 10.48.** Consider the following colored incidence diagram:

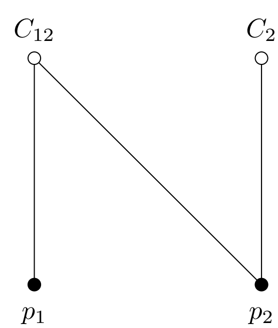



Figure 22. A colored incidence diagram $I(D)$ for $D = \mathcal{I} \cup \mathcal{J}$.

This represents the boundary stratification of a Baily-Borel compactification for which $\mathcal{I} = \{p_1, p_2\}$ consists of two points, $\mathcal{J} = \{C_{12}, C_2\}$ is two curves, where $p_1, p_2 \in \overline{C_{12}}$, $p_2 \in \overline{C_2}$, and $p_1 \notin \overline{C_2}$. This can be represented by the following configuration of curves and points:

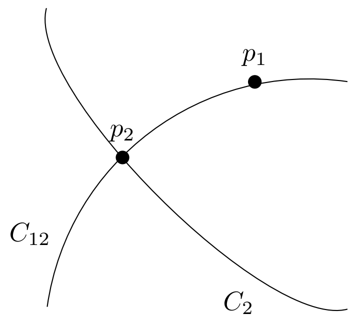



FIGURE 23. A configuration of curves and points representing $I(D)$ in Figure 23.

Spell out which root system this is attached to? Yes, that would be a good idea.

**Remark 10.49.** Each 0-cusp $p_i$ has an associated Vinberg diagram $\mathcal{D}(p_i)$ whose maximal parabolic subdiagrams enumerate the 1-cusps $C_{ij}$ adjacent to $p_i$ in the incidence diagram.

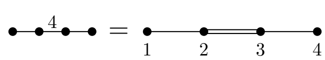
**Example 10.50.** The following figure shows the Vinberg diagram for the 0-cusp ???? associated to the lattice $N := (18, 0, 0)$:

This has the following two maximal parabolic subdiagrams:

*Vinberg1Parabolic1* *Vinberg1Parabolic2*

That there are exactly 2 such subdiagrams is reflected in the fact that the vertex ??? in the incidence diagram has valence 2.

## 11. Diagrams to surfaces




The interpretation of these Coxeter diagrams in terms of root systems:

Needs some notation from [**?**]:

| | |
|---|---|
| $V(M)$ | the light cone $V(M) = \left\{x \in M \otimes \mathbb{R} \mid x^2 > 0\right\}$ of a hyperbolic lattice $M$ |
| $V^+(X)$ | the half containing polarization of the light cone $V(S_X)$ |
| $\mathcal{L}(S) = V^+(S_X)/\mathbb{R}^+$ | the hyperbolic space of a surface $S$ |
| $W^{(2)}(M)$ | the group generated by reflections in all $f \in M$ with $f^2 = -2$ |
| $W^{(4)}(M)$ | the group generated by reflections in all $(-4)$ roots of $M$ |
| $W^{(2,4)}(M)$ | the group generated by reflections in all $(-2)$ and $(-4)$ roots of $M$ |
| $\mathcal{M}^{(2)}$ | a fundamental chamber of $W^{(2)}(S)$ in $\mathcal{L}(S)$ |
| $\mathcal{M}^{(2,4)}$ | a fundamental chamber of $W^{(2,4)}(S)$ in $\mathcal{L}(S)$ |
| $P^{(2)}\left(\mathcal{M}^{(2,4)}\right)$ | all $(-2)$-roots orthogonal to $\mathcal{M}^{(2,4)}$ |
| $P^{(4)}\left(\mathcal{M}^{(2,4)}\right)$ | all $(-4)$-roots orthogonal to $\mathcal{M}^{(2,4)}$ |
| $(X, \theta)$ | a K3 with involution $\theta$ |
| $X^\theta$ | the fixed locus of an involution |
| $P(X)_{+I}$ | the subset of exceptional classes of $(X, \theta)$ of type I |

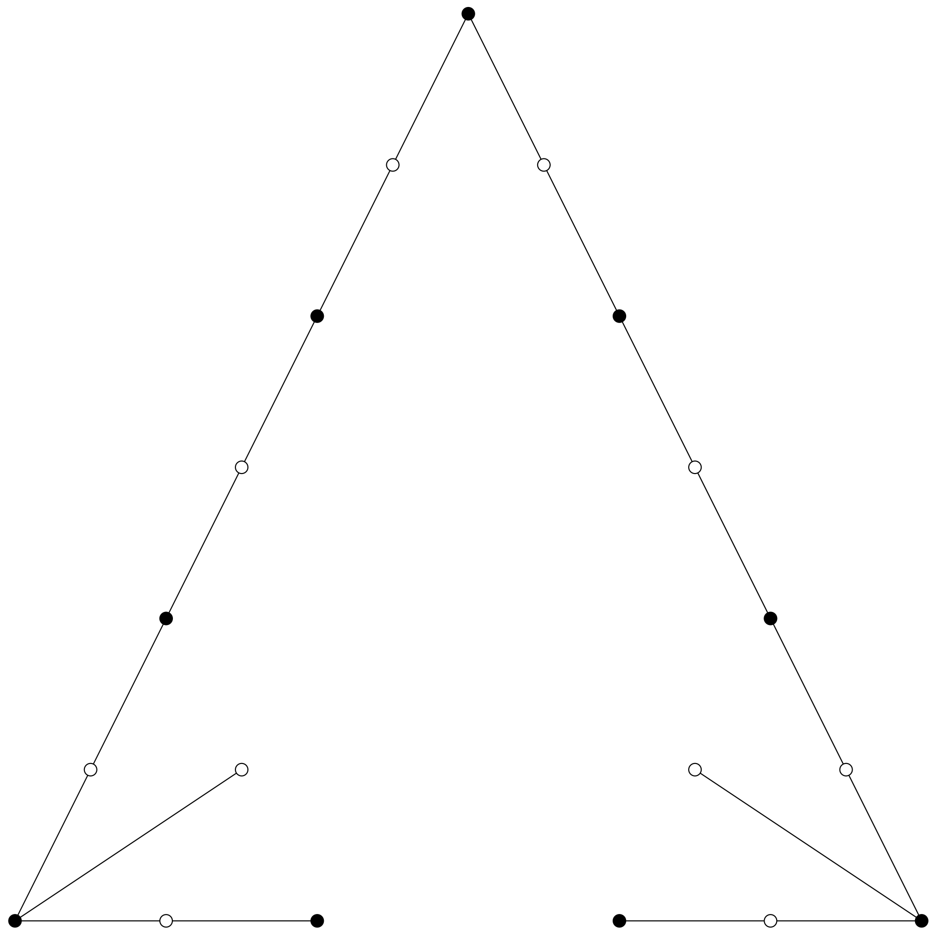
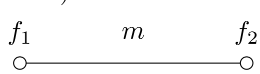


FIGURE 24. Coxeter-Vinberg diagram

### 11A. **Edge notation.**

- Vertices corresponding to different elements $f_1, f_2 \in P\left(\mathcal{M}^{(2,4)}\right)$ are not connected by any edge if $f_1 \cdot f_2 = 0$.
- Simple edges of weight $m$ (equivalently, by $m-2$ simple edges if $m > 2$ is small):

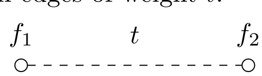



$$\Longrightarrow \qquad \frac{2f_1 \cdot f_2}{\sqrt{f_1^2 f_2^2}} = 2\cos\frac{\pi}{m}, \quad m \in \mathbb{N}$$

- Thick edges:


$$\Longrightarrow \qquad \frac{2f_1 \cdot f_2}{\sqrt{f_1^2 f_2^2}} = 2$$

- Broken edges of weight $t$:




$$\Longrightarrow \qquad \frac{2f_1 \cdot f_2}{\sqrt{f_1^2 f_2^2}} = t > 2$$

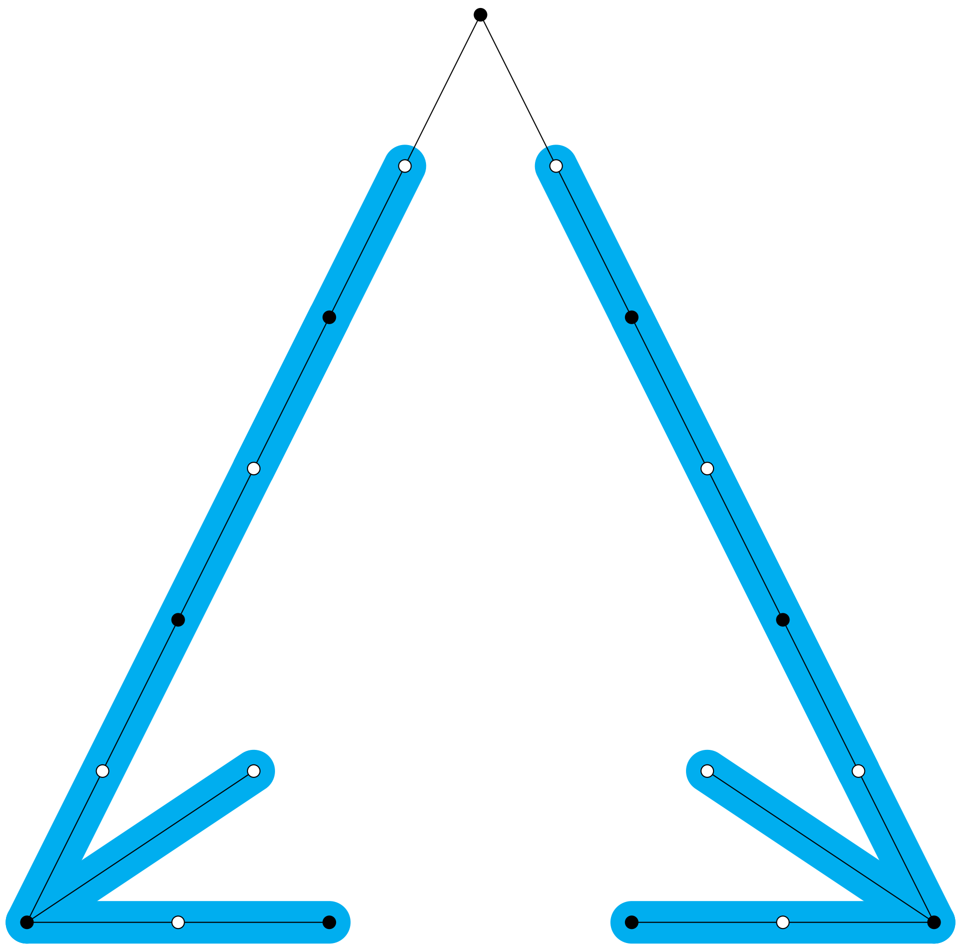


FIGURE 25. Caption

- A vertex corresponding to $f \in P^{(4)}\left(\mathcal{M}^{(2,4)}\right)$ is black:

$$\overset{f}{\bullet} \qquad \Longrightarrow \qquad f^2 = -4?$$

- It is white if $f \in P^{(2)}\left(\mathcal{M}^{(2,4)}\right)$:

$$\overset{f}{\circ} \qquad \Longrightarrow \qquad f^2 = -2?$$

- It is double-circled white if $f \in P(X)_{+I}$ (i.e. it corresponds to the class of a rational component of $X^{\theta}$).

$$\overset{f}{\circledcirc} \qquad \Longrightarrow \qquad ??$$

Interpreting this geometrically: consider the cycle of $2\bar{k}$ white vertices cycling between plain and double-circled:

- See [**?**]

- Each edge on the outer cycle corresponds to $\mathbb{P}^2$
- Single circle vertices (with odd $i$) corresponds to a line in $\mathbb{P}^2$
- Double-circled vertices (with even $i$) correspond to conics on the $\mathbb{P}^2$
- Explicit example worked out in [**?**, §5].
- It seems like that from the Coxeter diagram, you draw the fan of a toric surface, you compute the charge, and then, if this is not 24, you fix it by blowing up some non-torus fixed points along the toric boundary.

## 12. Hermitian symmetric domains

### 12A. **Cusp correspondence for Hermitian symmetric domains.**

**Definition 12.1** (Symmetric spaces)**.** A **locally symmetric space** is a connected Riemannian manifold such that every $x \in M$ is the fixed point of some involution $\gamma_x \in \operatorname{Isom}(U_x)$, the real algebraic Lie group of holomorphic automorphisms of an open subset $U_x \subseteq M$, which acts by $-1$ on $T_xM$. Equivalently, the covariant derivative of the curvature tensor vanishes, which is analogous to a constant curvature condition. It is a **symmetric space** if $\gamma_x$ extends from $\operatorname{Isom}(U)$ to $\operatorname{Isom}(M)$.

If $M$ is a symmetric space, then $M \cong G/K$ where $G = \operatorname{Isom}(M)$ is its group of isometries and $K \coloneqq \operatorname{Stab}_G(x)$ is the stabilizer of any point $x \in M$. We say a manifold $M$ is **homogeneous** if $M \cong G/K$ for some $G$ and $K$.

**Remark 12.2.** Idea: Hermitian symmetric manifolds are manifolds that are homogeneous spaces such that every point has an involution preserving the Hermitian structure. These were first studied by Cartan in the context of Riemannian symmetric manifolds. They show up often as orbifold covers of moduli spaces, e.g. polarized abelian varieties (with or without level structure), polarized K3 surfaces, polarized irreducible holomorphic symplectic manifolds, etc. There is a structure theorem: any Hermitian symmetric manifold $M$ decomposes as a product $M \cong \mathbb{C}^n/\Lambda \times M_c \times M_{nc}$ where $\Lambda$ is some lattice, $M_c$ is an HSM of compact type, and $M_{nc}$ is an HSM of non-compact type. Every HSM of compact type is a flag manifold $G/P$ for $G$ a semisimple complex Lie group and $P$ is a parabolic subgroup. Every HSM of non-compact type admits a canonical so-called Harish-Chandra embedding whose image is a bounded symmetric domain $D \subseteq \mathbb{C}^N$ for some $N$. Moreover, every HSM of non-compact type admits an associated Borel embedding into an associated HSM of compact type called its compact dual. Moreover, there is a Lie-theoretic classification of HSMs of compact and non-compact type – they are all of the form $G/K$ for $G$ a simple compact (resp. non-compact) Lie group and $K \leq G$ is a maximal compact subgroup with center isomorphic to $S^1 \cong \mathrm{U}_1(\mathbb{C})$.

By the Harish-Chandra embedding, non-compact HSMs can be realized as bounded domains $D \subseteq \mathbb{C}^N$ and admit a compactification by taking the closure $\bar{D} \supseteq D$ in $\mathbb{C}^N$. There is a partition of $\bar{D}$ by an equivalence relation related to being connected through chains of holomorphic discs, and each equivalence class is called a boundary component of $D$. Boundary components are in bijection with their normalizer subgroups, which are precisely maximal parabolic subgroups of $G \coloneqq \operatorname{Aut}(D)$.

A **Hermitian symmetric domain** is a Hermitian symmetric space of non-compact type.

**Example 12.3.** Some very basic examples of Hermitian symmetric manifolds:

- Tori $\mathbb{C}/\Lambda$ with Hermitian structure $g = dxdx + dydy$ induced from $\mathbb{R}^2$ (constant zero curvature).
- The upper half space $\mathcal{H}^1$ with Hermitian structure the hyperbolic metric $g = y^{-2}dxdy$ (constant negative curvature)
- $\mathbb{P}^1(\mathbb{C})$ with the Fubini-Study metric (constant positive curvature).

More advanced examples of symmetric spaces:

- $\mathbb{E}^n$, Euclidean space $\mathbb{R}^n$.
- $\mathbb{S}^n$, the spherical geometry,
- $\mathbb{H}^n$, hyperbolic space,
- $\mathrm{Sym}_{n\times n}^{>0}(\mathbb{R}) \leq \mathrm{SL}_n(\mathbb{R})$ the Riemannian manifold of positive-definite symmetric matrices with real entries
- $X$ defined in the following way: let $V$ be a Hermitian $\mathbb{C}-$module with Hermitian form $h$ of signature $(p,q)$ and let $X \subseteq \mathrm{Gr}_p(V)$ be the Grassmannian of $p-$dimensional subspaces $W$ such that $h|_W$ is positive definite.

We first record their isometry groups:

- $\mathrm{Isom}(\mathbb{E}^n) = \mathbb{R}^n \rtimes \mathrm{O}_n(\mathbb{R})$.
- $\mathrm{Isom}(\mathbb{S}^n) = \mathrm{O}_{n+1}(\mathbb{R})$
- $\mathrm{Isom}(\mathbb{H}^n) = \mathrm{O}_{n+1}^+(\mathbb{R})$ the index 2 subgroup of $\mathrm{O}_{n+1}(\mathbb{R})$ which preserves the upper sheet $(\mathbb{H}^n)^+$.[8]
- $\mathrm{Isom}(\mathrm{Sym}_{n\times n}^{>0}(\mathbb{R})) = \mathrm{SL}_n(\mathbb{R})$.
- $\mathrm{Isom}(X) = \mathrm{SU}_{p,q}(\mathbb{C})$?

Computing stabilizers of points, one can show

$$\mathbb{R}^n \cong \frac{\mathbb{R}^n \rtimes \mathrm{O}_n(\mathbb{R})}{\mathrm{O}_n(\mathbb{R})}$$

$$\mathbb{S}^n \cong \frac{\mathrm{O}_{n+1}(\mathbb{R})}{\mathrm{O}_n(\mathbb{R})}$$

$$\mathbb{H}^n \cong \frac{\mathrm{O}_{n+1}^+(\mathbb{R})}{\mathrm{O}_n(\mathbb{R})}$$

$$\mathrm{Sym}_{n\times n}^{>0}(\mathbb{R}) \cong \frac{\mathrm{SL}_n(\mathbb{R})}{\mathrm{SO}_n(\mathbb{R})}$$

$$X \cong \frac{\mathrm{SU}_{p,q}(\mathbb{C})}{\mathrm{SU}_p(\mathbb{C}) \times \mathrm{SU}_q(\mathbb{C})}$$

Note that taking $(p,q) = (1,1)$ yields $\mathbb{H}^2$.

In retrospect, maybe we don't need all of this background!

**Definition 12.4.**

A **Hermitian symmetric space** is a locally symmetric space $M$ which is additionally equipped with an integrable almost-complex structure whose Riemannian metric is Hermitian.

We say $M$ is **irreducible** if it is not the cartesian product of two symmetric Hermitian spaces; every irreducible such space is either $\mathbb{R}^n$ for some $n$ or a homogeneous space $G/K$ for $G$ a real Lie group and $K$ a maximal subgroup. We say $M$ is of **compact type** if $G$ is compact and $K$ is a maximal proper subgroup, and of

[8]Note that for $n = 1$, we can take the upper half-plane model which has isometry group $\mathrm{PSL}_2(\mathbb{R})$ or the disc model which has isometry group $\mathrm{PSU}_{1,1}(\mathbb{C})$. These are actually isomorphic as Lie groups.

**non-compact type** if $G$ is non-compact. If $M$ is an irreducible Hermitian symmetric domain of non-compact type, there is an open embedding $M \hookrightarrow \mathcal{D}_L \subseteq \mathbb{C}^n$ onto a bounded subset $\mathcal{D}$ of complex $n-$space, in which case we call $\mathcal{D}$ a **bounded Hermitian symmetric domain**.

The simplest example is the upper half plane $\mathbb{H} := G/K$ for $G = \mathrm{SL}_2(\mathbb{R})$ and $K = \mathrm{SO}_2(\mathbb{R})$, which is biholomorphic to the bounded domain $\Delta$ via the Cayley transformation, which is a homogeneous space for $(G, K) = (\mathrm{SU}_{1,1}, B)$ where $B$ is the subgroup of diagonal matrices. Note that $\mathbb{H} \cong \mathcal{H}_1$ is the Siegel upper half space of genus 1.

If $(V, q)$ is a real quadratic space where $V := L_{\mathbb{R}}$ for $L$ a lattice, we can define a corresponding domain

$$\mathcal{D}_L^{\pm} := \left\{ \mathbb{C}z \in \mathbb{P}(V_{\mathbb{C}}) \mid z^2 = 0,\, |z| > 0 \right\},$$

the set of lines spanned by isotropic vectors of positive Hermitian norm $|z| := z\bar{z}$ in $V_{\mathbb{C}}$. If $\mathrm{sgn}(L) = (2, n)$, so $L$ is hyperbolic, this has an irreducible component decomposition into two parts $\mathcal{D}_L^{\pm} = \mathcal{D}_L^{+} \coprod \mathcal{D}_L^{-}$ interchanged by conjugation $z \mapsto \overline{z}$. Each component is an irreducible Hermitian symmetric domain of type

$$(G, K) = (\mathrm{SO}(V) := \mathrm{SO}_{2,n}(\mathbb{R}), \mathrm{SO}_2(\mathbb{R}) \times \mathrm{SO}_n(\mathbb{R})).,$$

i.e. a Type IV domain for $\mathrm{SO}_{2,n}$. We let $\mathcal{D}_L := \mathcal{D}_L^{+}$ be a choice of one component and write $\mathrm{O}^{+}(L) \leq \mathrm{O}(L)$ for the subgroup which preserves $\mathcal{D}_L$ setwise. There is a distinguished divisor attached to $\mathcal{D}_L$, the **discriminant divisor**:

$$\mathcal{H}_L := \bigcup_{v \in R_2(L)} H_v \cap \mathcal{D}_L,$$

the hyperplane configuration defined by mirrors of roots. When Global Torelli is satisfied, there is a period map $\phi$ whose image is typically the complement of some hyperplane arrangement $\mathcal{H}$. In good cases, the relevant arrangement is precisely $\mathcal{H}_L$.

Note that $\mathcal{D}_L$ is isomorphic to a flag variety $G_{\mathbb{C}}/P$ for $P$ some parabolic subgroup, and thus the compact form $\widetilde{\mathcal{D}}_L$ is a projective algebraic variety containing $\mathcal{D}_L$.

We say $\mathcal{D}_L$ as above is a **Hermitian symmetric domain of orthogonal type** or a **type IV Hermitian symmetric domain** in Cartan's classification. The period domains of K3 and Enriques surfaces are examples of such Type IV domains for $1 \leq n \leq 19$.

**Definition 12.5.** Let $G$ be a simple linear algebraic group defined over $\mathbb{Q}$. We define

$$G(\mathbb{Z}) := \mathrm{GL}_n(\mathbb{Z}) \cap G(\mathbb{Q})$$

where we use the natural embedding of algebraic groups $G \hookrightarrow \mathrm{GL}_n$ over $\mathbb{Q}$. A subgroup $\Gamma \leq G(\mathbb{Q})$ is **arithmetic** if $\Gamma \cap G(\mathbb{Z})$ has finite index in both $\Gamma$ and $G(\mathbb{Z})$.

**Definition 12.6** (Parabolic subgroups)**.** Let $G$ be a linear algebraic group over $\mathbb{Q}$. We say $P \leq G$ is a parabolic subgroup if $G/P$ is a projective variety.

**Remark 12.7.** As the notation suggests, there are other types of irreducible Hermitian symmetric domains. The following are some typical examples of the form $\Gamma\backslash\Omega$ for various definitions of $\Omega$:

- Type III: Siegel modular varieties corresponding to $\Gamma \leq \mathrm{Sp}(\Lambda)$, the isometry group of a symplectic lattice, of rank $n \geq 3$.
- Type IV: Orthogonal modular varieties corresponding to $\Gamma \leq \mathrm{O}^+(\Lambda)$, a connected component of the isometry group of a lattice of signature $(2, n)$ for $n \geq 3$,
- Type $\mathrm{I}_{n,n}$: Hermitian modular varieties/Hermitian upper half spaces. These are attached to $\Gamma \leq \mathrm{U}(\Lambda)$ for $\Lambda$ a Hermitian form $q$ of signature $(n, n)$ with $n \geq 2$. The compact dual is the Grassmannian $\mathrm{Gr}_{n,2n}$.
- Type $\mathrm{II}_{2n}$: Quaternionic modular varieties/quaternionic upper half spaces. These are attached to $\Gamma \leq \mathrm{Sp}_{2n}(H)$ for $H$ Hamilton's quaternions, attached to a skew-Hermitian space of dimension $2n$ with $n \geq 2$. The compact dual is the orthogonal Grassmannian $\mathrm{O\,Gr}_{2n,4n}$

Where do $\mathcal{H}_g$ and $\Gamma\backslash\mathbb{H}^n$ fit in?

**Remark 12.8.** For $\Lambda$ a lattice of signature $(2, n)$, the Hermitian symmetric domain attached to $\Lambda$ is the following: define $Q \subseteq \mathbb{P}(\Lambda_{\mathbb{C}})$ be the quadric cut out by $(\omega, \omega) = 0$, then $\Omega_\Lambda$ is a choice of one of the two connected components of the open set $Q$ defined by $(\omega, \bar{\omega}) > 0$. Letting $\mathrm{O}^+(\Lambda) \leq \mathrm{O}(\Lambda)$ be the subgroup preserving the component $\Omega_\Lambda$ and $\Gamma \leq \mathrm{O}^+(\Lambda)$ be any finite index subgroup, we obtain

$$X_\Lambda(\Gamma) := \Gamma\backslash\Omega_\Lambda.$$

Embedding $\Omega_\Lambda$ in its compact dual, it has 0 and 1-dimensional boundary strata, corresponding to 1 and 2-dimensional isotropic subspaces of $\Lambda_{\mathbb{Q}}$. The BB compactification $\overline{X_\Lambda(\Gamma)}^{\mathrm{bb}}$ is the union of $\Omega_\Lambda$ and these rational boundary components, quotiented by the action of $\Gamma$, equipped with the Satake topology.

A toroidal compactification $\overline{X_\Lambda(\Gamma)}^{\mathrm{tor}}$ is specificed by a finite collection of suitable fans $\{F_I\}$, one for each 0-cusp (i.e. each $\Gamma-$orbit of isotropic lines $I$ in $\Lambda_{\mathbb{Q}}$).

For each $I$ there is a tube domain realization given by taking the linear projection from the boundary point, which defines an isomorphism

$$\Omega_\Lambda/U(I)_{\mathbb{Z}} \cong U \subseteq T_I := U(I)_{\mathbb{C}}/U(I)_{\mathbb{Z}}$$

an open subset of an algebraic torus, where $U(I)_{\mathbb{Z}} := \Gamma \cap U(I)_{\mathbb{Q}}$ and $U(I)_{\mathbb{Q}}$ is the unipotent part of $\mathrm{Stab}_{\mathrm{O}^+(\Lambda_{\mathbb{Q}})}(I)$. There is a canonical isomorphism $U(I)_{\mathbb{Q}} \cong (I^\perp/I) \otimes I$, and $F_I$ gives a polyhedral decomposition of the extended positive cone of $U(I)_{\mathbb{R}}$. Each $F_I$ defines a partial compactification $(\Omega_\Lambda/U(I)_{\mathbb{Z}})^{F_I}$ inside the torus embedding $T_I \hookrightarrow T_I^{F_I}$. The boundary points of this compactification which lie over the cusp corresponding to $I$ are comprised of a union of torus orbits $T_I \cdot \sigma$ for $\sigma \in F_I$ a cone which is not an isotropic ray.

The partial compactifications for the 1-cusps are completely canonical, so the overall compactification is defined by gluing onto the boundary of $X_\Lambda(\Gamma)$ certain natural quotients of all of these partial compactifications to obtain $\overline{X_\Lambda(\Gamma)}^{\mathrm{tor}}$. This yields a compact algebraic space which is proper over $\mathrm{Spec}\,\mathbb{C}$, and there is a natural morphism $\overline{X_\Lambda(\Gamma)}^{\mathrm{tor}} \to \overline{X_\Lambda(\Gamma)}^{\mathrm{bb}}$.

**Example 12.9.** Let $G := \mathrm{SL}_2$ defined over $\mathbb{Q}$ and let $\Gamma \leq \mathrm{SL}_2(\mathbb{Q})$ be an arithmetic subgroup. The (noncompact) modular curve attached to $\Gamma$ is

$$Y(\Gamma) := \Gamma\backslash\mathbb{H}^1.$$

In this case, rational boundary components are given by $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\} \subseteq \mathbb{P}^1(\mathbb{C})$, and a cusp of $Y(\Gamma)$ is a $\Gamma-$orbit in $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$, of which there are finitely many.

Adding them yields a compactification

$$X(\Gamma) \coloneqq \overline{Y(\Gamma)} \coloneqq Y(\Gamma) \cup \{\text{cusps}\}$$

topologized appropriately, where e.g. $\{\infty\}$ is one such cusp.

Note that one typically takes the following groups for moduli of elliptic curves with level structure:

- $Y(N) \coloneqq Y(\Gamma(N))$ where
$$\Gamma(N) \coloneqq \ker\left(\phi_N : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\right).$$
The level structure is a basis for $E[n]$.
- $Y_0(N) \coloneqq Y(\Gamma_0(N))$ where $\Gamma_0(N) \supseteq \Gamma(N)$ is the pullback $\phi_N^{-1}\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right)$.
The level structure is an identification $\mu_N \hookrightarrow E_{\text{tors}}$.
- $Y_1(N) \coloneqq Y(\Gamma_1(N))$ where $\Gamma_1(N)$ is the pullback $\phi_N^{-1}(1b01)$. The level structure is a point $p \in E$ of order $N$ in the group structure.

How parabolic subgroups appear here: for $G \coloneqq \mathrm{SL}_2$, parabolic subgroups are all conjugate to the subgroup $P$ of upper-triangular matrices, and $G(\mathbb{Q})/P(\mathbb{Q}) \cong \mathbb{P}^1(\mathbb{Q})$ parameterizes all such parabolic subgroups.

Why automorphic forms matter: consider $\Gamma \coloneqq \mathrm{SL}_2(\mathbb{Z})$. The graded ring of modular forms $\bigoplus_k M_k$ is graded-isomorphic to $\mathbb{C}[x, y]$ where $|x| = 4, |y| = 6$, and $\operatorname{Proj} \mathbb{C}[x, y] \cong \mathbb{P}^1(\mathbb{C})$. Letting $\{f_0, \cdots, f_N\}$ be a basis of $M_k$, we can write down a map

$$\begin{aligned} \phi_k : Y(\mathrm{SL}_2(\mathbb{Z})) &\to \mathbb{P}^n(\mathbb{C}) \\ z &\mapsto [f_0(z) : \cdots : f_N(z)] \end{aligned}$$

For $k = 12$ this separates points and tangent directions, giving a projective embedding. Explicitly, the morphism is

$$\phi_{12}(z) = [E_4(z) : E_4(z)^3 - E_6(z)^2] \approx j(z)$$

modulo some missing constants. In general, finding enough automorphic forms yields a projective embedding.

Would like to spell this out in terms of line bundles and linear systems too, in this easy case.

12B. **Misc.**

**Remark 12.10.** Let $L$ be a lattice of signature $(2, n)$ and the associated period domain $\Omega_L^{\pm} = \Omega_L^+ \amalg \Omega_L^-$. Let $\mathrm{O}(L)^+ \leq \mathrm{O}(L)$ be the finite index subgroup fixing $\Omega_L^+$, equivalently the subgroup of elements of spinor norm one. A modular variety of orthogonal type is a homogeneous space of the form $F_L(\Gamma) \coloneqq \Gamma \backslash \Omega_L^+$ for an arithmetic subgroup $\Gamma \leq \mathrm{O}(L_{\mathbb{Q}})^+$.

By general theory, such spaces admit BB compactifications $\overline{F_L(\Gamma)}^{\text{bb}}$ where rational maximal parabolic subgroups correspond to stabilizers of isotropic subspaces of $L_{\mathbb{Q}}$; since $\operatorname{sgn}(L) = (2, n)$ these are always isotropic lines or planes.

For period spaces of K3 surfaces, one takes $\Gamma \coloneqq \mathrm{O}(L_{2d}) \cap \ker\left(\mathrm{O}(L) \to \mathrm{O}(A_L)\right)$. Boundary strata correspond to central fibers of KPP models of Type II and Type III.

**Remark 12.11.** Let $L$ be a symplectic lattice of rank $2g$, ie.e. a free $\mathbb{Z}-$module with a nondegenerate alternating form $(-, -)$. Define the associated period space

$$D_L \coloneqq \left\{V \in \mathrm{Gr}_g(L_{\mathbb{C}}) \mid (V, V) = 0,\, i(V, \bar{V}) > 0\right\} \cong \mathrm{Sp}_{2g}(\mathbb{R})/\,\mathrm{U}_g(\mathbb{C}) \cong \mathcal{H}^g$$

which is a Hermitian symmetric domain of type III that can be identified with the Siegel upper half-space of dimension $g$. We can form the moduli space of PPAV as

$$\mathcal{A}_g \coloneqq \mathrm{Sp}_{2g}(\mathbb{Z})\backslash \mathcal{H}^g \cong \mathrm{Sp}_{2g}(\mathbb{Z})\backslash \mathrm{Sp}_{2g}(\mathbb{R})/\,\mathrm{U}_g(\mathbb{C})$$

Rational boundary components of $\overline{\mathcal{A}_g}^{\mathrm{bb}}$ correspond to $\Gamma \coloneqq \mathrm{Sp}_{2g}(\mathbb{Z})$ orbits of totally isotropic subspaces in $L_{\mathbb{Q}}$. Since $\Gamma$ acts transitively, such spaces are indexed by their dimension $i = 0, 1, \cdots, g$ and there is a stratification

$$\overline{\mathcal{A}_g}^{\mathrm{bb}} = \amalg_{k=0}^{g} \mathcal{A}_k \implies \partial \overline{\mathcal{A}_g}^{\mathrm{bb}} = \amalg_{k=0}^{g-1} \mathcal{A}_k$$

**Remark 12.12.** The BB compactification of a locally symmetric domain $D$: write $D = H/K$ as a homogeneous space where $H \coloneqq \mathrm{Hol}(D)^+$ and $K \leq H$ is a maximal compact subgroup. Then cusps in $\partial \overline{\Gamma\backslash D}^{\mathrm{bb}}$ correspond to rational maximal parabolic subgroups of $H$. To get boundary components: apply the Harish-Chandra embedding to $D$ to embed $HC : D \hookrightarrow D^{cd}$ and let $F_P \in \overline{HC(D)}$ be a boundary component. Its normalizer $N(F_P) \coloneqq \{g \in H \mid g(F_P) = F_P\} \leq H$ is a maximal parabolic in $H$. We say $F_P$ is **rational** if $N(F_P)$ can be defined over $\mathbb{Q}$. Since $\Gamma$ preserves such rational $F_P$, we can set $\partial D \coloneqq$ the disjoint union of all rational $F_P$ and set $\overline{\Gamma\backslash D}^{\mathrm{bb}} = \frac{D \amalg \partial D}{\Gamma}$.

12B.1. *Explicit realizations of symmetric spaces.*

**Remark 12.13.** The symmetric space associated with a Lie group $G$ is in some sense the most natural space $G$ acts on. For $G = \mathrm{O}_{p,q}(\mathbb{R})$, the symmetric space is $\mathrm{Gr}^+(\mathbb{R}^{p,q})$, the Grassmannian of maximal positive-definite subspaces of $\mathbb{R}^{p,q}$. The right choice of maximal compact subgroup here is $K \coloneqq \mathrm{O}_p(\mathbb{R}) \times \mathrm{O}_q(\mathbb{R})$, the subgroup fixing $\mathbb{R}^{m,0}$. When $(p,q) = (2,n)$, these symmetric spaces admit special descriptions. Note that $\mathrm{O}_{n+1}(\mathbb{R})$ is the group of isometries of $S^n$, so its projectivization $\mathrm{PO}_{n+1}(\mathbb{R})$ is the isometry group of an elliptic geometry. One can similarly obtain isometries of hyperbolic geometry:

Typo maybe.

- Start with $\mathbb{R}^{1,n}$
- Take the norm 1 vectors $H^{\pm} \coloneqq \left\{v \in \mathbb{R}^{1,n} \mid v^2 = 1\right\} = H^+ \amalg H^-$ to get a 2-sheeted hyperboloid; the pseudo-Riemannian metric on $\mathbb{R}^{1,n}$ restricts to a Riemannian metric on $H$.
- Take one sheet $H^+$; this is a model of $\mathbb{H}^n$

Indexing might be off here

The group of isometries of $H^+$ is now $\mathrm{PO}_{1,n}(\mathbb{R})$. Note that in $\mathrm{O}_{1,n}(\mathbb{R})$ there is an index 2 subgroup[9]

$$\mathrm{O}_{1,n}(\mathbb{R})^{\pm} = \left\{\gamma \in \mathrm{O}_{1,n}(\mathbb{R}) \mid \gamma(H^+) = H^+, \gamma(H^-) = H^-\right\}.$$

**Remark 12.14.** Forming the symmetric spaces for $\mathrm{O}_{2,n}(\mathbb{R})$: the maximal compact is $K = \mathrm{O}_2(\mathbb{R}) \times \mathrm{O}_n(\mathbb{R})$ and $\mathrm{O}_2(\mathbb{R})$ is similar enough to $\mathrm{U}_1(\mathbb{C})$ that we should expect the associated symmetric space to be Hermitian. It will be an open subset of a certain quadric:

- Start with $\mathbb{P}(\mathbb{C}^{2,n})$.
- Take the quadric of isotropic vectors $Y = \left\{z \in \mathbb{P}(\mathbb{C}^{2,n}) \mid z^2 = 0\right\}$.
- Take the open subset $U \coloneqq \{z \in Y \mid (z, \bar{z}) > 0\}$.

[9] Apparently, these are elements whose spinor norm equals their determinant.

Why this matches the previous description: write $z = x + iy$, then $x^2 = y^2 > 0$ and $(x, y) = 0$, so $V := \mathbb{R}x \oplus \mathbb{R}y$ are an orthogonal basis for a positive definite subspace of $\mathbb{R}^{2,n}$. Multiplying by a scalar only changes basis, so we essentially get a map $\mathbb{P}(U) \to \mathrm{Gr}^+(\mathbb{R}^{2,n})$ naturally. This symmetric space can also be identified with points $z \in \mathbb{C}^{1,n-1}$ with $\Im(z) \in C^+$, one of two cones of $\mathbb{R}^{1,n-1}$, realizing this as a tube domain generalizing $\mathbb{H}$.

## 13. Hyperbolic geometry

### 13A. Hyperbolic lattices.

Note: some of this mixes conventions, need to fix later.

**Warning 13.1.** There is a significant gap in the AG literature vs the physics literature for the terminology for hyperbolic spaces, and the traditional AG terminology can be "wrong" in some senses. For example, the AG literature will typically call $\{v \in L_{\mathbb{R}} \mid v^2 > 0\}$ a "light cone", but this is not quite correct: the actual *light cone* in general relativity is $\{v \in L_{\mathbb{R}} \mid v^2 = 0\}$. The following picture is the usual mnemonic:

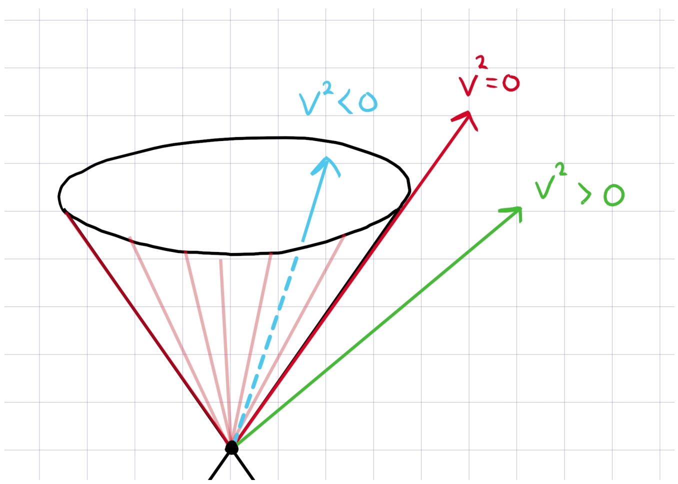



Figure 26. $\{v^2 = 0\}$ is the light cone, its interior is timelike and exterior spacelike.

**Definition 13.2** (Hyperbolic lattices)**.** An indefinite lattice $L$ is a **hyperbolic lattice**[10] if $\mathrm{sgn}(L) = (1, n_-)$ or $(n_+, 1)$ for some $n_-, n_+ \geq 1$. By convention, by twisting $L$ to $L(-1)$ if necessary, we assume hyperbolic lattices have signature $(1, n)$. In this convention, the single positive-definite direction is referred to as **timelike**, and the remaining directions are **spacelike**.

**Definition 13.3** (Time/light/spacelike vectors)**.** Let $L$ be a hyperbolic lattice of signature $(1, n)$. We say a vector $v \in L_{\mathbb{R}}$ is

- **timelike** if $v^2 < 0$,
- **lightlike** or **isotropic** if $v^2 = 0$.
- **spacelike** if $v^2 > 0$,

More generally, a subspace $W \subseteq \mathbb{E}^{1,n}$ with the restricted form $(-,-)_W$ is

[10] Also called a **Lorentzian lattice**.

- **timelike** if $(-,-)_W$ is negative-definite, or is indefinite and non-degenerate,
- **lightlike** or **isotropic** if $(-,-)_W$ is degenerate, or
- **spacelike** if $(-,-)_W$ is positive-definite.

Define

$$\begin{aligned} L^{<0} &:= \left\{v \in L_{\mathbb{R}} \mid v^2 < 0\right\} && \text{The timelike regime} \\ L^{=0} &:= \left\{v \in L_{\mathbb{R}} \mid v^2 = 0\right\} && \text{The lightlike regime} \\ L^{>0} &:= \left\{v \in L_{\mathbb{R}} \mid v^2 > 0\right\} && \text{The spacelike regime} \end{aligned}$$

**Remark 13.4.** [**?**] refers to the non-spacelike regime $L^{\geq 0} := \left\{v \in L_{\mathbb{R}} \mid v^2 \geq 0\right\}$ as the **round cone**; this is used for a model over $\overline{\mathbb{H}^n}$ with ideal points included, and is often used as the support of a semifan for a semitoroidal compactification.

**Definition 13.5** (Past and future light cones)**.** Let $L$ be a hyperbolic lattice of signature $(1, n)$. The spacelike regime $L^{>0}$ of $L$ has an irreducible component decomposition

$$L^{>0} := \left\{v \in L_{\mathbb{R}} \mid v^2 > 0\right\} = C_L^+ \amalg C_L^-,$$

whose components we refer to as the **future light cone** and **past light cone** of $L$ respectively, and can be distinguished by the sign of the coordinate in the negative-definite direction:

$$C_L^+ := \left\{v \in L^{>0} \mid v_0 > 0\right\}, \qquad C_L^- := \left\{v \in L^{>0} \mid v_0 < 0\right\}.$$

We write their closures in $L_{\mathbb{R}}$ as $\overline{C_L^+}$ and $\overline{C_L^-}$ respectively, and write $C_L := C_L^+$ for a fixed choice of a **future** light cone and $\overline{C_L}$ for its closure.

13B. **Models of hyperbolic space.**

**Definition 13.6** (Euclidean upper-half space)**.** The upper-half space in $\mathbb{E}^n$ is

$$\mathbb{E}^n_+ := \left\{(x_1, \cdots, x_n) \in \mathbb{E}^n \mid x_1 > 0\right\}.$$

**Definition 13.7** (Minkowski space)**.** The $n-$**dimensional Minkowski space** $\mathbb{E}^{1,n}$ is the real vector space $\mathbb{R}^{n+1}$ equipped with a bilinear form of signature $(1, n)$ which can be explicitly written as

$$vw := -v_0 w_0 + \sum_{i=1}^n v_i w_i$$

with the associated quadratic form

$$v^2 := Q(v) := -v_0^2 + \sum_{i=1}^n v_i^2.$$

This induces a metric

$$\rho(v, w) := \operatorname{arccosh}(-vw).$$

**Remark 13.8.** If $L$ is hyperbolic of signature $(1, n)$ then $L_{\mathbb{R}} \cong \mathbb{E}^{1,n}$ is a Minkowski space of dimension $n + 1 = \operatorname{rank}_{\mathbb{Z}} L$.

13B.1. *Half-plane models.*

**Definition 13.9** (de Sitter space and light cone of a lattice)**.** Let $L$ be a hyperbolic lattice and consider the squaring functional

$$\begin{aligned} f_L : L_{\mathbb{R}} &\to \mathbb{R} \\ v &\mapsto v^2. \end{aligned}$$

One can show that $\pm 1$ are regular values of $f_L$ and thus define two canonical "hyperbolic unit spheres" which are regular surfaces. We define the **de Sitter space of $L$** as

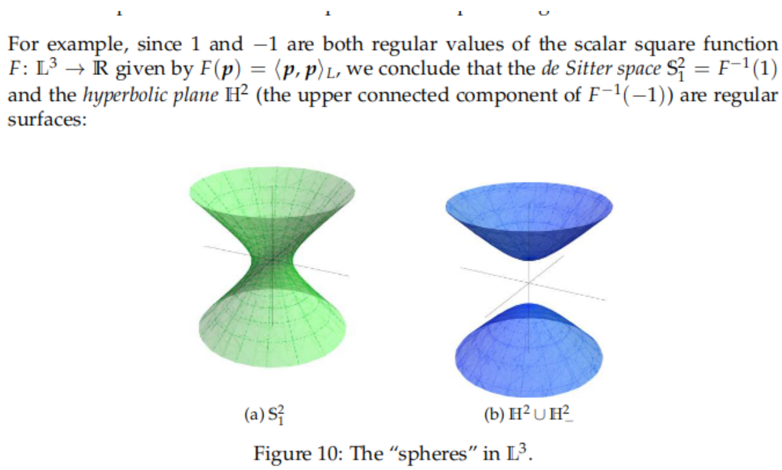
$$\mathrm{dS}_L \coloneqq f_L^{-1}(1) = \left\{ v \in L_{\mathbb{R}} \mid v^2 = 1 \right\} \subseteq L^{>0}$$

in the spacelike regime and the **unit hyperboloid of $L$** as the two-sheeted hyperboloid

$$H_L \coloneqq f_L^{-1}(-1) = \left\{ v \in L_{\mathbb{R}} \mid v^2 = -1 \right\} \subseteq L^{<0}$$

in the timelike regime.

**Example 13.10.** Figure 27 shows the de Sitter space and unit hyperboloid for a lattice $L$ of signature $(2,1)$ in $\mathbb{E}^{2,1}$, visualized in $\mathbb{R}^3$.


For example, since 1 and $-1$ are both regular values of the scalar square function $F\colon \mathbb{L}^3 \to \mathbb{R}$ given by $F(\boldsymbol{p}) = \langle \boldsymbol{p}, \boldsymbol{p} \rangle_L$, we conclude that the *de Sitter space* $\mathbb{S}^2_1 = F^{-1}(1)$ and the *hyperbolic plane* $\mathbb{H}^2$ (the upper connected component of $F^{-1}(-1)$) are regular surfaces:

(a) $\mathbb{S}^2_1$ (b) $\mathbb{H}^2 \cup \mathbb{H}^2_-$

Figure 10: The "spheres" in $\mathbb{L}^3$.

FIGURE 27. The hyperbolic unit spheres: the de Sitter space and light cone for $\mathbb{E}^{2,1}$.

**Definition 13.11** (Half-plane model/Lobachevsky space of a lattice)**.** Let $L$ be a hyperbolic lattice. The **half-plane model of $\mathbb{H}^n$ associated to $L$** or **Lobachevsky space of $L$** is the unit hyperboloid of $L$ intersected with its future light cone,

$$\mathbb{L}^n_L \coloneqq H_L \cap C_L \coloneqq \left\{ v \in L_{\mathbb{R}} \mid v^2 = -1, v_0 > 0 \right\},$$

given the metric restricted from $L_{\mathbb{R}} \cong \mathbb{E}^{n,1}$. This more simply be described as the future sheet of the unit hyperboloid $H_L$, using the irreducible component decomposition

$$H_L = H_L^+ \amalg H_L^- = \{ v \in H_L \mid v_0 > 0 \} \amalg \{ v \in H_L \mid v_0 < 0 \}$$

and setting $\mathbb{L}^n_L \coloneqq H_L^+$.

**Remark 13.12.** Note that $H_L^+$ is in the timelike regime. This gives a model of the hyperbolic space $\mathbb{H}^n$ which we often denote $\mathbb{H}_L$ when we do not fix a specific choice of model, or simply by $\mathbb{H}^n$ when the dependence on $L$ is not important.

**Remark 13.13** (The isometry group of hyperbolic spaces)**.** It can be shown that the isometries of the timelike regime $L^{<0}$ are restrictions of isometries of the ambient Minkowski space $\mathbb{E}^{1,n}$, and thus

$$\mathrm{Isom}(L^{<0}) \cong \mathrm{Isom}(\mathbb{E}^{1,n}) \cong \mathrm{O}_{1,n}(\mathbb{R}).$$

Using the half-plane model, we can thus naturally identify

$$\mathrm{Isom}(\mathbb{L}^n) \cong \mathrm{O}_{1,n}^+(\mathbb{R}) \coloneqq \mathrm{Stab}_{\mathrm{O}_{1,n}(\mathbb{R})}(C_L),$$

the index 2 subgroup which stabilizes the future light cone $C_L^+$ of $L$. These are precisely the isometries of $\mathbb{E}^{1,n}$ of positive spinor norm.

13B.2. *Ball models.*

**Definition 13.14** (The Poincaré ball model)**.** Let $L$ be a hyperbolic lattice. The **Poincaré ball model of $\mathbb{H}^n$ associated to** $L$ is defined as

$$\mathbb{B}_L^n \coloneqq \mathbb{P}(L^{<0}),$$

the projectivization of the timelike regime of $L$, where

$$\begin{aligned} \mathbb{P}(-) : \mathbb{E}^{1,n} \setminus \{x_n \neq 0\} &\to \mathbb{E}^n \\ (x_0, \cdots, x_{n-1}, x_n) &\mapsto \left(\frac{x_0}{x_n}, \cdots, \frac{x_{n-1}}{x_n}\right). \end{aligned}$$

In this model, there is a natural compactification $\overline{\mathbb{H}^n}$ in $\mathbb{P}(S^n)$ such that the interior is given by $\mathbb{B}_L^n$ as above and the boundary by $\partial\overline{\mathbb{H}^n} = \mathbb{P}(L^{=0})$, i.e. ideal points correspond to (the projectivization of) the lightlike regime.

**Remark 13.15** (An alternative construction)**.** It can be explicitly constructed by considering the future light cone $C_L$ described in Theorem 13.5. Letting $\mathbb{R}_{>0}$ act on $L_\mathbb{R} \cong \mathbb{E}^{1,n}$ by scaling along the timelike direction (i.e. in the coordinate $v_0$), the ball model can be formed as the quotient

$$\mathbb{B}_L^n \cong C_L/\mathbb{R}_{>0} \subset \mathbb{P}(\mathbb{S}^n).$$

**Remark 13.16.** The advantage of $\mathbb{B}_L^n$ over $\mathbb{L}_L^n$ is that the former provides a natural compactification in $\mathbb{P}(\mathbb{S}^n)$. Moreover, it can be easier to work with hyperplanes in the ball model: let $\pi : \mathbb{E}^{1,n} \to \mathbb{P}(\mathbb{S}^n)$ be the natural projection, then every hyperplane $H_v \coloneqq v^\perp$ for $v \in \mathbb{B}_L^n$ is of the form

$$H_v = \{\pi(x) \mid x \in C_L,\, xv = 0\}$$

One can also concretely interpret the bilinear form geometrically in the ball model in the following way:

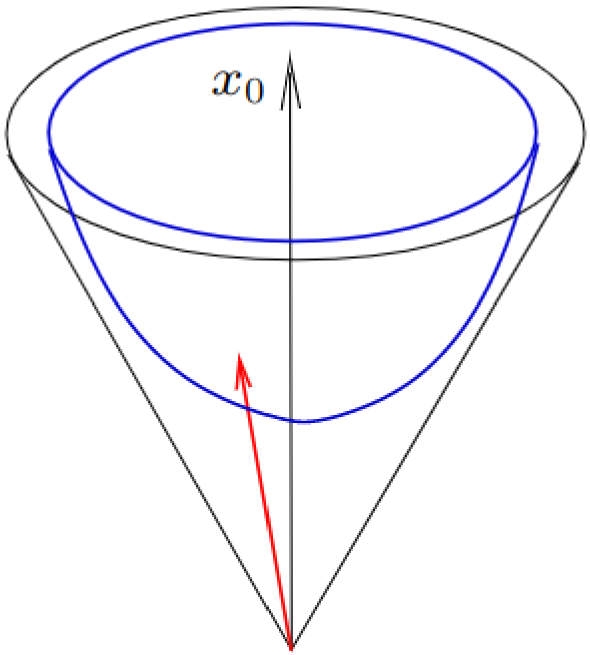
$$\begin{aligned} H_v \pitchfork H_w &\implies |vw| < 1 \implies -vw = \cos(\angle(H_v, H_w)) \\ H_v \parallel H_w &\implies |vw| = 1 \implies -vw = \cos(\angle(H_v, H_w)) \\ H_v \,)( \, H_w &\implies |vw| > 1 \implies -vw = \cosh(\rho(H_v, H_w)), \end{aligned}$$

where $\rho$ is the hyperbolic metric described in Definition 13.7.

**Remark 13.17.** $\mathrm{Isom}(\mathbb{B}^n) = \mathrm{PO}_{1,n}(\mathbb{R})$.

13B.3. *Ideal points.*

**Remark 13.18.** Let $H_L \cong \mathbb{H}^n$ be a model of hyperbolic space associated to a hyperbolic lattice $L$ of signature $(1,n)$. Boundary points $\partial\overline{H_L}$ correspond to ideal points in $\mathbb{H}^n$, i.e. points "at infinity", which in turn correspond to 1-dimensional isotropic subspaces of $L$.




In this model, points in $\mathbb{H}^n$ are points in the interior of the cone and on the hyperboloid. Moreover points on $\partial\overline{\mathbb{H}^n}$ correspond to points on the surface of the cone:

$$\partial\overline{\mathbb{H}^n} \cong \{v = (v_0, \cdots, v_{n+1}) \in L_{\mathbb{R}} \mid v_0 > 0\} \cap \left\{v \in L_{\mathbb{R}} \mid v^2 = 0\right\}.$$

We interpret $uv = -\cos(\angle(H_u H_v))$, so $uv = -1$ means $H_u \cap H_v \in \partial\overline{\mathbb{H}}^n$, i.e. they are "parallel" planes. Hyperplanes in $\mathbb{H}^n$ correspond to branches of hyperbolas obtained by slicing the hyperboloid by a plane in $L_{\mathbb{R}}$.

2

**Remark 13.19.** Define Minkowski space as $\mathbb{E}^{1,n}$, which is $\mathbb{R}^n$ with the form $vw = v_0 w_0 - \sum v_i w_i$. Define Lobachevsky space $\mathbb{L}^n$ as the hyperboloid model of hyperbolic space, a certain "hyperbolic unit sphere":

$$\mathbb{L}^n := \left\{v \in \mathbb{E}^{1,n} \mid v^2 = 1, v_0 > 1\right\}.$$

The geodesic curves are precisely intersections of the form $H_2 \cap \mathbb{L}^n$ where $H_2 \in \mathrm{Gr}_2(\mathbb{R}^{n+1})$ is a standard 2-plane passing through the origin in the ambient space. The hyperbolic metric on $\mathbb{L}^n$ is gotten by computing the length in the standard metric in $\mathbb{R}^{n+1}$ of any geodesic curve between two points. The associated Poincare ball model is contained in the standard Euclidean ball $\mathbb{B}^n \subset \mathbb{R}^{n+1}$ and is the projection of $\mathbb{L}^n$ onto the hyperplane $\{x_0 = 0\} \subset \mathbb{R}^{n+1}$ using rays passing through $(-1, 0, 0, \cdots, 0)$. Explicitly, the projection is

$$\phi : \mathbb{L}^n \to \mathbb{B}^n$$
$$(v_0, \cdots, v_n) \mapsto \frac{1}{1+v_0}(v_1, \cdots, v_n)$$

Geodesics are now straight lines through the origin or arcs of Euclidean circles intersecting $\partial\mathbb{B}^n$ orthogonally. Define the hyperbolic upper-half-space as

$$\mathbb{H}^n := \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}$$

which is obtained by taking inversions through certain spheres centered on $\partial\mathbb{B}^n$. Geodesics are now straight lines orthogonal to $\partial\mathbb{H}^n$ or half-circles centered on $\partial\mathbb{H}^n$.

## 13C. Root Systems.

**Definition 13.20** (Primitive vectors)**.** Let $L$ be any lattice. A finite set $S := \{s_1, \cdots, s_n\} \subseteq L$ of elements in $L$ is **primitive** if $S$ is $\mathbb{R}-$linearly independent and $L \cap \mathbb{R}S = \bigoplus_{i=1}^n Ls_i$, i.e. no $s_i$ can be replaced with a small vector in the same 1-dimensional subspace which is also in $L$. A primitive set of size one is called a **primitive element**, and we write $L_{\text{prim}}$ for the set of such.

**Definition 13.21** (Roots and $k-$roots in lattices)**.** Let $L$ be any lattice. For $k \in \mathbb{Z}_{>0}$, define the set of $k-$**roots in** $L$ as

$$\Phi^k(L) := \left\{v \in L_{\text{prim}} \mid v^2 = k,\, 2(v, L) \subseteq k\mathbb{Z}\right\}$$

A **root** is by definition a $2-$root. We write the set of roots in $L$ as $\Phi^2(L)$, and the complete set of roots as

$$\Phi^\infty(L) := \bigcup_{k \geq 1} \Phi^k(L).$$

**Remark 13.22.** In the theory of 2-elementary lattices, the roots consist of all $(-2)-$vectors along with any $(-4)-$vector $v$ with $\operatorname{div}(v) = 2$.

**Definition 13.23** (Reflections)**.** Let $L$ be any lattice and $L_\mathbb{R}$ its associated $\mathbb{R}-$module. An element $s \in \operatorname{GL}(L_\mathbb{R})$ is a **reflection** if there exists a vector $v \in L_\mathbb{R}$ and an $\mathbb{R}-$linear functional $f \in \operatorname{Hom}_\mathbb{R}(L_\mathbb{R}, \mathbb{R})$, both depending on $s$,

$$s(x) = x - f(x)v \quad \forall x \in L_\mathbb{R}, \qquad f(v) = 2.$$

Concretely, $s$ is an isometry of $L_\mathbb{R}$ which pointwise fixes a hyperplane and is an involution satisfying $\det(s) = -1$. Every reflection can be written in the form

$$s(u) = s_v(u) = u - \frac{uv}{v^2/2}v$$

for some $v^2 \neq 0$ in $L_\mathbb{R}$ determined up to scaling. The reflection in $v$ is only well-defined when $2\operatorname{div}(v) \in v^2\mathbb{Z}$ where $\operatorname{div}(v)$ is the divisibility of $v$ defined in Theorem 17.11. The **reflection hyperplane** associated to $s$ is the fixed subspace

$$H_v := \ker(f) = \ker(\mathrm{id} - s) \cong v^\perp.$$

**Definition 13.24** (Mirrors in hyperbolic lattices)**.** Any root $v \in \Phi^2(L)$ defines a reflection $s_v$ through the mirror

$$H_v := v^\perp := \left\{x \in C_L^+ \mid xv = 0\right\}.$$

If $H_v$ is the reflection hyperplane of a root, we say it is a **mirror** in $L$. Note that $H_v$ is nonempty if and only if $v^2 < 0$.

**Definition 13.25** (Weyl group)**.** Let $L$ be any lattice. The **Weyl group of** $L$ is defined as the group generated by reflections in 2−roots,

$$W(L) := \left\{ s_v \mid v \in \Phi^2(L) \right\} \leq \mathrm{O}_L(\mathbb{R})$$

**Definition 13.26** (The discriminant locus)**.** For $L$ a hyperbolic lattice, define the **discriminant locus of** $L$ as the union of all mirrors of 2−roots,

$$D(L) := \bigcup_{v \in \Phi^2(L)} v^\perp := \bigcup_{v \in \Phi^2(L)} H_v.$$

**Definition 13.27** (Weyl chambers)**.** The **chamber decomposition** of $C_L$ is defined as

Forgot to write down what is $C_L$.

$$C_L^\circ := C_L \setminus D(L) = C_L \setminus \left( \bigcup_{\delta \in \Phi^2(L)} \delta^\perp \right),$$

the complement of all mirrors. This further decomposes into connected components called **Weyl chambers**: fixing a chamber $P$, there is a decomposition into orbits

$$C_L^\circ = \amalg_{s_v \in W(L)} s_v(P).$$

**Remark 13.28.** Any Weyl chamber $P$ is a simplicial cone, so the orbit decomposition yields a decomposition of $C_L^\circ$ into simplicial cones. Since $W$ acts on the set of Weyl chambers $\pi_0 C_L^\circ$ simply transitively and the closure $\overline{P}$ of any chamber is a fundamental domain for this action, there is a homeomorphism $\overline{P} \cong C_L^\circ / W$.

**Definition 13.29** (Fundamental chamber???)**.** Let $P$ be a Weyl chamber of $L$, define

$$\Phi^2(L)^+ := \left\{ v \in \Phi^2(L) \mid (v, P) > 0 \right\}$$
$$\Phi^2(L)^- := \left\{ v \in \Phi^2(L) \mid (v, P) < 0 \right\} = -\Phi^2(L)^+$$

which induces a decomposition

$$\Phi^2(L) = \Phi^2(L)^+ \amalg \Phi^2(L)^-.$$

**Remark 13.30.** Thus $P$ can be written as

$$P = \left\{ v \in C_L \mid (v, \Phi^2(L)^+) > 0 \right\} = \{\}.$$

This realizes $P$ as an intersection of positive half-spaces and thus as a polytope.

**Definition 13.31** (Walls)**.** Let $\overline{P}$ be the closure in $L_\mathbb{R}$ of $P$. We say a mirror $H_v \subseteq L_\mathbb{R}$ for $v \in \Phi^2(L)^+$ is a **wall of** $P$ if $\mathrm{codim}_{L_\mathbb{R}}(H_v \cap \overline{P}) = 1$.

**Definition 13.32** (Simple systems)**.** Let $P$ be a Weyl chamber of $P$ and let

$$\Pi(L, P) := \left\{ v \in \Phi^2(L) \mid H_v \text{ is a wall of } P \right\}$$

be the set of walls of $P$. We can more economically define $P$ by

$$P = \{ v \in C_L \mid (v, \Pi(L, P)) > 0 \},$$

where no inequality is redundant. Moreover, $(P, \Pi(L, P))$ forms a Coxeter system, and $\overline{P}$ is a fundamental domain for $W(L) \curvearrowright L_\mathbb{R}$.

Todo, messed up notation here a bit.

**Definition 13.33** (Chambers and $\mathrm{O}^+(L)$)**.** The connected components of

$$V_L^+ := \left\{x \in L^\pm \mid (\Phi^2(L), x) \neq 0\right\}$$

are called **chambers** of $L$. Any positive isometry preserves $L^+$ and $L^-$ set-wise, motivating the definition of the **group of positive isometries of** $L$

$$\mathrm{O}^+(L) := \left\{\gamma \in \mathrm{O}(L) \mid \gamma(L^+) = L^+, \gamma(L^-) = L^-\right\}$$

**Definition 13.34** (Positive isometries)**.** We say an isometry $\gamma \in \mathrm{O}(L)$ is **positive** if it preserves a chamber (i.e. a connected component of $V_L^+$)

**Definition 13.35** (Roots, root systems, root lattices)**.** A vector $v \in L$ is a **root** if $v^2 = -2$[11], and we write $\Phi(L)$ for the set of roots in $L$. If $L$ is negative definite and $L = \mathbb{Z}\Phi(L)$[12], we say $L$ is a **root lattice**. Any root lattice decomposes as a direct sum of root lattices of ADE type.

**Definition 13.36** (Weyl group)**.** The **Weyl group** of $L$ is the maximal subgroup of the orthogonal group of $L$ generated by hyperplane reflections in roots,

$$W(L)_2 := \langle s_v \mid v \in \Phi(L)\rangle_{\mathbb{Z}} \leq \mathrm{O}(L).$$

One can similarly define the group of reflections in *all* vectors,

$$W(L) := \langle s_v \mid v \in L\rangle_{\mathbb{Z}} \trianglelefteq \mathrm{O}(L).$$

Since conjugating a reflection by any automorphism is again a reflection, this is a normal subgroup. If $L$ is a hyperbolic lattice, we replace $\mathrm{O}(L)$ in the above definition by $\mathrm{O}^+(L)$, the isometries that preserve the future light cone.

**Definition 13.37** (Mirrors/walls and chambers)**.** The **mirror** or **wall** associated with a root $v \in \Phi(L)$ is the hyperplane $H_v := v^\perp$. As $v$ ranges over $\Phi(L)$, these partition $L_{\mathbb{R}}$ into subsets called **chambers**. The Weyl group acts on $L_{\mathbb{R}}$ by isometries and acts simply transitively on chambers, and we often distinguish a fundamental domain for this action called the **fundamental chamber**. We write $D_L$ for the closure in $L_{\mathbb{R}}$ of a fundamental chamber. A **cusp** of $L$ is a primitive isotropic lattice vector $e \in D_L \cap L$.

## 13D. General Period Domains.

Very useful convention: $\Omega^S$ involves $S^\perp$, while $\Omega_S$ involves just $S$.

**Remark 13.38.** Define $G^L := G_{L^\perp}$ for $G$ any algebraic group determined by $L$.

Let $L \leq \mathrm{II}_{3,19}$ be a sublattice of signature $(1, r-1)$ so $\mathrm{sgn}(L^\perp) = (2, 19-r+1)$. One can always form the period domain corresponding to $L-$polarized K3 surfaces as

$$\Omega^L := \Omega_{L^\perp} := \left\{x \in (L^\perp)_{\mathbb{C}} \mid x^2 = 0, x\overline{x} > 0\right\},$$

The period domain can be described as a Hermitian symmetric space:

$$\Omega^L \cong \frac{\mathrm{SO}^L(\mathbb{R})}{\mathrm{SO}_2(\mathbb{R}) \times \mathrm{SO}_{20-r}(\mathbb{R})}$$

For any arithmetic subgroup $\Gamma \leq \mathrm{O}^L(\mathbb{R})$ there is a complex-analytic isomorphism

$$\Gamma \backslash \Omega^L \cong \left(\Gamma \cap \mathrm{SO}^L(\mathbb{R})\right) \backslash \Omega^L.$$

[11] One occasionally calls any time-like vector $v^2 < 0$ a "root", in which case distinguishes between e.g. $(-2)-$roots $\Phi_2(L)$ and $(-4)-$roots $\Phi_4(L)$.

[12] i.e. if the roots form a $\mathbb{Z}-$generating set for $L$

In particular, for $L$ a primitive sublattice of $\mathrm{II}_{3,19}$, letting $F_L$ be the stack of $L-$polarized K3 surfaces, the period map $\tau_L$ yields an open immersion

$$\tau_L : F_L(\mathbb{C}) \hookrightarrow \widetilde{\mathrm{SO}^L}(\mathbb{Z})\backslash\Omega^L$$

where $\widetilde{\mathrm{SO}^L}$ are isometries of $L$ which extend to an isometry of $\mathrm{II}_{3,19}$ which fixes $L$.

For $\operatorname{sgn} L = (2, n)$ let $G_L \coloneqq \mathrm{SO}_{L_\mathbb{Q}}$ be its associated rational isometry group and let $\mathbb{X} \coloneqq \Omega_L$ the associated Hermitian symmetric space as above, forming a Shimura datum $(\mathbb{X}, G) \coloneqq (\Omega^L, \mathrm{SO}_{L_\mathbb{Q}})$. We can then realize

$$\mathrm{Sh}_L(\mathbb{C}) \coloneqq \mathrm{Sh}_{\mathbb{K}_L}[G_L, \mathbb{X}_L](\mathbb{C}) \cong \widetilde{\mathrm{SO}_L}(\mathbb{Z})\backslash\mathbb{X}_L$$

where $\mathbb{K}_L \coloneqq \ker\left(G_L \to \mathrm{Aut}(A_L)\right)(\widehat{\mathbb{Z}})$, the admissible morphism in $G_L(\mathbb{A}_f)$; the stack $\mathrm{Sh}_\mathbb{K}[G, \mathbb{X}]$ is a certain well-known quotient stack attached to a Shimura datum $(G, \mathbb{X})$ and a choice of a compact open subgroup $\mathbb{K} \leq G(\mathbb{A}_f)$ of the finite adeles.

Defining the compact dual:

$$\Omega^{L,\mathrm{cd}} \coloneqq \left\{x \in (L^\perp)_\mathbb{C} \mid x^2 = 0\right\}.$$

## 14. Integral Affine Structures

### 14A. Integral affine geometry.

#### 14A.1. *Affine groups.*

**Definition 14.1** (Affine algebraic groups for vector spaces)**.** When $V$ is a vector space and $\rho : G \hookrightarrow \mathrm{GL}(V)$ a group acting by linear transformations on $G$, define

$$\mathrm{AG}_V \coloneqq G \rtimes_\rho V.$$

In particular, if $G = \mathrm{GL}(V)$ acts on $V$ in the natural way, we define

$$\mathrm{AGL}_V = \mathrm{GL}(V) \rtimes V$$

where $\mathrm{GL}(V)$ acts on on $V$ by linear transformations. For $W \leq V$ a subspace, we can refine a relative affine group

$$\mathrm{AGL}_{V;W} \coloneqq \mathrm{GL}(V) \rtimes W$$

where one allows arbitrary linear transformations of $V$ but only translations along subspaces of $W$. Concretely, note that a choice of $\mathbb{R}-$basis for $V$ determines an isomorphism $\mathrm{AGL}_V \cong \mathrm{GL}_n(\mathbb{R}) \rtimes \mathbb{R}^n$.

**Definition 14.2** (Affine algebraic groups for lattices)**.** For $L$ a lattice and $S, R$ two $\mathbb{Z}-$modules, we similarly define

$$\mathrm{AO}_L(S) \coloneqq \mathrm{O}_L(S) \rtimes (L \otimes_\mathbb{Z} S)$$
$$\mathrm{AO}_L(S; R) \coloneqq \mathrm{O}_L(S) \rtimes (L \otimes_\mathbb{Z} R)$$

where $\mathrm{O}_L(S)$ acts on $L \otimes_\mathbb{Z} R$ by base change and linear transformations. More generally, if $G_L$ is a group defined over $\mathbb{Z}$ acting on $L_\mathbb{R}$ via a linear representation $\rho : G_L \to \mathrm{GL}(L_\mathbb{R})$ (e.g. if $G_L$ is a subgroup of $\mathrm{GL}(L_\mathbb{R})$) define the $G_L-$**affine group of** $L$ as

$$\mathrm{AG}_L(S) \coloneqq G_L(S) \rtimes_\rho (L \otimes_{\mathbb{Z}} S)$$
$$\mathrm{AG}_L(S; R) \coloneqq G_L(S) \rtimes_\rho (L \otimes_{\mathbb{Z}} R).$$

**Example 14.3** (Explicit affine groups)**.** Note that if $L$ is a lattice, we have $\mathrm{AO}_L(\mathbb{R}) \cong \mathrm{O}_L(\mathbb{R}) \rtimes L_{\mathbb{R}}$. Concretely, if $L$ is isometric to $\mathbb{Z}^n$ with the standard bilinear form, note that $L_{\mathbb{R}} \cong \mathbb{R}^n$ and $\mathrm{GL}(L_{\mathbb{R}}) \cong \mathrm{GL}_n(\mathbb{R})$. Elements of $\mathrm{AGL}_L(\mathbb{R})$ can be concretely written as affine$-\mathbb{R}-$linear maps of the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ with $A \in \mathrm{GL}_n(\mathbb{R})$ and $\mathbf{b}$ a real translation. This construction can be used to recover the following standard groups:

$$\mathrm{AGL}_{\mathbb{Z}^n}(\mathbb{Z}) \cong \mathrm{GL}_n(\mathbb{Z}) \rtimes \mathbb{Z}^n$$
$$\mathrm{AGL}_{\mathbb{Z}^n}(\mathbb{R}) \cong \mathrm{GL}_n(\mathbb{R}) \rtimes \mathbb{R}^n$$
$$\mathrm{AGL}_{\mathbb{Z}^n}(\mathbb{Z}; \mathbb{R}) \cong \mathrm{GL}_n(\mathbb{Z}) \rtimes \mathbb{R}^n$$
$$\mathrm{AGL}_{\mathbb{Z}^n}(\mathbb{R}; \mathbb{Z}) \cong \mathrm{GL}_n(\mathbb{R}) \rtimes \mathbb{Z}^n$$

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$$\mathrm{ASL}_{\mathbb{Z}^n}(\mathbb{Z}) \cong \mathrm{SL}_n(\mathbb{Z}) \rtimes \mathbb{Z}^n$$
$$\mathrm{ASL}_{\mathbb{Z}^n}(\mathbb{R}) \cong \mathrm{SL}_n(\mathbb{R}) \rtimes \mathbb{R}^n$$
$$\mathrm{ASL}_{\mathbb{Z}^n}(\mathbb{Z}; \mathbb{R}) \cong \mathrm{SL}_n(\mathbb{Z}) \rtimes \mathbb{R}^n$$
$$\mathrm{ASL}_{\mathbb{Z}^n}(\mathbb{R}; \mathbb{Z}) \cong \mathrm{SL}_n(\mathbb{R}) \rtimes \mathbb{Z}^n$$

---

$$\mathrm{AO}_{\mathbb{Z}^{p,q}}(\mathbb{R}) = \mathrm{O}_{p,q}(\mathbb{R}) \rtimes \mathbb{R}^{p,q}$$

Is the orthogonal group of the standard lattice $\mathbb{Z}^n$ actually $\mathrm{SL}_n(\mathbb{Z})$?

14A.2. *$G-$manifolds.* For this section, let $X$ be a connected real $C^\infty$ manifold of real dimension $n$, and write $U_{ij} \coloneqq U_i \cap U_j$ for the intersection of two open sets.

**Definition 14.4** ($G-$manifolds)**.** Suppose $X$ admits an action of a Lie group $G \leq \mathrm{Diff}(X)$ by self-diffeomorphisms. We say $X$ is a $G-$**manifold** if there exists a $C^\infty$ real atlas on $X$ with transition functions valued in $G$. Thus there is a covering $X = \bigcup_i U_i$ with open embeddings $\phi_i : U_i \to \mathbb{R}^n$ such that $\phi_i \circ \phi_j^{-1} \in G$ for all pairs $i, j$, and for $x \in U_{ij}$ one has $(\phi_i \circ \phi_j)(x) = g_{ij}(x)$ for some $g_{ij} \in G$ depending on $i$ and $j$.

14A.3. *Integral affine manifolds and lattice manifolds.*

**Definition 14.5** (Affine and integral affine manifolds)**.** We say $X$ is an **affine manifold** if $X$ is equipped with the structure of $G-$manifold for $G \coloneqq \mathrm{AGL}_n(\mathbb{R})$, and an **integral affine manifold** if one instead takes $G \coloneqq \mathrm{AGL}_n(\mathbb{Z})$. [13] For $L$ a $\mathbb{Z}-$lattice, an $L-$**lattice manifold** is an integral affine manifold together with a choice of a discrete 0-dimensional submanifold $S \subseteq X$ which are preimages under the charts $\phi_i$ of some collection of lattice points $P \subseteq L \subseteq \mathbb{R}^n$.

**Remark 14.6.** In our situation, we'll consider integral affine surfaces relative to the standard lattice $\mathbb{Z}^n$, i.e. manifolds of real dimension two whose charts have transition functions in $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$.

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[13] In the literature one typically takes an integral affine structure on a real oriented surface to additionally preserve orientation and a volume form, so one takes $G \coloneqq \mathrm{ASL}_2(\mathbb{Z})$.

14B. **Lagrangian torus fibrations.** For this section, let $(M, \omega)$ be a symplectic manifold with $\dim_{\mathbb{R}} M = 2n$ and $\omega \in H^0(\Omega^2_M)$ a globally defined symplectic form.

**Warning 14.7.** In discussions of LTFs, a "singularity" means a singularity of the fibration $\pi : (X, \omega) \to B$ and not necessarily a singularity of $X$ itself. It is often the case that $X$ is a smooth symplectic manifold and $\pi$ has nodal singularities.

**Remark 14.8.** The base $B$ of a LFT describes a fibration in the following way:

- Fibers over interior points are smooth Lagrangian tori $F_x \cong (S^2)^2$,
- Fibers over non-vertex edge points (except the origin) are isotropic circles $F_x \cong S^1$.
- Fibers over toric vertices (vertices with no attached monodromy ray) are points $F_x \cong \text{pt}$
- Fibers over non-toric vertices (vertices with an attached monodromy ray) are circles $S^1$ of corank 1 elliptic singularities $F_x \cong S_1$.
- Fibers over a red x are singular Lagrangian tori with a single node, i.e. a torus $S^1 \times S^1$ with a cycle collapsed.
- Fibers over dotted lines encode branch cuts.

**Definition 14.9.** If $M$ is a symplectic manifold and $\mu : (M, \omega) \to B$ is a topological fibration, we say $\mu$ is a **regular Lagrangian (torus) fibration** or **regular LTF** if all fibers $F_x$ of $\mu$ are Lagrangian[14] submanifolds. It is simply a **Lagrangian fibration** if $\mu$ restricts to a regular LTF on the open dense subset $\widetilde{B} \subseteq B$ of regular values in the base.

**Remark 14.10.** By the Liouville-Mineur-Arnold theorem, any fiber $F$ of regular LTF admits a neighborhood in $M$ that is fiber-preserving symplectomorphic to a projection $(V \times (S^1)^n, \omega_{\text{std}}) \to V$ for some $V \subseteq \mathbb{R}^n$. Thus the regular fibers are necessarily maximal-dimension tori.

**Remark 14.11.** Why Lagrangian fibrations are relevant to integral affine geometry: a consequence of the Liouville-Mineur-Arnold theorem on completely integrable systems along with a result of Duistermaat shows that the base $B$ and the fibers $F_x$ over regular points $x \in B$ admit an IAS. In fact, more is true: every IA manifold can be realized as the base of some Lagrangian fibration. Parallel transport in $B$ around the image of a nodal critical point yields a notion of monodromy for any such IAS which is a shear with respect to some eigenline of the monodromy operator associated with each node.

**Definition 14.12.** An **almost toric fibration** of a symplectic 3-manifold is a Lagrangian fibration $\pi : (X, \omega) \to B$ such that every point admits a Darboux neighborhood $(x_1, y_1, x_2, y_2) \in (\mathbb{B}^4, \omega_{\text{std}} := dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$ such that $\pi$ is of one of the following forms:

(1) Regular points: $\pi(x, y) = (x_1, x_2)$,
(2) Elliptic of corank 1: $\pi(x, y) = (x_1, x_2^2 + y_2^2)$,
(3) Elliptic of corank 2: $\pi(x, y) = (x_1^2 + y_1^2, x_2^2 + y_2^2)$,
(4) Nodal or focus-focus: $\pi(x, y) = (x_1 y_1 + x_2 y_2, x_1 y_2 - x_2 y_1)$, or in complex coordinates $\pi(x, y) = \overline{x} y$.

The fibers are Lagrangian, and singular fibers are either toric (corank 2 elliptic singularities or an $S^1$ of corank 1 elliptic singularities) or nodal.

[14]Maximally isotropic submanifolds with respect to the restriction of the symplectic form $\omega$, so $\omega|_{F_x} = 0$ and $\dim F = \frac{1}{2} \dim M$.

## 14C. Integral affine spheres.

**Definition 14.13** (Pseudofans)**.** Given an ACP $(V, D)$, define the **pseudofan** of the pair $\mathcal{F}(V, D)$ as ... [?, §6.10]

Todo, AESsymp 6.10

**Definition 14.14** (Corner blowup equivalence classes (cbecs))**.** Two pseudofans $\mathfrak{F}(V', D')$ and $\mathfrak{F}(V'', D'')$ are in the same **corner blowup equivalence class (cbec)** if, after potentially allowing for corner blowups, they correspond to different toric models of a single toric pair $(V, D)$. Call this class $\mathfrak{C}(V, D)$.

**Definition 14.15** (Integral affine spheres)**.** An **integral affine sphere** or $\mathrm{IAS}^2$ is an integral affine structure defined on $X := S^2 \setminus \{p_1, \cdots, p_n\}$, the complement of finitely many (possibly singular) points of a 2-sphere, such that $X$ is a lattice manifold for $L := \langle n \rangle \cong \mathbb{Z}^n$, so the transition functions lie in $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$, and for each singularity $p_i$ a choice of cbec $\mathfrak{C}(V_i, D_i)$ is fixed and there exists a neighborhood $U_i \ni p_i$ which is isomorphic to an open subset of a pseudofan $\mathfrak{F}(V_i, D_i) \in \mathfrak{C}(V_i, D_i)$ in this cbec. A **triangulated** $\mathrm{IAS}^2$ is a pair $(B, \mathcal{T})$ where $B$ is an $\mathrm{IAS}^2$ as above and $\mathcal{T}$ is a triangulation. The **charge** of a singularity $p_i$ on an $\mathrm{IAS}^2$ is $Q(p_i) := \cdots$ By [?], $\sum_{i=1}^{n} Q(p_i) = 24$. An $\mathrm{IAS}^2$ is **generic** if there are precisely 24 singularities of type $\mathrm{I}_1$.

Not super sure about this, I can't find AE's definition of the pair $(B, \mathcal{T})$. It's used in [?, §6.12]

Todo, define

## 14D. Type III Kulikov models $\rightleftharpoons$ $\mathrm{IAS}^2$.

**Remark 14.16.** We follow the treatment in [?, §6].

$$\mathcal{X}_0 \text{ Type III} \to (\mathrm{IAS}^2, Q = 24):$$

Let $\mathcal{X}_0$ be a $d-$semistable Type III Kulikov surface $\mathcal{X}_0$. One can construct a triangulated $\mathrm{IAS}^2$ in the following way: write $\mathcal{X}_0 = \bigcup_i V_i$, and define $B := \coprod_i \mathfrak{F}(V_i, D_i)$. Most of the pairs $(V_i, D_i)$ are toric pairs, and the nontoric pairs will be encoded in the singularities of the integral affine structure. One can further triangulate $B$ by letting $\mathcal{T} := \Gamma(\mathcal{X}_0)$.

$$\{(V_i, D_i\}_{i \in I} \to \mathcal{X}_0 \text{ Type III}$$

Given a collection of ACPS $\{(V_i, D_i)\}_{i \in I}$ along with gluing maps $\phi_{ij} : D_{ij} \xrightarrow{\sim} D_{ji}$, one can form the pushout $\mathcal{X}_0 := \coprod_{\phi_{ij}} (V_i, D_i)$. If the glued surface satisfies

- $\Gamma(\mathcal{X}_0) \cong S^2$,
- $D_{ij}^2 + D_{ji}^2 = -2$, and
- the gluing is $d-$semistable in the sense of Friedman,

then $\mathcal{X}_0$ is a Type III surface, which additionally admits a smoothing to a family of K3 surfaces $\mathcal{X}$.

Following [?], we standardize this choice of $\mathcal{X}$ by fixing the numerical types and ordered toric models of all pairs $(V_i, D_i)$, performing all interior blowups in copies of $\mathbb{P}^1$ at their origins, and each gluing morphism $\phi_{ij}$ is chosen to match origins and triple points.

## 14E. Integral Affine Surgeries.

**Remark 14.17.** There is a very easy formula for the self-intersection numbers on a toric surface. Let $D_i$ be the divisor corresponding to a ray $\rho_i$ and let $\mathbf{w}$ be a primitive generator of $\rho_{i+1}$ and $\mathbf{v}$ of $\rho_{i-1}$, taken in counterclockwise order. Then

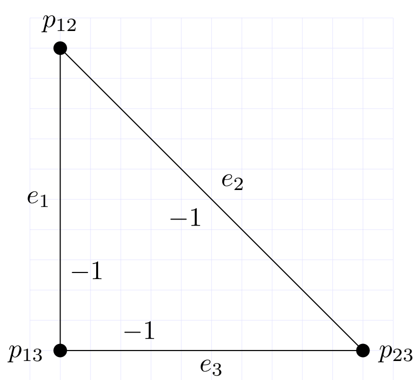
$$-D_i^2 = \det \begin{bmatrix} u_1 & w_1 \\ u_2 & w_2 \end{bmatrix}.$$

**Definition 14.18.** A 2-dimensional **shear** is any matrix of the form $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$.

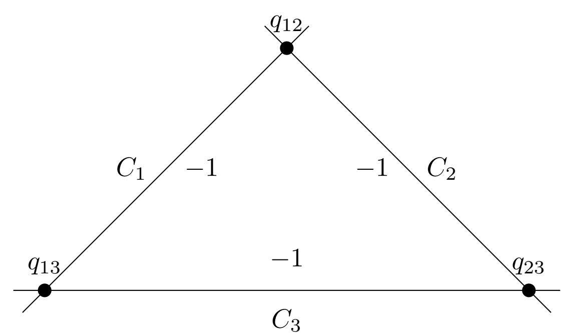
**Remark 14.19** (Running example: $\mathbb{P}^2$)**.** In this section, we will refer to the standard moment polytope and toric boundary divisor in Figure 28 for $\mathbb{P}^2$ as a running example. The polytope $P$ is a 10-fold dilate of the standard simplex in $\mathbb{R}^2$ with edges $e_i$ corresponding to curves $C_i$ on the toric boundary divisor $D$ with $d_i := -C_i^2 = -1$ for $i = 1, 2, 3$; the $C_i$ are thus rational and isomorphic to copies of $\mathbb{P}^1$ in $\mathbb{P}^2$ by classification of curves. The polytope has three lattice vertices $p_{ij} := e_i \cap e_j$ at their intersections corresponding to three toric points $q_{ij} := C_i \cap C_j$. We can compute the charge as

$$Q(D) = 12 - 3(-1 - 3) = 0,$$

verifying that $(\mathbb{P}^2, D)$ is a toric ACP.




(A) The moment polytope.




(B) The toric boundary.

FIGURE 28. The moment polytope and corresponding toric boundary divisor $D$ for $\mathbb{P}^2$. The numbers record $d_i := -C_i^2$, noting that each $C_i$ is a line in $\mathbb{P}^2$ and thus satisfies $C_i^2 = 1$.

**Remark 14.20.** There are several surgeries on can perform on an IAS, following Engel's thesis. The most important ones are:

- Corner (toric) blowups. Changes cycle by
$$(\dots, d_i, d_{i+1}, \dots) \mapsto (\dots, d_i + 1, 1, d_{i+1} + 1, \dots) \qquad Q' = Q + 1$$
- Internal (nontoric/almost toric) blowups. Changes cycle by
$$(\dots, d_i, \dots) \mapsto (\dots, d_i + 1, \dots), \qquad Q' = Q$$
- Node smoothing/nodal trades. Changes cycle by
$$(\dots, d_i, d_{i+1}, \dots) \mapsto (\dots, d_i + d_{i+1} - 2, \dots), \qquad Q' = Q + 1$$
.
- Nodal slides. Leaves cycle and charge unchanged?

Nodal trades and slides modify the IAS $B \to B'$ but correspond to symplectomorphisms $X \to X'$.

Loops around singular fibers have nontrivial monodromy which is a shear in a direction (dual to) the vanishing cycle of the nodal fiber.

For this section, let $P$ be a Symington polytope in $L_{\mathbb{R}}$ where $\partial P = \bigcup_{i=1}^n P_i$ is decomposed as a union of straight line segments $P_i$ of maximal length with endpoints in $L$. Write $\mathbf{v}_{i,i+1} \coloneqq P_i \cap P_{i+1} \in L$ for the integral vertices of $P$ and let $(\mathbf{x}_i, \mathbf{y}_i)$ be a primitive integral ordered basis aligned with $P_{i+1}$ and $P_i$ respectively. By convention, we take $d_{ij} = -D_{ij}^2 \geq 0$. Define $P^\circ = P \setminus \partial P$ and $P_{\text{sing}}$ as the set of singular points of an IAS. Define $P_{\text{smooth}} \coloneqq P \setminus P_{\text{sing}}$ for the smooth locus.

**Definition 14.21** (Self-intersection of an edge of a Symington polytope)**.** For $P$ a Symington polytope and $P_i$ an edge, define the **negative self-intersection number of** $P_i$ to be $d_i$ defined by the following equation:

$$d_i \mathbf{y}_i = y_{i-1} - \mathbf{x}_i = y_{i-1} + y_{i+1}.$$

Note that in the toric case, $d_i = -D_i^2$, generalizing the standard computation of the self-intersection number of a toric curve.

**Example 14.22** (Computing a self-intersection number of a polytope edge)**.**

14E.1. *Corner/toric blowups.*

**Definition 14.23** (Corner/toric blowups)**.** A **corner (or toric)** blowup of a Symington polytope at a vertex $v_{i,i+1} = e_i \cap e_{i+1}$ is....

One inserts a new edge $f_i$ with self-intersection $-1$, intersecting $e_i$ and $e_{i+1}$, and then decreases the self-intersections of $e_i$ and $e_{i+1}$ by 1.

**Example 14.24** (Corner/toric blowups)**.** Starting with Figure 28, we'll perform three toric blowups in sequence at the three toric points $q_{ij}$ and compute the charge at each step.

(1) We first blow up $q_{12} = C_1 \cap C_2$ by removing a basis triangle emanating from the vertex $p_{12}$ in the polytope. This introduces a new edge $f_{12}$ corresponding to an exceptional curve $E_{12}$, as well as two new vertices $p_{12}^1 \coloneqq f_{12} \cap e_1$ and $p_{12}^2 \coloneqq f_{12} \cap e_2$, geometrically corresponding to two new points $q_{12}^1, q_{12}^2$ on $E_{12} \cap C_1$ and $E_{12} \cap C_2$ respectively. Since $E_{12}^2 = -1$, the edge $f_{12}$ gets a weight of 1 and the adjacent edges $e_1$ and $e_2$ have their weights **increased** by 1. The charge is

$$Q = 12 + (1 - 3) + (0 - 3) + (-1 - 3) + (0 - 3) = 0.$$

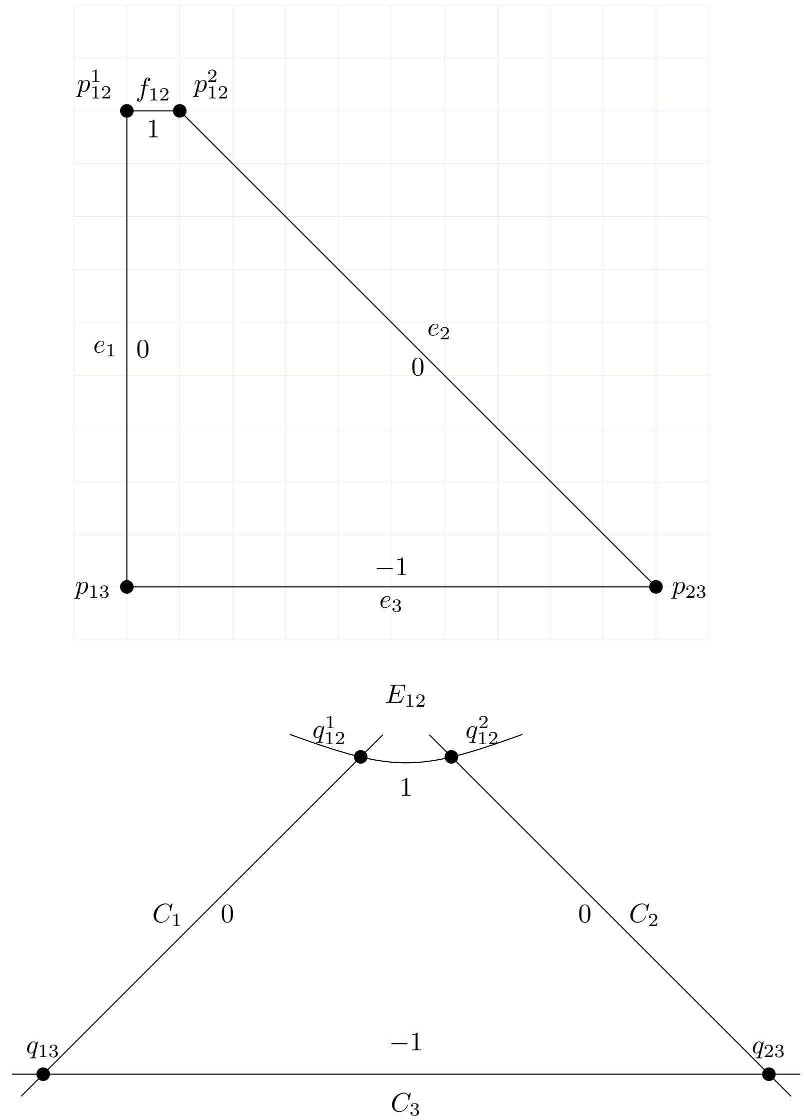



FIGURE 29. $\mathbb{P}^2$ with a single toric blowup at $q_{12}$.

(2) We now blow up $q_{23} \coloneqq C_2 \cap C_3$ by performing a similar procedure at vertex $p_{23}$. This **increases** the weights of $e_2$ and $e_3$ by one and introduces a new edge $f_{23}$ of weight 1 with endpoints $p_{23}^2 \coloneqq f_{23} \cap e_2$ and $p_{23}^3 \coloneqq f_{23} \cap e_3$. Geometrically, this similarly introduces an exceptional curve $E_{23}$ and two new points $q_{23}^i \coloneqq E_{23} \cap C_i$ for $i = 2, 3$. The charge is

$$Q = 12 + 3(1 - 3) + 2(0 - 3) = 0.$$

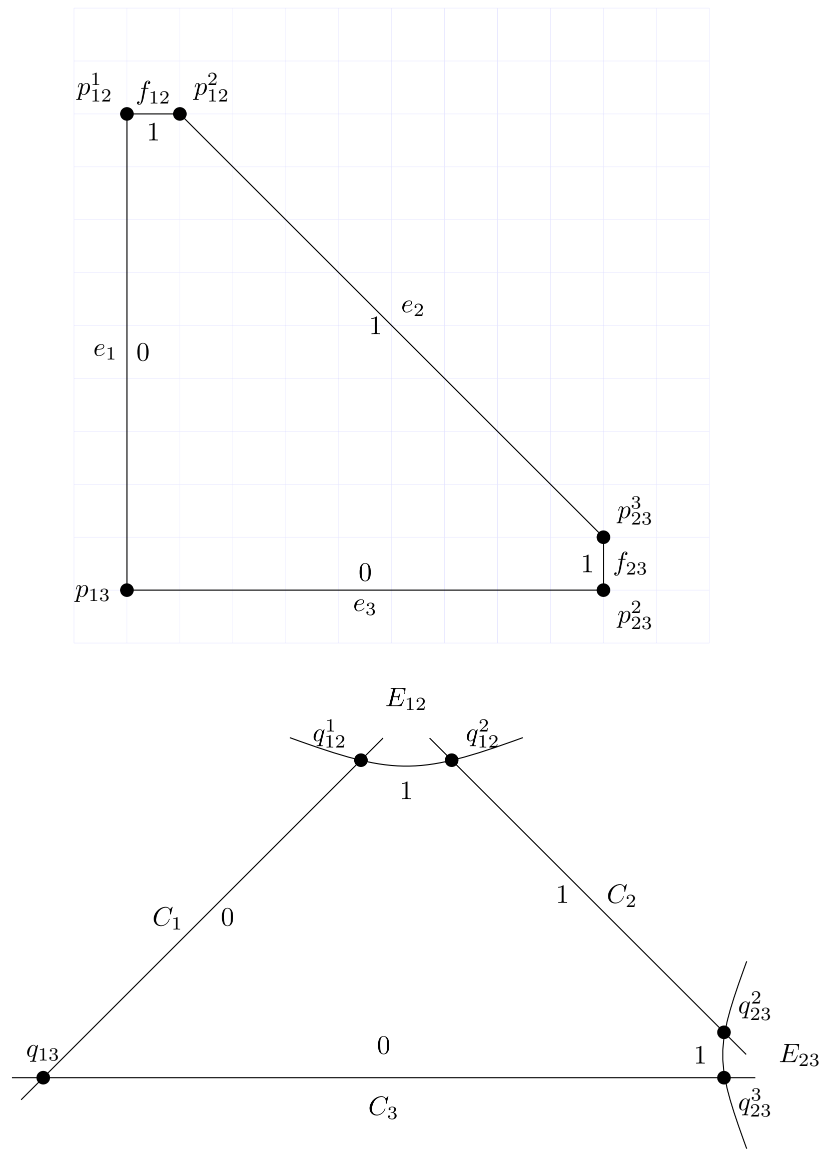



FIGURE 30. $\mathbb{P}^2$ with two toric blowups at $q_{12}$ and $q_{23}$.

(3) Finally, we blow up $q_{13}$, introducing a new edge $f_{13}$ of weight 1 and increasing the weights of $e_1$ and $e_3$ by 1. This introduces the exceptional curve $E_{13}$ and $q_{13}^i$ for $i = 1, 2$ in a similar way. The resulting polytope is a non-regular hexagon with 3 short edges and 3 long edges, all of weight 1, corresponding to a cycle of six $(-1)$–curves. The charge is

$$Q = 12 + 6(1 - 3) = 0.$$

**Remark 14.25.** If $(V, D)$ is an ACP with a deformation $(\widetilde{V}, \widetilde{D})$ where $\widetilde{D}$ is an internal blowup of $D$ on edge $D_i$, then the deformation is the result of a surgery on the pseudofan of $(V, D)$ by replacing the weight $d_i$ on the edge $e_i$ with $d_i - 1$.

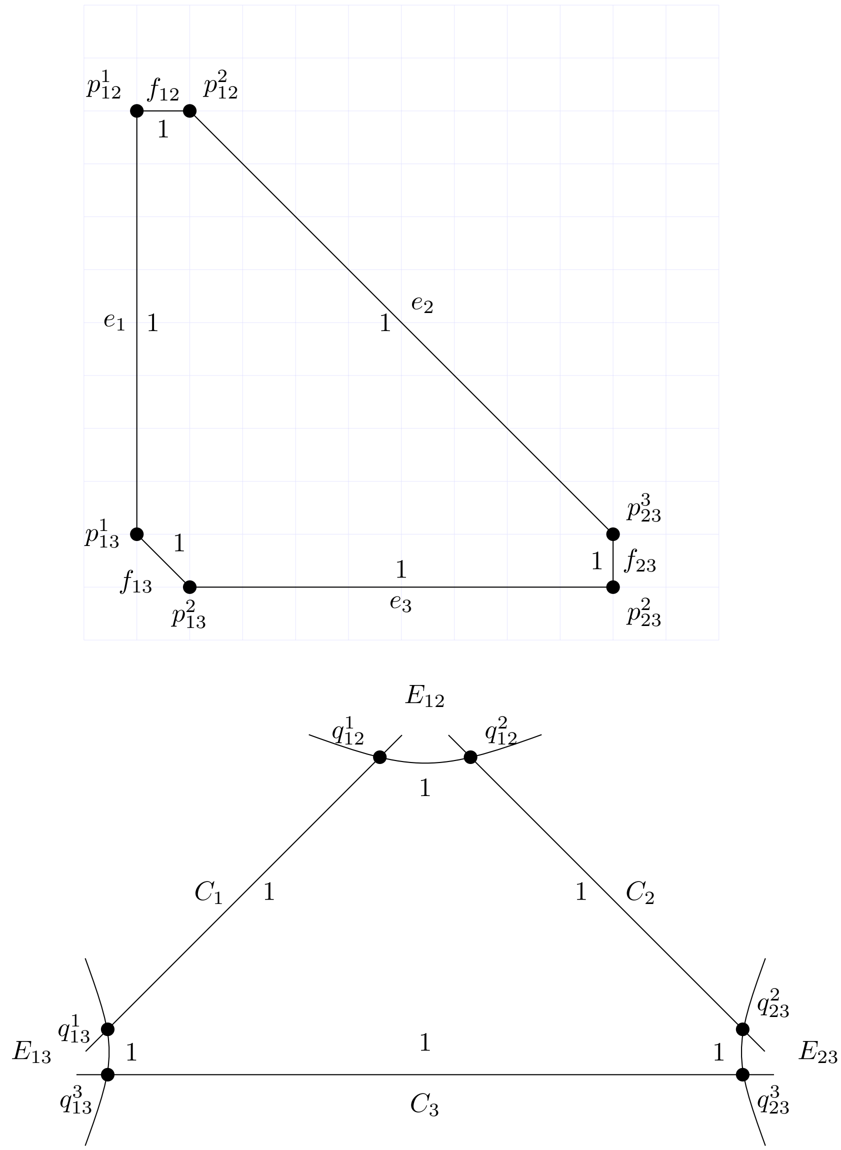



FIGURE 31. $\mathbb{P}^2$ with all three toric points blown up, resulting in a toric variety with an anticanonical cycle of six $(-1)-$curves whose polytope is a non-regular hexagon.

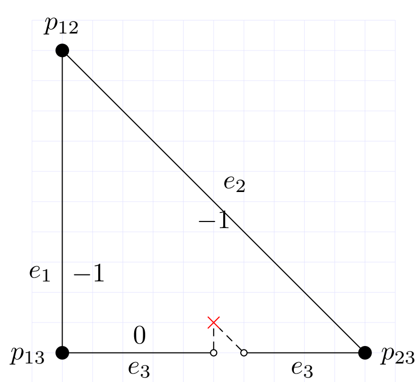
14E.2. *Edge/nontoric/internal blowups.*

**Example 14.26** (Internal blowup and self-intersection numbers)**.** Starting with Figure 28, we'll perform three nontoric blowups in sequence and compute the charge at each step. To blow up a nontoric point, we take the Symington polytope and cut an integer dilate $n$ of a basis triangle; figure Figure 32a shows such one such surgery of size $n = 1$. Recall that we define $d_i := -D_i^2$ and $Q(D) := 12 + \sum_{i=1}^{n}(d_i - 3)$ when $D$ is a cycle of $n \geq 2$ rational curves.

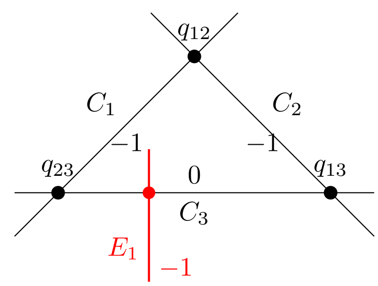
(1) We first blow up a nontoric point $C_3$ by performing a size 1 surgery, cutting out a basis triangle as in Figure 32a. This leaves the self-intersection $d_3 = C_3^2 = -1$ unchanged, and introduces an exceptional curve $E_1$ with $E_1^2 = -1$. Note that although the edge $e_3$ appears to be broken into two parts in this picture, the dotted edges are glued together such that $e_3$ remains a single contiguous edge. The boundary divisor is often represented in one of the two equivalent ways, with either an additional orthogonal curve as in Figure 32b or a single extra dot as in Figure 32c to record the exceptional divisor $E_1$. For simplicity, we adopt the latter convention.

Note that in these diagrams, the red numbers indicate the *positive* self-intersection numbers $E_i^2$ directly, as opposed to the *negative* self-intersection numbers for the $d_i$. This is because the red numbers do not contribute to the charge. The charge of the resulting polytope and boundary divisor is thus

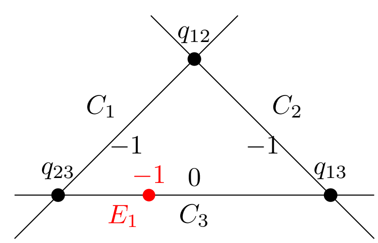
$$Q = 12 + (0 - 3) + 2(-1 - 3) = 1.$$




(A) The Symington polytope: a (dilated) standard simplex with one nontoric blowup.




(B) The boundary divisor.




(C) Simplified notation for the boundary divisor.

FIGURE 32. $\mathbb{P}^2$ with a single nontoric blowup at a point contained in $C_3$.

(2) We can continue by blowing up a nontoric point on $C_2$ by cutting a basis triangle out of $e_2$, introducing an exceptional curve $E_2$ intersecting $C_2$. The Symington polytope and corresponding boundary divisor are shown in

Figure 33 and have charge

$$Q = 12 + 2(0-3) + (-1-3) = 2.$$

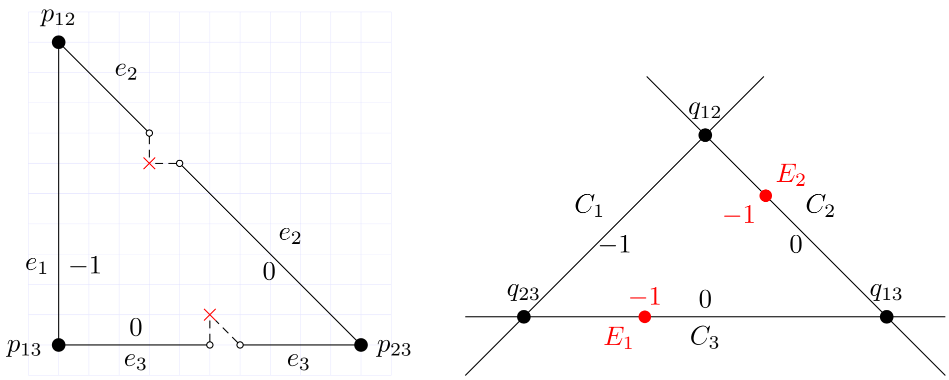



FIGURE 33. $\mathbb{P}^2$ with two nontoric blowups at points in the curves $C_3$ and $C_2$.

(3) Finally, we blow up a nontoric point on $C_1$ by cutting a basis triangle out of $e_1$, introducing an exceptional curve $E_3$ intersecting $C_1$. The Symington polytope and boundary divisor are shown in Figure 34 and have charge

$$Q = 12 + 3(0-3) = 3.$$

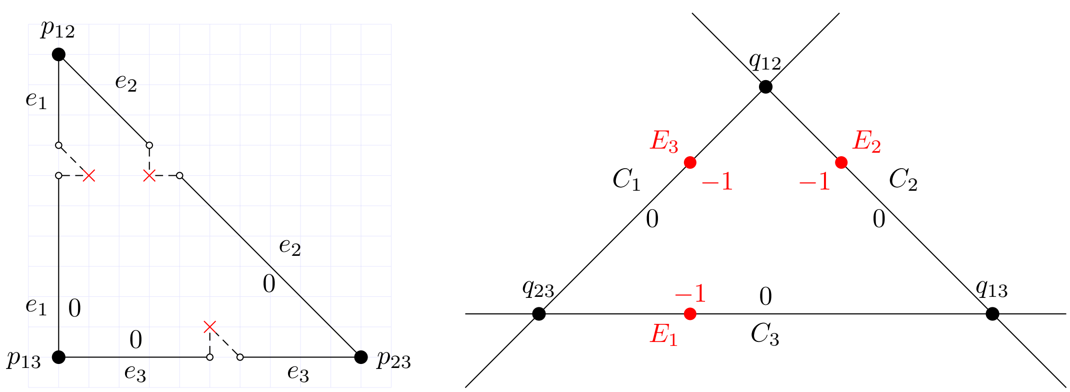



FIGURE 34. $\mathbb{P}^2$ with three nontoric blowups at points in the curves $C_2, C_3, C_1$.

and

14E.3. *Nodal slides.*

**Definition 14.27** (Nodal slides)**.** Two Symington polytopes $(B, \Lambda)$ and $(B, \Lambda')$ with the same underlying polytope but different IASs are related by a **nodal slide** if there is a curve $\gamma \subseteq B$ such that $(B \setminus \gamma, \Lambda) \cong (B \setminus \gamma, \Lambda')$, $\gamma$ contains exactly one node of $(B, \Lambda)$ and one node of $(B, \Lambda')$, and $\gamma \subseteq L_i$, the monodromy eigenline through the node $n_i$. This operation literally slides a node along the monodromy direction, changing the length of the corresponding slit. See Example 14.33 for an explicit example.

**Remark 14.28.** Symington's results and an application of Moser's argument show that if $P_1, P_2$ are related by a nodal slide or a nodal trade, the two corresponding almost-toric manifolds are diffeomorphic and preserve the symplectic structure up to isotopy, and are thus symplectomorphic.

14E.4. *Node smoothing/nodal trades.*

**Definition 14.29** (Node smoothing/nodal trade)**.** Two distinct Symington polytopes $P_1 := (B_1, \Lambda_1)$ and $P_2 := (B_2, \Lambda_2)$ are related by a **node smoothing (or nodal trade)** if each polytope contains a curve $\gamma_1$ and $\gamma_2$ respectively such that $P_1 \cong P_2$ and $P_1$ contains precisely one less vertex than $P_2$. A **node smoothing (or nodal trade) of size** $n$ on a Symington polytope $P$ at the vertex $\mathbf{v}_{i,i+1}$ is node smoothing $P'$ of $P$ obtained in the following way:

- Let $S_i(n) \subseteq P^\circ$ be a slit with endpoints $\mathbf{v}_{i,i+1}$ and $\mathbf{v} := \mathbf{v}_{i,i+1} + n(\mathbf{x}_i + \mathbf{y}_i)$.
- Let $L_i$ be the line containing $S_i(n)$ and let $M_i$ be the unique shear which fixes $L_i$ pointwise and satisfies $M_i(\mathbf{x}_i) = -\mathbf{y}_i$.
- Look along $S_i(n)$ from the perspective of $\mathbf{v}_{i,i+1}$ and glue the clockwise edge of $S_i(n)$ to its counterclockwise edge by the shearing map $M_i$.

**Remark 14.30.** This introduces an $I_1$ singularity into the fibration where the node is in the interior of the polytope. The fiber over the node is a pinched torus.

**Remark 14.31.** This defines a new IAS $P'$ such that

$$P'_{\text{sing}} = P_{\text{sing}} \cup \{\mathbf{v} := \mathbf{v}_{i,i+1} + n(\mathbf{x}_i + \mathbf{y}_i)\}$$

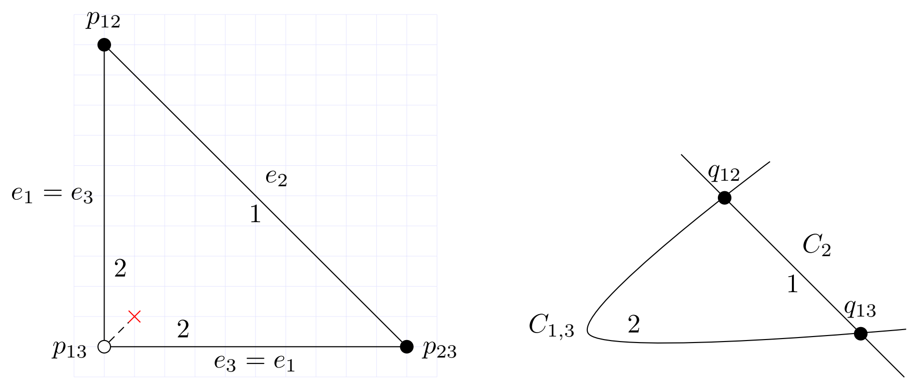
which straightens $P_i$ and $P_{i+1}$ into a single edge and deletes the vertex $\mathbf{v}_{i,i+1}$. Crossing the slit imposes a nontrivial change-of-frame[15]: in the basis $\mathbf{e}_i^1 := -\mathbf{x}_i - \mathbf{y}_i$, completed to an orthogonal, oriented lattice basis $\left\{\mathbf{e}_i^1, \mathbf{e}_i^2\right\}$, the change-of-frame matrix is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Thus a node smoothing introduces a new nodal singularity in $B$ marked with a red $x$ and a slit marked by a dotted line into the polytope. The fibration $\pi : X \to B$ acquires a nodal singular fiber: it replaces an elliptic corank 2 fiber with a nodal fiber in the neighborhood of an elliptic corank 1 fiber. This trades a node on the boundary divisor (corresponding to a vertex of the moment polytope) for a nodal singularity of an almost-toric fibration above the red $x$. Topologically, this is a surgery that removes a small ball around a $T-$fixed point and glues in a local model of a focus-focus singularity for the moment map.

**Example 14.32** (Node smoothings of size 1)**.** Consider the following sequence of node-smoothing surgeries, with the base polytope $B$ pictured on the left and the boundary divisor on the right:

(1) Perform a node smoothing at $p_{13}$ (corresponding to $q_{13}$). This has the effect of introducing a node into the interior of $P$, deletes vertex $p_{13}$, and merges edges $e_1$ and $e_3$ while increasing the self-intersection number by 1. Geometrically, this smooths the intersection $C_1 \cap C_3 = q_{13}$, produces a new

[15] I.e. for any paths crossing the slit, their tangent vectors undergo this transformation when they pass through the slit.

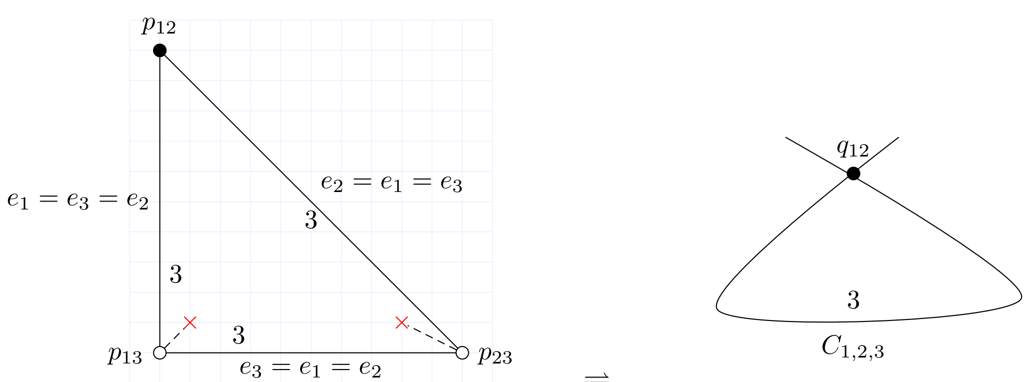
curve $C_{1,3}$ of self-intersection 2, i.e. a conic in $\mathbb{P}^2$, while $C_2$ still a line in $\mathbb{P}^2$ of self-intersection 1, which now intersects $C_{1,3}$ at two distinct points $q_{12}, q_{13}$:



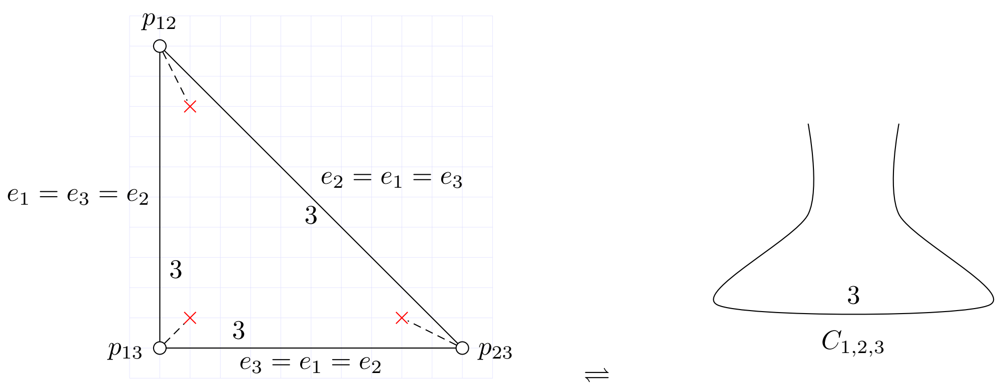

(A) The moment polytope. (B) The boundary divisor.

FIGURE 35. An almost-toric base $B$ for $\mathbb{P}^2$ with one node smoothing/nodal trade surgery of size $n = 1$ performed.

(2) Perform a node smoothing at $p_{23}$ (corresponding to $q_{23}$). This involves merging edges $e_2$ and $e_3$ from the original diagram for $P$ and deleting vertex $p_{23}$, yielding a polygon with a single edge of self-intersection 3 and one vertex $p_{12}$. One then puts a singularity in the interior of the polytope emanating from where $p_{23}$ was, with a dotted edge. Geometrically, this smooths the node $q_{13} = C_2 \cap C_3$, which became a node in $C_2 \cap C_{1,3}$ in the previous surgery. We thus get a nodal cubic curve of self-intersection 3, with node $q_{12}$:




(3) Perform a node smoothing at $p_{12}$ (corresponding to $q_{12}$). This involves deleting $p_{12}$, inserting a red singularity in the interior, and connecting it with a slit to where $p_{12}$ once was. Geometrically, this smooths the node $q_{12}$, yielding a smooth cubic curve:




Note that in all cases, all surgeries were of minimal size $n = 1$, taken in the direction of the eigenline of the local monodromy about each vertex. One can compute a primitive integral vector in the monodromy eigenline in the following way: at $p_{ij}$, let $e^1_{ij}, e^2_{ij}$ be the primitive integral positively-oriented basis of vectors along the adjacent edges. Let $M_{ij}$ be the matrix that transforms the standard basis to this basis; the eigenline is then along

$$\varepsilon_{ij} \coloneqq M_{ij} \cdot (1,1).$$

Explicitly, the following calculations are used to determine the directions of the shears in the above diagrams:

(1) At $p_{13}$ we have $e^1_{13} = (1,0)$ and $e^2_{13} = (0,1)$, so

$$M_{13} = \left[e^1_{13}, e^2_{13}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies \varepsilon_{13} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1,1).$$

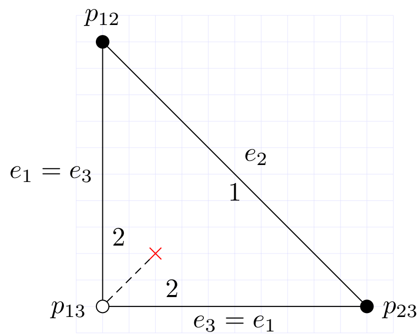
(2) At $p_{12}$, we have $e^1_{12} = (0,-1)$ and $e^2_{12} = (1,-1)$ so

$$M_{12} = \left[e^1_{12}, e^2_{12}\right] = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \implies \varepsilon_{12} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1,-2).$$

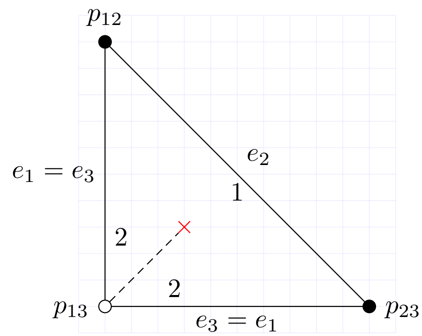
(3) At $p_{23}$, we have $e^1_{23} = (-1,1)$ and $e^2_{23} = (-1,0)$ so

$$M_{23} = \left[e^1_{23}, e^2_{23}\right] = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \implies \varepsilon_{23} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-2,1).$$

**Example 14.33** (Node smoothings of larger sizes and nodal slides)**.** Consider again the moment polytope and toric boundary for $\mathbb{P}^2$ shown in Figure 28. The first surgery shown in Figure 35 is a node smoothing at $p_{13}$ of size $n = 1$; the following are node smoothings of sizes $n = 2$ and $n = 3$ respectively:




(A) Size $n = 2$ node smoothing.




(B) Size $n = 3$ node smoothing.

These two surgeries are related by a nodal slide along the indicated eigenline associated to $p_{13}$.

**Remark 14.34.** If $(V, D)$ is an ACP with a deformation $(\widetilde{V}, \widetilde{D})$ where $\widetilde{D}$ is a node smoothing of $D$ at $\mathbf{v}_{i,i+1}$, then the deformation is the result of a surgery on the pseudofan of $(V, D)$ where the triangular face with edges $e_i$ and $e_{i+1}$ are collapsed and replaced with a single edge of self-intersection $d_i + d_{i+1} + 2$.

## 15. Toric geometry of degenerations and KPP models

**Remark 15.1.** One tends to say a pair $(X, D)$ is **maximal**, e.g. the pair corresponding to the central fiber of a degeneration if the intersections of irreducible components of $D$ are all 0-dimensional.

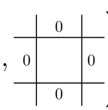
**Definition 15.2.** A curve $F$ is **exceptional** if $F$ is irreducible and $F^2 < 0$. By [**?**, Lemma 4.2], for $\pi : \mathcal{X} \to \mathcal{Y}$, exceptional curves on $\mathcal{Y}$ come in three types: $F^2 = -4, -2, -1$.

**Definition 15.3** (Anticanonical pairs)**.** todo

### 15A. Toric vs. semitoric.

**Example 15.4** (A nontoric blowup)**.** An example of a toric anticanonical pair is

$$(V, D) = \left( \mathbb{P}^1 \times \mathbb{P}^1, \right)$$




The following figure depicts $\mathrm{Bl}_{p_1,\ldots,p_9}(\mathbb{P}^1 \times \mathbb{P}^1)$:

### 15B. Kulikov-Persson-Pinkham models.

**Definition 15.5** (Modifications)**.** A proper birational morphism $f : X \dashrightarrow Y$ of algebraic varieties is called a *modification*.

**Definition 15.6** (Semistability)**.** A family $\pi : \mathcal{X} \to \Delta$ is **semistable** if

(1) $\mathcal{X}$ is smooth,
(2) $\mathcal{X}_0 = \bigcup_i V_i$ is a union of RNC divisors,
(3) $K_{\mathcal{X}} \sim_{\Delta} 0$, or equivalently $\omega_{\mathcal{X}/\Delta} \cong \mathcal{O}_{\mathcal{X}}$ is trivial.

**Definition 15.7** (Smoothing)**.** Let $X$ be a singular surface. A **smoothing** of $X$ is a smooth proper family $\mathcal{X} \to \Delta$ such that $\mathcal{X}_0 \cong X$ and $\mathcal{X}_t$ is smooth for $t \neq 0$.

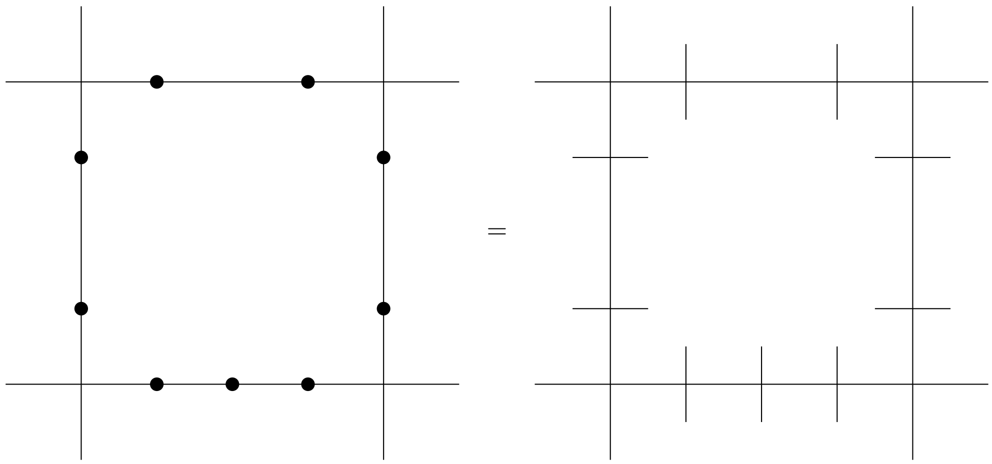



Figure 37. Nontoric blowups

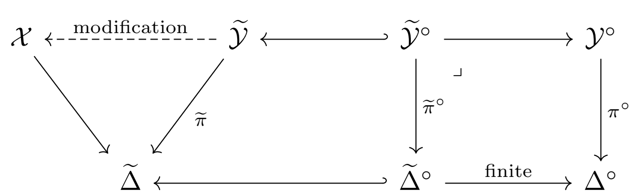
**Definition 15.8** (Recognizable divisors, [**?**])**.** Let $F_S^q$ be the moduli stack of quasipolarized K3 surfaces. A canonical choice of polarizing divisor is a choice of ample divisor $R$ on the generic K3 surface in $F_S^q$. It is **recognizable** for $F_S$ if every $S-$quasipolarized KPP surface $\mathcal{X}_0$ contains a divisor $R_0 \subset \mathcal{X}_0$ which, for any $S-$quasipolarized smoothing $\mathcal{X} \to \Delta$ of $\mathcal{X}_0$ with $\Delta^\circ \subset U$ has the property that $R_0$ is the flat limit of $R_t \subset \mathcal{X}_t$ for $t \neq 0$ up to the action of $\mathrm{Aut}^0(\mathcal{X}_0)$.

**Remark 15.9.** This captures the notion of a divisor on the generic fiber of a family which can be extended to any choice of central fiber, or conversely, there is a choice of divisor on any degenerate $\mathcal{X}_0 \in \partial F_{2d}$ which is independent of smoothing directions by KPP models. [**?**] shows that any choice of recognizable divisor for $F_{2d}$ produces (up to normalization) a semitoroidal compactification of the period domain, and produces recognizable divisors for $F_{2d}$ for all $d$.

**Definition 15.10** (Kulikov-Persson-Pinkham (KPP) models)**.** Let $\pi^\circ : \mathcal{X}^\circ \to \Delta^\circ$ be a family of smooth complex K3 surfaces over $\Delta^\circ$. A *Kulikov-Persson-Pinkham model* or **KPP model** is a proper semistable threefold $\pi : \mathcal{X} \to \Delta$. The central fiber $\mathcal{X}_0$ of a KPP model is called a *Kulikov surface*, and $\mathcal{X}$ is by definition a smoothing of $\mathcal{X}_0$.

Reference: see AET symplectic involutions §6a – they cite [Kul77, PP81]

A theorem of Kulikov and Persson-Pinkham says that for any punctured family of algebraic K3 surfaces $\mathcal{Y}^\circ \to \Delta^\circ$, there is a finite ramified base change $\widetilde{\Delta}^\circ \to \Delta^\circ$ with pullback $\widetilde{\mathcal{Y}}^\circ$, a completed family $\widetilde{\pi} : \mathcal{Y} \to \widetilde{\Delta}$ extending over the origin, and a modification $\mathcal{X} \dashrightarrow \widetilde{\mathcal{Y}}$ such that $\mathcal{X} \to \widetilde{\Delta}$ is a KPP model:



**Definition 15.11** ($d$−semistability)**.** Let $\mathcal{X} \to \Delta$ be a Type III KPP model. The central fiber $\mathcal{X}_0$ is $d$−**semistable** if

$$\mathrm{T}^1_{\mathcal{X}_0} \coloneqq (\Omega^1_{\mathcal{X}_0})^\vee \coloneqq \mathcal{E}xt^1_{\mathcal{O}_{\mathcal{X}_0}}(\Omega^1_{\mathcal{X}_0}, \mathcal{O}_{\mathcal{X}_0}) \cong \mathcal{O}_{\mathcal{X}_0^{\mathrm{sing}}}.$$

This is an analytic condition.

## 15C. **Type I/II/II surfaces.**

**Remark 15.12** (Classification of KPP models)**.** Let $\mathcal{X} \to \Delta$ be a KPP model and consider the Picard-Lefschetz transformation on integral singular cohomology induced by a simple closed loop $\gamma$ generating $\pi_1(\Delta^*, t)$:

$$T_\gamma : H^2(\mathcal{X}_t; \mathbb{Z}) \to H^2(\mathcal{X}_t; \mathbb{Z})$$

Since $\mathcal{X}_0$ is reduced normal crossings by definition, $T_\gamma$ is unipotent, so let $N \coloneqq \log T_\gamma$ be its logarithm. Then $N$ is nilpotent of some minimal order $k$, i.e. $N^{k-1} \neq 0$ but $N^k = 0$, and by [**?**, **?**] we have the following cases:

Find + cite types of curves that occur in each type? E.g. type II with rational ends

- $k = 0$: $\mathcal{X}$ is Type I with $\mathcal{X}_0$ smooth
- $k = 1$: $\mathcal{X}$ is Type II where $\mathcal{X}_0$ is a chain of elliptic ruled surfaces with rational surfaces at each end, glued along intersections in smooth elliptic double curves with no triple points,
- $k = 2$: $\mathcal{X}$ is Type III where $\mathcal{X}_0$ is a chain of rational surfaces whose intersections are cycles of rational curves such that the dual complex $\Gamma(\mathcal{X}_0)$ is a triangulation of the sphere $S^2$.

Furthermore, $N$ is integral, and there exist explicit elements $\delta, \lambda \in H^2(\mathcal{X}_t; \mathbb{Z})$ such that the log monodromy action is given by

$$Nx = (x \smile \lambda)\delta - (x \smile \delta)\lambda.$$

**Remark 15.13.** Importantly, the central fibers $\mathcal{X}_0$ of KPP models corresponding to semistable degenerations of K3 surfaces admit a classification by monodromy. The monodromy operator $T$ is quasi-unipotent, so $(T^k - I)T^n = 0$ for some $n$. By semistability, we can take $k = 1$, and since $T$ is an operator on $H^2$ we can choose $n = 3$.

## 15D. **Charge.**

**Definition 15.14** (Charge)**.** Let $(V, D)$ be a *rational anticanonical pair*, so $K_V + D \sim_{\mathbb{Q}} 0$ and $D = \sum D_j$ is a chain of rational curves $D_j$ in $V$. Note that $(V, D)$ is log Calabi-Yau pair. Call $(V, D)$ a *toric pair* if $V$ is a toric variety with $D = \partial V$ its toric boundary.

If $D$ is reduced, one of three cases can occur:

(1) $D$ is a smooth elliptic curve,
(2) $D$ is a cycle of $n$ smooth rational curves if $n \geq 2$, or if $n = 1$ then $D$ is a single irreducible nodal curve,
(3) $D$ is one of three exceptional cases.

The *charge* $Q(V, D)$ is computed by the following formula:

$$Q(V,D) = \begin{cases} 12 + \sum_j \left(-D_j^2 - 3\right) & \text{if } D \text{ is nodal with } \geq 2 \text{ components} \\ 11 - D^2 & \text{if } D \text{ is irreducible nodal} \end{cases}$$

If we are in case (ii) and $n \geq 2$, this formula simplifies to

$$Q(V, D) \coloneqq 12 - \sum_j \left(D_j^2 + 3\right).$$

**Remark 15.15.** A corner blowup does not change the charge, while an internal blowup increases the charge by 1.

15E. **Toric models.**

**Definition 15.16** (Toric models)**.** A *corner blowup* is a pair $(V', D')$ where $V'$ is a blowup $f : V' \to V$ at a torus fixed point, and $D'$ is the anticanonical divisor of $V'$. An *internal*, or *almost toric blowup* $V'' \to V$ is a pair $(V'', D'')$ where $V''$ is blowup at an interior point of a curve $D_j$, i.e. at a point $P \in D_j \setminus \cup_{k \neq j} D_k$, whose anticanonical divisor $D''$ is the strict transform of $D$.

Choosing an order for such blowups, we define an (ordered) *toric model* of $V$ to be a span of pairs

$$\begin{array}{ccccc} & & (V', D') & & \\ & \overset{\text{Corner}}{\swarrow} & & \overset{\text{Internal}}{\searrow} & \\ (V, D) & & & & (\bar{V}, \bar{D}) \end{array}$$

where $V' \to V$ is an ordered sequence of corner blowups and $V' \to \bar{V}$ an ordered sequence of internal blowups.

> Do we get a map $V \dashrightarrow \overline{V}$?

> Yes, $V$ and $\overline{V}$ are birational to each other. On the other hand, I am confused about the map $V' \to \overline{V}$. After interiors blow ups, we do not get a toric variety, but $V'$ is a toric variety.

**Remark 15.17.** [**?**] shows that any rational anticanonical pair admits a toric model. One can show that the charge is an obstruction to $(V, D)$ being a toric pair, and measures the number of internal blowups in a toric model. Type III surfaces occurring as the central fibers $\mathcal{X}_0$ of Kulikov degenerations are 24 steps away from being toric in the following sense: by a theorem of [**?**], if $(V, D) = \cup_i (V_i, D_i)$ is a rational anticanonical pair then

$$\sum_i Q(V_i, D_i) = 24.$$

## 16. KSBA compactifications

**Remark 16.1.** Recall the Deligne-Mumford compactification of $\mathcal{M}_g$:

1. (1) Given a family of curves $\mathcal{X} \to \Delta$, after a base change and a birational transformation, we can assume this is a semistable family: $\mathcal{X}$ is smooth and $\mathcal{X}_0$ is RNC.
2. (2) Consider the relatively minimal model, which here involves contracting $(-1)-$curves. In this case, $\mathcal{X} \to \Delta$ is still smooth.
3. (3) Take the relative canonical model where one contracts $(-2)-$curves, making $\mathcal{X} \to \Delta$ singular.

Note that in the first two steps, $\mathcal{X}_0$ is not unique, but in the third step it becomes unique (at the cost of having a singular total space). As a result, one obtains $\overline{\mathcal{M}_g}$ by adding stable curves: $\mathcal{X}_0$ is at worst nodal, and every $\mathbb{P}^1$ component contains 3 distinguished points. This is equivalent to $\mathrm{Aut}(\mathcal{X}_0) < \infty$ and also equivalent to having an ample dualizing sheaf $\omega_{\mathcal{X}_0} > 0$. The last characterization is the one that generalizes to higher dimensional varieties and pairs.

**Remark 16.2.** A general question one can ask: given a moduli space $M$ that admits a KSBA and a Hodge-theoretic compactification, how do these compare? Is there a lift of the period map?

$$\begin{array}{ccc} \overline{M}^{\mathrm{KSBA}} & \overset{}{\underset{\exists ?\, \widetilde{\varphi}}{\dashrightarrow}} & \overline{\Gamma\backslash\Omega}^{\mathrm{bb}}, \overline{\Gamma\backslash\Omega}^{\mathrm{tor}}, \cdots \\ \uparrow & & \uparrow \\ M & \xrightarrow{\varphi} & \Gamma\backslash\Omega \end{array}$$

This is generally hard: given a smooth family $\mathcal{X} \to \Delta$ with $\mathcal{X}_0$ RNC one has the Clemens-Schmidt exact sequence, but there is not a clear analog when $\mathcal{X}$ is singular.

**Remark 16.3.** Since one can realize a K3 surface of degree 2 as ramified double cover $\pi : X \to \mathbb{P}^2$ over a sextic curve with branch divisor $B$, one can identify $F_2$ as moduli of pairs $(\mathbb{P}^2, B)$ with $B$ such a sextic. More generally, for $X$ a surface and $B$ an effective $\mathbb{Q}-$divisor, a pair $(X, B)$ is a stable pair of degree 6 if $(X, \frac{1+\varepsilon}{2}B)$ is slc and $K_X + \frac{1+\varepsilon}{2}B > 0$ with $2K_X + B \sim 0$ which smooths to $\mathcal{X}$ to a pair $\mathcal{X}_t = (\mathbb{P}^2, B_t)$ with $B_t$ a sextic. This ensures that the double cover is a K3 surface. Hacking shows there is a coarse moduli space of such pairs which contains $F_2$, providing a compactification whose boundary has geometric significance but is not explicitly described.

16A. **Stable pairs and KSBA compactifications.**

**Definition 16.4** (RNC)**.**

todo, modern terminology for simple normal crossings seems to be reduced normal crossings

**Definition 16.5** (dlt pairs)**.** A log pair $(X, D)$ is **divisorial log terminal (dlt)** if there is a smooth open subset $U \subseteq X$ for which the restriction $D|_U$ is RNC and for any divisorial valuation $E$ with center $X \setminus U$ we have $\alpha(E, X, D) > -1$.

**Definition 16.6** (KSBA stable pairs)**.** A pair $(X, D)$ with $X$ a variety and $D$ a $\mathbb{Q}-$ divisor with coefficients in $[0, 1]$ is **KSBA stable** if

- "Nodal/local": $(X, D)$ is a proper semi-log-canonical (slc) pair
- "Stable/global": $K_X + D > 0$, i.e. the log canonical class is ample.

This generalizes the notion of a nodal curve to higher dimensions. The first is a singularity condition, enforcing having at worst nodal singularities in codimension 1. The second condition generalizes a $g = 0$ curve having at least 3 marked points and a $g = 1$ curve having at least 1 marked point, where nodes are counted as points.

If $(X, 0)$ is a stable pair, we say $X$ is a stable variety.

**Remark 16.7.** A theorem of KSBA shows that if certain numerical invariants are fixed, a projective coarse moduli space of stable pairs exists. An example application: taking the closure in the space of stable pairs for $\mathcal{M}_{g,n}$ exactly recovers the Deligne-Mumford compactification.

**Definition 16.8** (Stable pairs)**.** A **stable pair** is a pair $(X, \varepsilon R)$ where $X$ is a seminormal slc surface with trivial dualizing sheaf and $R$ is an ample Cartier divisor containing no log canonical centers of $X$. For $0 < \varepsilon \ll 1$, this yields a KSBA stable pair.

**Remark 16.9.** There is a well-formulated notion of taking the closure of KSBA stable pairs for a moduli space $M$, yielding compactifications $\overline{M}^{\mathrm{KSBA}}$. .

Need to find a good source for this material, just general theory of KSBA compactification. Could also put here just how the theory works for $F_S$, c/o [?]

## 17. Lattice theory

### 17A. Definitions.

**Definition 17.1** (Basic of lattices)**.** For $R$ an integral domain and $k$ its field of fractions, an $R-$**lattice** $L$ is a finitely-generated projective $R-$module equipped with a nondegenerate symmetric bilinear form

$$\beta : L \otimes_R L \to k$$

As a matter of notation, we write $\beta(v, w)$ as $(v, w)$ or $vw$ or $v.w$ or $v \cdot w$, and similarly $v^2 \coloneqq \beta(v, v)$. The **quadratic form** associated to $\beta$ is

$$\begin{aligned} Q : L &\to k \\ v &\mapsto v^2 \coloneqq \beta(v, v) \end{aligned}$$

Note that given $Q$, one can recover $\beta$ by the standard formula

$$\beta(v, w) \coloneqq \frac{1}{2}(Q(v, w) - Q(v) - Q(w))$$

For a fixed set of $R-$module generators $\{v_1, \cdots, v_n\}$ of $L$, the **Gram matrix** $G(L)$ of $L$ is the matrix $G(L)_{ij} \coloneqq \beta(v_i, v_j) \in \mathrm{GL}_n(R)$, which is unique up to conjugation by matrices in $\mathrm{GL}_n(R)$ and is symmetric if $\beta$ is. The **discriminant** of $L$ is $\mathrm{disc}(L) \coloneqq \det(G(L))$. We say $L$ is **integral** if $\beta(L, L) \subseteq R$. An integral lattice is **even** if $\beta(L, L) \subseteq 2R$ and odd otherwise.

**Remark 17.2.** In our applications, we take $R \coloneqq \mathbb{Z}$ and $k \coloneqq \mathbb{Q}$, noting that since $\mathbb{Z}$ is a PID, projective $\mathbb{Z}-$modules are free and thus a $\mathbb{Z}-$lattice is equivalently a free $\mathbb{Z}-$module equipped with a bilinear form.

**Definition 17.3** (Morphisms of lattices)**.** An **embedding** of $R-$lattices $L_1$ in $L_2$ is an injective morphism of $R-$modules $\iota : L_1 \to L_2$ which is equivariant with respect to the bilinear forms, i.e.

$$\iota(\beta_L(l_1, l_2)) = \beta_M(\iota(l_1), \iota(l_2)).$$

The embedding is **primitive** if $\operatorname{coker} \iota$ is torsionfree. Any non-primitive embedding can be promoted to a primitive embedding by taking the **saturation** of $\iota(L_1)$ in $L_2$, the intersection of all primitive sublattices of $L_2$ which contain $\iota(L_1)$. For a fixed primitive embedding $L_1 \hookrightarrow L_2$, regarding $L_1 \leq L_2$ as a sublattice, define the **orthogonal complement** of $L_1$ in $L_2$ as

$$L_1^{\perp L_2} \coloneqq \{x \in L_2 \mid \beta_{L_2}(x, \iota(L_1)) = 0\}.$$

The **dual lattice**[16] of an integral lattice $L$ is

$$L^\vee \coloneqq \mathrm{Hom}_R(L, R) = \{x \in L_k \mid \beta_{L_k}(x, L) \subseteq R\} \subseteq L_k.$$

Todo: orthogonal sums of lattices.

Todo: genus of a lattice.

Smith-Minkowski-Siegal mass formula, used in Scattone.

**Remark 17.4** (Numerical invariants)**.** We now suppose $R = \mathbb{Z}$ and define $L_\mathbb{R} \coloneqq L \otimes_\mathbb{Z} \mathbb{R}$ and $L_\mathbb{C} \coloneqq L \otimes_\mathbb{Z} \mathbb{C}$ to be the real and complex vector spaces associated with $L$. The **rank** of $L$ is its rank as a free $\mathbb{Z}-$module $\mathrm{rank}(L) \coloneqq \mathrm{rank}_\mathbb{Z} L$, or equivalently $\dim_\mathbb{Q} L_\mathbb{Q}$. There is an induced bilinear form on $L_\mathbb{Q}$ whose eigenvalues are $\pm 1$ or 0. Write $n_+, n_-, n_0$ for the multiplicities of the eigenvalues $1, -1, 0$ respectively; we

[16] Note that the natural extension of the bilinear form $\beta_L$ on $L$ to $\beta_{L_k}$ on $L_k$ may no longer be $R-$valued.

say $L$ is **nondegenerate** if $n_0 = 0$, or equivalently if the following map sending vectors to linear functionals defines an isomorphism of $\mathbb{Q}-$modules

$$\begin{aligned} \phi : L_{\mathbb{Q}} &\to \mathrm{Hom}_{\mathbb{R}}(L_{\mathbb{Q}}, \mathbb{Q}) \\ x &\mapsto f_x := (x, -) \end{aligned}$$

Define the **signature** of $L$ as $\mathrm{sgn}(L) := (n_+, n_-)$. A positive-rank lattice is **positive (resp. negative) definite** if it is of signature $(n_+, 0)$, resp. $(0, n_-)$.

**Remark 17.5** (Discriminant forms)**.** The dual lattice $L^\vee$ admits a canonical $\mathbb{Q}-$valued bilinear form extending the $\mathbb{Z}-$valued bilinear form $\beta$ on $L$,

$$\begin{aligned} \beta^\vee : L^\vee \otimes_{\mathbb{Z}} L^\vee &\to \mathbb{Q} \\ f_x \otimes f_y &\mapsto \beta_L(x, y) \end{aligned}$$

If $L$ is integral, there is an injection $L \hookrightarrow L^\vee$ realizing $L$ as a finite index sub $\mathbb{Z}-$module of $L^\vee$, thus the cokernel

$$A_L := \mathrm{coker}(L \hookrightarrow L^\vee) \cong L^\vee / L$$

is a finite abelian group called the **discriminant group**. It is called this because its order is the discriminant of $L$. The discriminant group is canonically equipped with a well-defined quadratic form

$$\begin{aligned} q_L : A_L &\to \mathbb{Q}/\mathbb{Z} \\ \ell + L &\mapsto \frac{1}{2}\ell^2 + \mathbb{Z} \end{aligned}$$

If $L$ is even, this takes values in $\mathbb{Q}/2\mathbb{Z}$. There is an associated bilinear form on $A_L$ defined by the standard formula,

$$\beta_{A_L} : A_L \otimes_{\mathbb{Z}} A_L \to \mathbb{Q}/\mathbb{Z}.$$

We call the pair $(A_L, q_L)$ the **discriminant form** of $L$. We define the **length** $\ell(A_L)$ of $A_L$ as the minimal number of $\mathbb{Z}-$module generators of $A_L$.

If $A_L = 0$, or equivalently $L \cong L^\vee$ or $\mathrm{disc}(L) = \pm 1$, we say $L$ is **unimodular**. If $L$ is unimodular and $S \hookrightarrow L$ is a primitive sublattice, then there is a canonical isomorphism of discriminant forms

$$(A_S, q_S) \cong (A_{S^{\perp L}}, -q_{S^{\perp L}})$$

where $S^\perp$ is the orthogonal complement of $\iota(S)$ in $L$. If $A_L \cong (\mathbb{Z}/n\mathbb{Z})^{\oplus^m}$ decomposes as a sum of cyclic groups of the same order $n$, we say $L$ is $n-$**elementary.**

**Remark 17.6** (Overlattices)**.** An **overlattice** $\widetilde{L}$ of $L$ is submodule of $L^\vee$ such that $L \subseteq \widetilde{L} \leq L^\vee$ which is integral, i.e. the restriction $\beta^\vee|_{\widetilde{L}}$ has image contained in $\mathbb{Z}$. Equivalently, their index $[\widetilde{L} : L]$ as $\mathbb{Z}-$modules is finite. Note that if $L_1 \hookrightarrow L_2$ is an embedding of lattices with $\mathrm{rank}\, L_1 = \mathrm{rank}\, L_2$, then $L_2 \hookrightarrow {L_1}^\vee$ is an overlattice of $L_1$ in ${L_1}^\vee$. For two lattices $L$ and $L'$, we write $L \oplus L'$ for the direct sum which is orthogonal with respect to the combined bilinear form $\beta_L \oplus \beta_{L'}$.

**Remark 17.7** (Orthogonal group and reflections)**.** The **orthogonal group of** $L$ is the set of isometric $\mathbb{Z}-$linear operators on $L$:

$$\mathrm{O}(L) := \{\gamma \in \mathrm{Aut}_{\mathbb{Z}}(L) \mid \gamma(v).\gamma(w) = v.w \,\forall v, w \in L\}.$$

Similarly, we define the **orthogonal group of** $A_L$ as operators which preserve the discriminant quadratic form:

$$\mathrm{O}(A_L) = \{\gamma \in \mathrm{Aut}_{\mathbb{Z}}(A_L) \mid q_L(\gamma(v)) = q_L(v)\}.$$

There is a group homomorphism $\mathrm{O}(L) \to \mathrm{O}(A_L)$.

**Remark 17.8.** For $v \in L$ with $v^2 \neq 0$, define its **spinor norm** as $\|v\|_{\mathrm{spin}} := \frac{1}{2}v^2$, its **spinor normalization** as $\widehat{v} := v/\|v\|_{\mathrm{spin}}$, and its corresponding **hyperplane reflection** as

$$s_v(x) := x - (x, v)\widehat{v}.$$

Note that $s_v \in \mathrm{O}(L)$ for any $v \in L$. If $(V, Q)$ is a finite-dimensional nondegenerate quadratic space over a characteristic zero field, every isometry of $(V, Q)$ is a product of such hyperplane reflections.

**Remark 17.9** (Invariants and classification)**.** We define an invariant $\delta \in \mathbb{F}_2$ of $A_L$ called the **coparity** by $\delta = 0$ if $q_L(A_L) = 0 \mod \mathbb{Z}$ and $\delta = 1$ otherwise. By a result of Nikulin, integral even indefinite $2-$elementary lattices $L$ are determined up to isomorphism by $\mathrm{sgn}(L)$ and the tuple $(r, a, \delta)$ where $r := \mathrm{rank}_{\mathbb{Z}} L$ and $a := \mathrm{rank}_{\mathbb{F}_2} A_L$. Note that if $L$ is hyperbolic, $\mathrm{sgn}(L)$ is uniquely determined by $r$, so the triple $(r, a, \delta)$ suffices to uniquely classify 2-elementary hyperbolic lattices.

**Remark 17.10.** If $M$ is an even lattice of signature $(s_+, s_-)$, one can always find a primitive embedding into an even unimodular lattice of higher rank. In particular, if $L$ is such a lattice of signature $(r_+, r_-)$, there is a unique primitive embedding $M \hookrightarrow L$ if

- $s_+ < r_+$,
- $s_- < r_-$, and
- $\ell(A_M) \leq \mathrm{rank}\, L - \mathrm{rank}\, M - 2$.

**Definition 17.11** (Divisibility)**.** For $v \neq 0$ in a lattice $L$, its **divisibility** is defined by

$$vL = \mathrm{div}(v)\mathbb{Z},$$

the dilation factor of the one-dimensional lattice generated by $v$, measuring how far $v$ is from generating a standard sublattice $\mathbb{Z}$.

17B. **Hyperbolic lattices.**

**Definition 17.12** (The future orthogonal group of a hyperbolic lattice)**.** Let $L$ be a hyperbolic lattice of signature $(1, n)$, define

$$\mathrm{O}^+(L) := \mathrm{O}(L) \cap \mathrm{O}^+(L_{\mathbb{R}}).$$

17C. **Constructions of specific lattices.**

**Remark 17.13** (Twists of a lattice)**.** If $L$ is a lattice with bilinear for $\beta_L$, define $L(n)$ to be the twist of $L$ by $n$, which has the same underlying $\mathbb{Z}-$module but is equipped with the scaled bilinear form

$$\beta_{L(n)}(x, y) := n\beta_L(x, y).$$

**Remark 17.14** (The lattice $\langle n \rangle$)**.** The lattice $\langle n \rangle$ is defined as the rank 1 lattice $\mathbb{Z}$ with one generator $f$ satisfying $\beta(f, f) = n$. The Gram matrix is the $1 \times 1$ matrix $[n]$, and the associated quadratic form is $Q(x) = nx^2$.

**Remark 17.15** (The lattices $\mathrm{I}_{p,q}$ and $\mathrm{II}_{p,q}$)**.** $\mathrm{II}_{p,q}$ when $p-q \equiv 0 \pmod 8$: the unique even indefinite integral lattice, determined uniquely by its rank and signature, with presentation

$$\mathrm{II}_{p,q} := \begin{cases} \mathrm{II}_{1,1}{}^{\oplus^p} \oplus E_8{}^{\oplus^{\frac{-(p-q)}{8}}}, & p<q \\ \mathrm{II}_{1,1}{}^{\oplus^q} \oplus E_8(-1)^{\oplus^{\frac{p-q}{8}}}, & p>q. \end{cases}$$

DZG: I haven't worked out whether the original source for this used positive or negative definite $E_8$, so might need to work this out later to match everything up.

**Definition 17.16.** Let $L$ be a lattice and let $B = \{e_1, e_{-1}, \cdots, e_n, e_{-n}\}$ be an ordered set of vectors in $L$. We say $B$ is **hyperbolic** if $e_i e_{-i} = 1$ and $e_i e_j = 0$ for $i \neq -j$. If $\mathbb{Z}B = L$, we say $B$ is a **hyperbolic basis** of $L$.

**Remark 17.17** (The hyperbolic lattice)**.** In rank 2, there are two unimodular hyperbolic lattices: the odd $\mathrm{I}_{1,1} := \langle 1\rangle \oplus \langle -1\rangle$, and the even $U := \mathrm{II}_{1,1}$. The latter is what we call the **hyperbolic lattice**, and can be realized as $\mathrm{II}_{1,1} = \mathbb{Z}f \oplus \mathbb{Z}g$ with $f, g$ a hyperbolic basis, i.e. $f^2 = g^2 = 0$ and $fg = 1$, with Gram matrix

$$G(\mathrm{II}_{1,1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

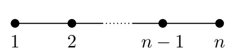
**Remark 17.18** (Type $A_n$)**.** The root lattice $A_n$ can be constructed as

$$A_n := \left\{\lambda \in \mathbb{Z}^{n+1}; \lambda_1 + \ldots + \lambda_{n+1} = 0\right\}, \operatorname{disc} A_n = n+1$$

Choosing the standard basis $\{e_i - e_{i-1} \mid 0 \le i \le n-1\}$, the Gram matrix is

$$G(A_n) = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix} \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$$

The discriminant group is cyclic of order $n+1$: $A_{A_n} \cong C_{n+1}$. The Weyl group is the symmetric group $S_n$. The corresponding Dynkin diagram is the following:

$A_n$ :


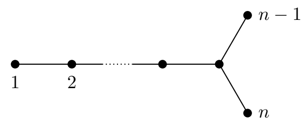

**Remark 17.19** (Type $D_n$)**.** The root lattice $D_n$ can be constructed as

$$D_n := \{\lambda \in \mathbb{Z}^n; \lambda_1 + \ldots + \lambda_n \equiv 0 (\mathrm{mod} 2)\}, \operatorname{disc} D_n = 4$$

In the basis $\{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, e_{n-1} + e_n\}$, the Gram matrix is

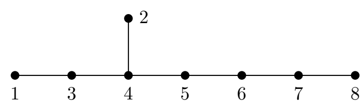
$$G(D_n) = \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix} \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$$

The discriminant group is $C_4$ is $n$ is even and ${C_2}^{\oplus^2}$ if $n$ is odd. The corresponding Dynkin diagram is the following:

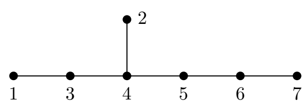
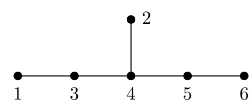
$D_n$ : 

**Remark 17.20** (The $^+$ and $^-$ constructions)**.** Let $v_+ = \frac{1}{2}(1, 1, 1, \cdots, 1)$ and $v_- = \frac{1}{2}(-1, 1, 1, \cdots, 1)$, and define $D_n^+ := D_n \oplus \mathbb{Z}v_1$ and $D_n^- := D_n \oplus \mathbb{Z}v_2$. If $n$ is odd, $D_n^+ \cong D_n^- \cong {D_n}^\vee$. If $n$ is even, $D_n^+ \ncong D_n^-$ and both contain $D_n$ with index 2 and are thus unimodular and are even integral unimodular lattices if and only 8 divides $n$.

**Remark 17.21** (Types $E_8, E_7, E_6$)**.** The root lattice $E_8$ can be constructed as $E_8 = D_8 \oplus \mathbb{Z}v_+ = D_*^+ \cong \mathrm{II}_{0,8}$. It is the rank 8 negative definite even unimodular lattice defined by the Cartan matrix of the $E_8$ Dynkin diagram:

$E_8$ : 

The lattice $E_7 = \mathrm{Ann}_{E_8}(L_1)$ for $L_1 \leq E_8$ any sublattice isometric to $A_1$, and similarly $E_6 = \mathrm{Ann}_{E_8}(L_2)$ for any $L_2$ isometric to $A_2$. These have the following Dynkin diagrams:

$E_7$ :  , $E_6$ : 

17D. **Neimeier lattices.**

**Definition 17.22.** The *Niemeier lattices* are the definite unimodular lattices of rank 24. We take them to be negative definite. These were classified by Niemeier in his doctoral thesis and published in 1973. One of these is the Leech lattice $\Lambda_{24}$, discovered by Leech in 1967, the unique Niemeier lattice with no roots. So it cannot be described as a unimodular overlattice of a rank 24 root lattice.

The following is a list of all such rank 24 lattices:

| | | | |
|---|---|---|---|
| $E_8^{\oplus 3}$ | $D_8^{\oplus 3}$ | $D_5^{\oplus 2} \oplus A_7^{\oplus 2}$ | $A_6^{\oplus 4}$ |
| $E_8 \oplus D_{16}$ | $D_9 \oplus A_{15}$ | $A_8^{\oplus 3}$ | $A_4^{\oplus 6}$ |
| $E_7^{\oplus 2} \oplus D_{10}$ | $E_6^{\oplus 4}$ | $A_{24}$ | $A_3^{\oplus 8}$ |
| $E_7 \oplus A_{17}$ | $E_6 \oplus D_7 \oplus A_{11}$ | $A_{12}^{\oplus 2}$ | $A_2^{\oplus 12}$ |
| $D_{24}$ | $D_6^{\oplus 4}$ | $D_4^{\oplus 6}$ | $A_1^{\oplus 24}$ |
| $D_{12}^{\oplus 2}$ | $D_6 \oplus A_9^{\oplus 2}$ | $D_4 \oplus A_5^{\oplus 4}$ | $\emptyset$ |

Conway studied $\mathrm{Aut}(\Lambda_{24})$ in order to produce three new sporadic simple groups, the Conway groups $C_{o_1}, C_{o_2}, C_{o_3}$.

17E. **Lattices in K3 theory.**

**Remark 17.23.** The following are sources of lattices that arise when studying moduli of K3s and Enriques surfaces:

- The K3 **lattice** $\mathrm{II}_{3,19}$ is defined by
$$\mathrm{II}_{3,19} \coloneqq \mathrm{II}_{1,1}^{\oplus 3} \oplus E_8^{\oplus 2}$$
which satisfies $\mathrm{rank}(\mathrm{II}_{3,19}) = 22$ and $\mathrm{sgn}(\mathrm{II}_{3,19}) = (3, 19)$. This is the lattice structure on $H^2(X;\mathbb{Z})$ for $X$ any K3 surface.
- The **Enriques lattice** $\mathrm{II}_{1,9}$ is defined by
$$\mathrm{II}_{1,9} \coloneqq \mathrm{II}_{1,1} \oplus E_8$$
which satisfies $\mathrm{rank}(\mathrm{II}_{1,9}) = 10$ and $\mathrm{sgn}(\mathrm{II}_{1,9}) = (1, 9)$.
- The **degree** $d$ **polarization lattice for K3 surfaces** is
$$\mathrm{II}_{3,19}\langle d\rangle \coloneqq {\mathrm{II}_{1,1}}^{\oplus^2} \oplus {E_8}^{\oplus^2} \oplus \langle -2d\rangle$$
- The **Borcherds lattice** is
$$\mathrm{II}_{2,26} = {\mathrm{II}_{1,1}}^{\oplus^2} \oplus {E_8}^{\oplus^3}$$
There is a primitive embedding $\mathrm{II}_{3,19} \hookrightarrow \mathrm{II}_{2,26}$.
- For $S$ a K3 surface, the **Néron-Severi lattice** $\mathrm{NS}(S)$ is defined as
$$\mathrm{NS}(S) \coloneqq H^{1,1}(S;\mathbb{R}) \cap H^2(S;\mathbb{Z}).$$
- The **transcendental lattice** of $S$, $\mathrm{T}(S)$, is defined as
$$\mathrm{T}(S) \coloneqq \mathrm{NS}(S)^{\perp},$$
where the orthogonal complement is taken in $H^2(S;\mathbb{Z})$.

**Example 17.24.** For $d = 2$,
$$\mathrm{II}_{3,19}\langle 2\rangle \coloneqq {\mathrm{II}_{1,1}}^{\oplus^2} \oplus {E_8}^{\oplus^2} \oplus \langle -4\rangle$$
which seems to be isometric to ${\mathrm{II}_{1,1}}^{\oplus^2} \oplus D_{16}$.

**Remark 17.25** (Some stuff maybe needed for automorphisms)**.** For a free finitely-generated $\mathbb{Z}-$module $A$, write $\ell(A)$ for the minimal number of generators of $A$. For $G \leq \mathrm{O}(L)$, defined the invariant lattice $L^G \coloneqq \{v \in L \mid \gamma(v) = v \,\forall \gamma \in G\}$ and the coinvariant lattice as its orthogonal complement in $L$, $L_G \coloneqq (L^G)^{\perp}$.

**Remark 17.26.** There are several unexpected isometries of lattices worth noting:

- $D_1 \cong \langle -4\rangle$
- $D_2 \cong {A_2}^{\oplus^2}$

17F. **Orthogonal and automorphism groups of lattices.**

**Remark 17.27.** In this section, we generalize the theory of orthogonal groups with respect to quadratic forms to orthogonal groups of lattices, using the fact that any symmetric bilinear form has an associated quadratic form when the characteristic of the underlying field is not 2. These constructions recover the standard real forms of orthogonal groups, the real Lie groups that are usually written $\mathrm{O}(n)$ and $\mathrm{O}(p, q)$, by tensoring the lattice up to $\mathbb{R}$. This hopefully also clears up some ambiguity in the literature, because many papers use the terms 'isometry group', 'orthogonal group', and 'automorphism group' of lattice interchangeably. This is a slight issue when defining integral affine spheres, since it's unclear which of $\mathrm{GL}_2(\mathbb{Z}), \mathrm{O}_2(\mathbb{Z}), \mathrm{SO}_2(\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z})$ one should use for the allowed group of transformations for the standard lattice $\mathbb{Z}^2$ that exists in the charts of the atlas.

**Definition 17.28** (General linear group of a $k-$vector space)**.** For $V$ a fixed $k-$vector space, we define the general linear group of $V$ by its functor of points:

$$\begin{aligned} \mathrm{GL}_V : k\text{-}\mathsf{Alg} &\to \mathsf{Groups} \\ R &\mapsto \mathrm{GL}_V(R) \coloneqq \mathrm{Aut}_R(V \otimes_k R) \end{aligned}$$

Fixing a $k-$basis for $V$, we make the following abuse of notation:

$$\mathrm{GL}(V) \coloneqq \mathrm{GL}_V(k) = \mathrm{GL}_n(k)$$

**Definition 17.29** (General linear group of a lattice)**.** For $L$ a fixed $\mathbb{Z}-$lattice, we define the general linear group of $L$ by its functor of points: for a $\mathbb{Z}-$algebra $R$, it is

$$\mathrm{GL}_L(R) \coloneqq \mathrm{Stab}_{\mathrm{GL}_{L_{\mathbb{R}}}(R)}(L) = \left\{\gamma \in \mathrm{GL}_{L_{\mathbb{R}}}(R) \mid \gamma(L) = L\right\}.$$

Fixing a lattice basis for $L$, we make the following abuse of notation:

$$\mathrm{GL}(L) \coloneqq \mathrm{GL}_L(\mathbb{Z}) \coloneqq \{\gamma \in \mathrm{GL}_n(\mathbb{Z}) \mid \gamma(L) = L\} = \mathrm{Stab}_{\mathrm{GL}_n(\mathbb{Z})}(L)$$

**Definition 17.30** (Orthogonal group of a quadratic $k-$module)**.** Let $(V, Q)$ be a fixed quadratic $k-$module.[17] We define the orthogonal group of $(V, Q)$ by its functor of points: on a $k-$algebra $R$, we set

$$\mathrm{O}_{(V,Q)}(R) \coloneqq \left\{\gamma \in \mathrm{GL}_V(R) \mid Q(\gamma v) = Q(v)\, \forall v \in V\right\}.$$

If $k = \mathbb{R}$, letting $\mathrm{sgn}(V, Q) = (p, q)$, fixing a $\mathbb{R}-$basis for $V$ we write $V \cong \mathbb{R}^{p,q}$, we make the following abuse of notation:

$$\mathrm{O}(V) \coloneqq \mathrm{O}_{(V,Q)}(\mathbb{R}) = \mathrm{O}_{p,q}(\mathbb{R}).$$

**Remark 17.31.** If $V$ is definite, so wlog $(p, q) = (n, 0)$, we write $\mathrm{O}(V) = \mathrm{O}_n(\mathbb{R})$. Note that if $k = \mathbb{R}$, $V = \mathbb{R}^n$, and $Q_{\mathrm{std}}$ is the standard positive definite form $Q_{\mathrm{std}}(\vec{x}) = \sum x_i^2$, the real points recover

$$\mathrm{O}_{\mathbb{R}^n, Q_{\mathrm{std}}}(\mathbb{R}) = \mathrm{O}_n(\mathbb{R})$$

the standard real orthogonal Lie group.

**Definition 17.32** (Orthogonal/isometry group of a lattice)**.** Let $L$ be a lattice with $(L_{\mathbb{R}}, q)$ its associated quadratic space. The orthogonal group or group of isometries of $L$ is defined on $\mathbb{R}-$algebras $R$ as

$$\mathrm{O}_L(R) \coloneqq \mathrm{O}_{(L_{\mathbb{R}},q)}(R) = \left\{\gamma \in \mathrm{GL}_{L_{\mathbb{R}}}(R) \mid (\gamma v, \gamma w) = (v, w)\, \forall v, w, \in L\right\}.$$

If a lattice basis is fixed, we make the following identification/abuse of notation:

$$\mathrm{O}(L) \coloneqq \mathrm{O}_L(\mathbb{Z}) = \{\gamma \in \mathrm{GL}_n(\mathbb{Z}) \mid (\gamma v, \gamma w) = (v, w)\, \forall v, w \in L\}$$

**Remark 17.33.** After choosing a basis, one can realize $\mathrm{O}(L) \leq \mathrm{GL}_n(\mathbb{Z})$ as a discrete compact (and thus finite) subgroup where $n \coloneqq \mathrm{rank}_{\mathbb{Z}} L$. Taking $L \coloneqq \mathbb{Z}^n$ to be the standard lattice or any positive-definite lattice of signature $(n, 0)$, we have $L_{\mathbb{R}} \cong \mathbb{R}^{n,0} \cong \mathbb{R}^n$ and this construction yields $\mathrm{O}_L(\mathbb{R}) = \mathrm{O}_n(\mathbb{R})$ recovering the standard orthogonal group. If $L$ is indefinite with signature $(p, q)$ then this construction recovers $\mathrm{O}_L(\mathbb{R}) = \mathrm{O}_{p,q}(\mathbb{R})$, the isometries of $\mathbb{R}^{p,q}$.

[17] I.e. a $k-$vector space equipped with a quadratic form.

**Definition 17.34** (Automorphism groups of lattices)**.** Let $L$ be a $\mathbb{Z}-$lattice, we define

$$\begin{aligned}\mathrm{Aut}_L(R) &\coloneqq \mathrm{Stab}_{\mathrm{O}_L(R)}(L)\\ &= \{\gamma \in \mathrm{O}_L(R) \mid \gamma(L) = L\}\\ &= \{\gamma \in \mathrm{GL}_{L_\mathbb{R}}(R) \mid \gamma(L) = L,\ (\gamma v, \gamma w) = (v, w)\}\\ &= \mathrm{GL}_L(R) \cap \mathrm{O}_{(L_\mathbb{R}, q)}(R).\end{aligned}$$

If a lattice basis is fixed, we make the following identification/abuse of notation:

$$\mathrm{Aut}(L) \coloneqq \mathrm{Aut}_L(\mathbb{R}) = \{\gamma \in \mathrm{O}_n(\mathbb{R}) \mid \gamma(L) = L,\ (\gamma v, \gamma w) = (v, w)\}.$$

**Definition 17.35** (Isotropic subspaces)**.** Let $L$ be a lattice with bilinear form $\beta_L$ with $\beta_{L_\mathbb{R}}$ its natural extension to $L_\mathbb{R}$. An **isotropic subspace** associated to $L$ is any subspace $W \leq L_\mathbb{R}$ such that $\beta_{L_\mathbb{R}}|_W = 0$, or equivalently $W \subseteq W^\perp$. A vector $e \in L$ is isotropic if $e^2 = 0$.

**Definition 17.36** (Isotropic Grassmannian)**.** Let $L$ be a lattice with bilinear form $\beta_L$, with $\beta_{L_\mathbb{R}}$ its natural extension to $L_\mathbb{R}$. The **Grassmannian** $\mathrm{Gr}_k(L)$ **of** $L$ is defined to be the algebraic variety of $k-$dimensional subspaces in $L_\mathbb{R}$ generated by vectors in $L$. The **isotropic Grassmannian** $\mathrm{Gr}_k^{\mathrm{iso}}(L)$ **of** $L$ [18] is defined as (an irreducible component of) the subvariety of $k-$dimensional subspaces of $L_\mathbb{R}$ which are isotropic with respect to $\beta_{L_\mathbb{R}}$.

**Remark 17.37.** Thus an isotropic line spanned by an isotropic vector $e \in L$ defines an element $\mathbb{Z}e \in \mathrm{Gr}_1^{\mathrm{iso}}(L)$. Similarly, an isotropic plane spanned by $e, f$ yields $\mathbb{Z}e \oplus \mathbb{Z}f \in \mathrm{Gr}_2^{\mathrm{iso}}(L)$.

**Remark 17.38.**

17G. **Discriminant quadratic forms.**

**Lemma 17.39.** *Let $L$ be a lattice that decomposes as $L \cong \bigoplus_i L_i$. Then $q_L = \sum_i q_{L_i}$, i.e. the discriminant form on $L$ decomposes as a sum of the discriminant forms on the $L_i$.*

Proof and example?

17G.1. *For* $\mathrm{II}_{1,1}(2)$. The lattice $\mathrm{II}_{1,1}(2)$ is generated by $e, f$ such that $e^2 = f^2 = 0$ and $e \cdot f = 2$. The elements $e/2$ and $f/2$ belong to the dual $\mathrm{II}_{1,1}(2)^\vee$ and their classes in $A_{\mathrm{II}_{1,1}(2)} = \mathrm{II}_{1,1}(2)^\vee / \mathrm{II}_{1,1}(2)$ give a set of generators. Their intersection matrix is $\left(\begin{smallmatrix} 0 & 1/2 \\ 1/2 & 0 \end{smallmatrix}\right)$, which gives the discriminant quadratic form $A_{\mathrm{II}_{1,1}(2)} \to \mathbb{Q}/2\mathbb{Z}$.

17G.2. *For* $D_{16}$. Let $D_{16} = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_{16}$ as in Figure 38. The discriminant group $A_{D_{16}}$ is known to be isomorphic to $\mathbb{Z}_2^2$, and two generators are given by

$$\frac{1}{2}(e_1 + e_3 + e_5 + e_7 + e_9 + e_{11} + e_{13} + e_{15}),$$
$$\frac{1}{2}(e_1 + e_3 + e_5 + e_7 + e_9 + e_{11} + e_{13} + e_{16}).$$

The intersection matrix of these two vectors is given by $\left(\begin{smallmatrix} -4 & -7/2 \\ -7/2 & -4 \end{smallmatrix}\right)$, which describes the discriminant quadratic form $q\colon A_{D_{16}} \to \mathbb{Q}/2\mathbb{Z}$. Note that adding or

[18] $\mathrm{OGr}_k$ is used when one has a nondegenerate *symmetric* bilinear form, and $\mathrm{SGr}_k$ for an alternating/skew-symmetric form.

subtracting an even integer (resp. an integer) along the diagonal (resp. off the diagonal) gives the same quadratic form. So the discriminant quadratic form of $D_{16}$ is $\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$, which is the same as the one for $\mathrm{II}_{1,1}(2)$.

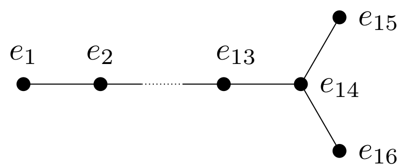



FIGURE 38. $D_{16}$ Dynkin diagram.

## 18. Valery's notes

### 18A. Notes from Valery.

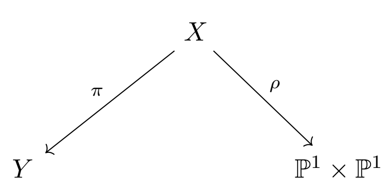
**Remark 18.1.** A K3 surface is a smooth projective surface $X$ with $K_X = 0$ and $h^1(\mathcal{O}_X) = 0$. An Enriques surface $Y$ is a smooth projective complex surface satisfying $h^1(\mathcal{O}_X) = 0$ and $K_X \neq 0$ but $2K_X = 0$. One can construct $X \coloneqq \mathrm{Spec}_{\mathcal{O}_X}(\mathcal{O}_X \oplus \mathcal{O}_X(K_X))$ to obtain a 2-to-1 ramified cover $\pi : X \to Y$ where $Y \cong X/\iota$ for $\iota$ a basepoint-free involution of $X$, often called the Enriques involution. It can be shown that $\iota$ is anti-symplectic, so $\iota^*\Omega_Y = -\Omega_Y$ where $\Omega_Y = ?$. Since a K3 surface is simply-connected, this realizes $X$ as the universal cover of $Y$, yielding $\pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$.

**Remark 18.2** (Lattices)**.** If $Y$ is Enriques with canonical K3 cover $\pi : X \to Y$, note that $\mathrm{Num}(Y)$ is isometric to $\mathrm{II}_{1,9}$ and the pullback $\pi^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ yields a sublattice isometric to $\mathrm{II}_{1,9}(2) \leq \mathrm{Pic}(X) \leq \mathrm{II}_{3,19}$. Thus we obtain a lattice polarized K3 surface, polarized by $\mathrm{II}_{1,9}(2)$, and thus a bijection between marked Enriques surfaces and $\mathrm{II}_{1,9}(2)-$polarized K3 surfaces.

A marked Enriques surface is a pair $(Y, \phi)$ where $\phi : \mathrm{Num}(Y) \to \mathrm{II}_{1,9}$ is an isometry of lattices.

**Remark 18.3** (Moduli)**.** For K3 surfaces, it is well-known that there is a 20-dimensional coarse moduli space of unpolarized surfaces $X$ and a 19-dimensional coarse moduli space $F_{2d}$ of polarized surfaces $(X, L)$ with $L^2 \coloneqq c_1(L)^2 = 2d$. For Enriques surfaces, both the unpolarized and polarized moduli spaces are 10-dimensional.

We review Horikawa's construction: one can obtain a K3 surface $X$ as a 2-to-1 cover $\rho : X \to \mathbb{P}^1 \times \mathbb{P}^1$ branched over a divisor $D \in -2K_{\mathbb{P}^1 \times \mathbb{P}^1}$ of bidegree $(4, 4)$. If $\pi : X \to Y$ also covers an Enriques surface as above, it is known that the involutions $\pi, \rho$ commute.

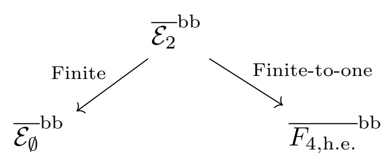


Defining $L := \rho^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$, we have $L^2 = 2\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)^2 = 4$, and so the pair $(X, L) \in F_4$ and yields what is called a **hyperelliptic degree 4 K3 surface**. Generally, for polarized degree 4 K3 surfaces $(X, L)$,

- A generic $(X, L)$ yields $\phi_{|L|} : X \hookrightarrow \mathbb{P}^3$, dimension of moduli = 19
- A hyperelliptic $(X, L)$ yields $\phi_{|L|} : X \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^1$, dimension of moduli = 18
- A unigonal $(X, L)$ has $|L|$ = a section $s$ plus an elliptic pencil $|E|$

**Remark 18.4** (Cusps)**.** Sterk described the Baily-Borel compactification of moduli of Enriques surfaces with a degree 2 numerical polarization $\overline{\mathcal{E}_2}^{\text{bb}}$, which is 10-dimensional with 5 0-cusps and 9 1-cusps. It is known that this maps to $\overline{\mathcal{E}_\emptyset}^{\text{bb}}$, which is dimension 10 and the map is finite, and to $\overline{F_{4,\text{h.e.}}}^{\text{bb}}$ which is dimension 18 and the map is finite-to-one onto its image. The Kulikov degenerations of $\overline{\mathcal{E}_\emptyset}^{\text{bb}}$ are described in [?], and $\overline{F_{4,\text{h.e.}}}^{\text{bb}}$ is described in the same paper.

**Question 18.5.** The main question is, how do the cusps of $\overline{\mathcal{E}_2}^{\text{bb}}$ map to the cusps of $\overline{\mathcal{E}_\emptyset}^{\text{bb}}$ and $\overline{F_{4,\text{h.e.}}}^{\text{bb}}$ respectively?




The answer to this question is lattice-theoretic and amounts to understanding the following diagram:

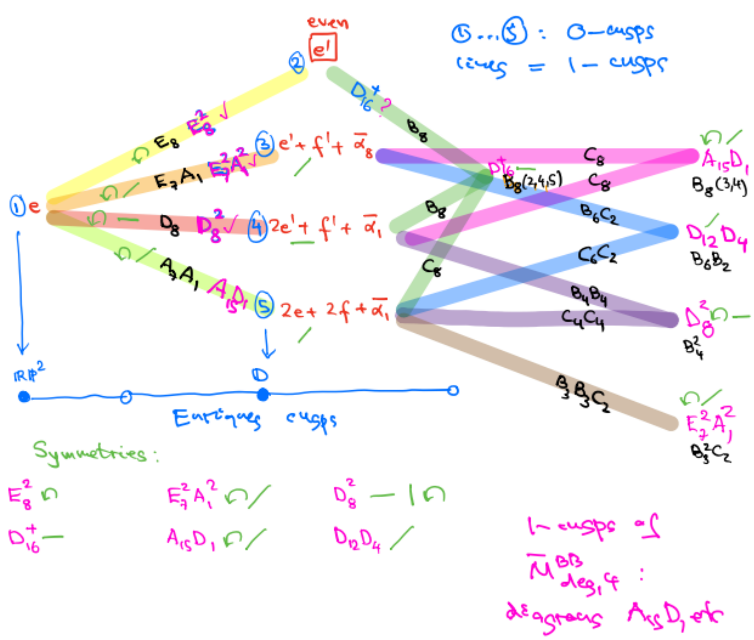



FIGURE 39. A cusp correspondence

**Remark 18.6** (Main task)**.** The main task is to describe an $\mathrm{IAS}^2$ with 2 involutions $\pi$ and $\rho$ of Enriques type and degree 4 hyperelliptic type respectively. Each of these is done individually in [**?**]. This amounts to finding a picture of an integral affine structure for which both symmetries are present, for each 0-cusp of $\overline{\mathcal{E}_2}^{\mathrm{bb}}$.

Todo: what are the IASs for degree 4 hyperelliptic and Enriques type individually?

### 18B. **Misc context.**

**Definition 18.7** (Elliptic pencils)**.** An **elliptic pencil** $|E|$ on a surface is a 1-dimensional linear system whose generic element is an irreducible elliptic curve.

It is well-known that an elliptic pencil $|E|$ on an Enriques surfaces $Y$ is basepoint-free, yielding an elliptic fibration $\phi_E : Y \to \mathbb{P}^1$. There are exactly two multiple fibers, $F$ and $F'$, and one can realize $|E| = |2F| = |2F'|$.

## 19. Mirror Symmetry

based on

**Remark 19.1.** Some ideas based on conference discussions and questions:

- Disambiguate the many different types of mirror symmetry to more explicitly say which type we use (SYZ, HMS, Batyrev, the one involving superpotentials and Landau-Ginzberg models, etc)
- Can we produce specific examples of toric varieties and their mirrors? E.g. what is the mirror of $\mathbb{P}^2$?
- 

**Remark 19.2.** For $\mathcal{X}$ a family of K3 surfaces polarized by a lattice $L$. Then we expect a mirror family $\check{\mathcal{X}}$ polarized by $\check{L}$ such that $\operatorname{rank} L + \operatorname{rank} \check{L} = 20$ and $L^\perp = \check{L} \oplus nU$

Todo: figure out if this is $U(n)$ or $U^{\oplus n}$.

## 20. Appendix

| $L$ | $A_L$ | $\sharp\Phi^2(L)$ | $W(L)$ |
|---|---|---|---|
| $A_n, n \geq 1$ | $C_{n+1}$ | $n^2 + n$ | $S_{n+1}$ |
| $D_n, n \geq 4$ | $\begin{cases} C_2^2 & n \text{ even} \\ C_4 & n \text{ odd} \end{cases}$ | $2(n^2 - n)$ | $C_2^{n-1} \rtimes S_n$ |
| $E_6$ | $C_3$ | 72 | |
| $E_7$ | $C_2$ | 126 | |
| $E_8$ | 1 | 240 | |

Table 8. Invariants of ADE lattices

### 20A. **Lattices.**

## 20B. Coxeter-Vinberg diagrams.

**Remark 20.1** (Diagram conventions)**.** Diagram conventions are shown in Table 1.

| Description | Diagram | Notation | $m_{ij}$ | $\angle(H_i, H_j)$ | $w_{ij}$ |
|---|---|---|---|---|---|
| Labeled simple edge | $H_1$ —$m_{ij}$— $H_2$ | $H_i \pitchfork H_j$ | $m_{ij}$ | $\pi/m_{ij}$ | $\cos\left(\frac{\pi}{m_{ij}}\right)$ |
| No Edge | $H_1$ ○ ○ $H_2$ | $H_i \perp H_j$ | 2 | $\pi/2$ | 0 |
| Simple Edge | $H_1$ ○—○ $H_2$ | $H_i \pitchfork H_j$ | 3 | $\pi/3$ | $\frac{1}{2}$ |
| Double Edge | $H_1$ ○═○ $H_2$ | $H_i \pitchfork H_j$ | 4 | $\pi/4$ | $\frac{\sqrt{2}}{2}$ |
| Triple Edge | $H_1$ ○≡○ $H_2$ | $H_i \pitchfork H_j$ | 5 | $\pi/5$ | $\frac{1+\sqrt{5}}{4}$ |
| Thick/bold edge | $H_1$ ○━○ $H_2$ | $H_i \parallel H_j$ | $\infty$ | 0 | 1 |
| Dotted Edge | $H_1$ ○- - -$w_{ij}$- - -○ $H_2$ | $H_i )( H_j$ | 0 | $\infty$ | $\cosh(\rho(H_i, H_j))$ |
| Simple vertex | ○ | $h_i^2 = -1$ | | | 1 |
| Black vertex | ● | $h_i^2 = -2$ | | | 2 |
| Double-circled vertex | ◎ | $h_i^2 = -4$ | | | 4 |

Table 9. A summary of conventions for Coxeter-Vinberg diagrams

| Cusp Type | Type II, $\mathcal{J}$ | Type III, $\mathcal{I}$ |
|---|---|---|
| Boundary Strata | 1-cusps/curves $C_i$ | 0-cusps/points $p_j$ |
| Vertex type | $C_i$ ○ | $p_j$ ● |
| Sublattice Type | Isotropic lines $[\mathbb{Z}e] \in \mathrm{Gr}_1^{\mathrm{iso}}(L)/\Gamma$ | Isotropic planes $[\mathbb{Z}e \oplus \mathbb{Z}f] \in \mathrm{Gr}_2^{\mathrm{iso}}(L)/\Gamma$ |
| Subdiagram Type | Maximal parabolic | Elliptic |

Table 10. Cusp types

## 20C. Boundary cusp correspondences.

**20D. Cusp diagrams.** The three relevant moduli spaces:

$$\overline{\mathcal{E}_2}^{\mathrm{bb}}, \overline{\mathcal{E}_\emptyset}^{\mathrm{bb}}, \overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}}$$

20D.1. *Unpolarized Enriques surfaces* $\overline{\mathcal{E}_\emptyset}^{\mathrm{bb}}$. Cusps for $\overline{\mathcal{E}_\emptyset}^{\mathrm{bb}}$, unpolarized Enriques surfaces, are shown in Figure 3.

$$\overline{\mathcal{E}_\emptyset}^{\mathrm{bb}}$$


Figure 40. Cusp diagram for $\overline{\mathrm{O}(N)\backslash\Omega_N}^{\mathrm{bb}}$ (unpolarized Enriques surfaces).

20D.2. *Degree 2 polarized Enriques surfaces* $\overline{\mathcal{E}_2}^{\mathrm{bb}}$. Cusps for $\overline{\mathcal{E}_2}^{\mathrm{bb}}$, unpolarized Enriques surfaces, are shown in Figure 4.

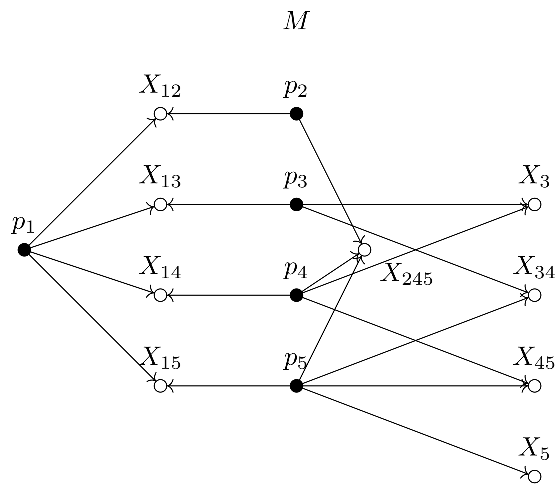



FIGURE 41. Cusp diagram for $\overline{\Gamma\backslash\Omega_N}^{\mathrm{bb}}$ (degree 2 polarized Enriques surfaces).

20D.3. *Numerically polarized Enriques surfaces* $\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{bb}}$. Cusps for $\overline{\mathcal{E}_2}^{\mathrm{bb}}$, numerically polarized Enriques surfaces, are shown in Figure 2.

Type

$$T = (20, 2, 0)_2 = U \oplus U(2) \oplus {E_8}^{\oplus^2} = U^{\oplus^2} \oplus D_{16}.$$

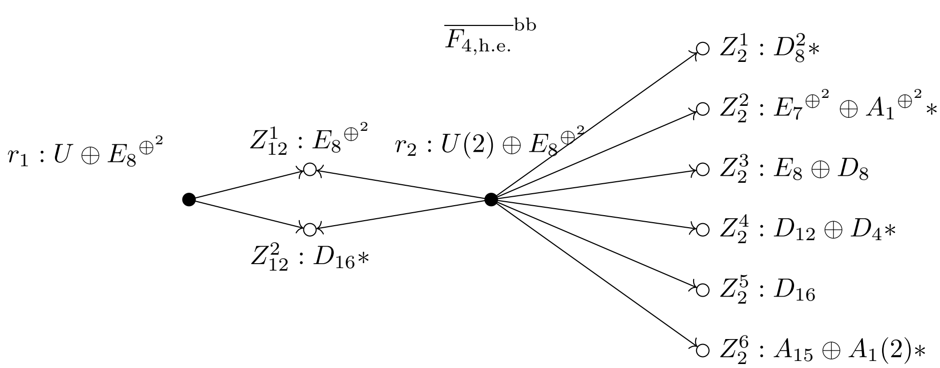



FIGURE 42. Cusp diagram for $\overline{O^+(\Lambda)\backslash\Omega_N}^{\mathrm{bb}}$ (numerically polarized Enriques surfaces).

20D.4. *Degree 2 K3 surfaces* $F_2$. Cusps for $F_2$, degree 2 K3 surfaces, are shown in **??**.

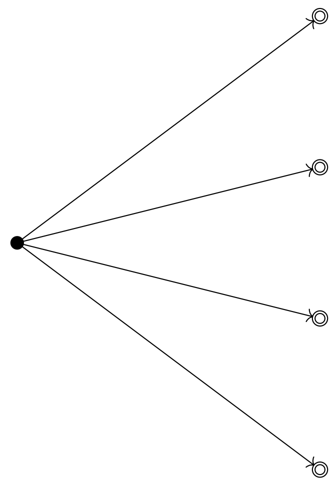
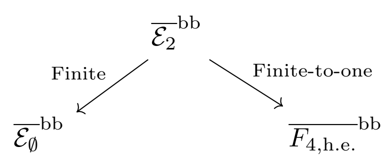


FIGURE 43. Cusp diagram for $\overline{F_2}^{\mathrm{bb}}$, degree 2 K3 surfaces

20E. **Cusp correspondence diagrams.** The correspondence we want to find:




20E.1. *Polarized to unpolarized correspondence.* Polarized $\{X_i, p_i\}$ to unpolarized $\{Y_i, q_i\}$ is shown in **??**.

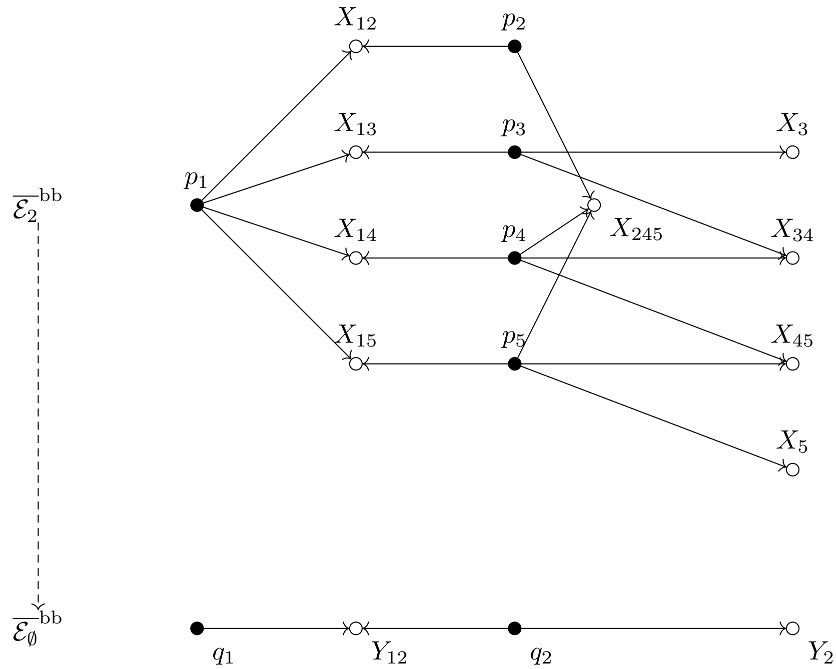



FIGURE 44. Goal 1

20E.2. *Polarized to hyperelliptic.* Polarized $\{X_i, p_i\}$ to hyperelliptic $\{Z_i, r_i\}$ is shown in **??**.

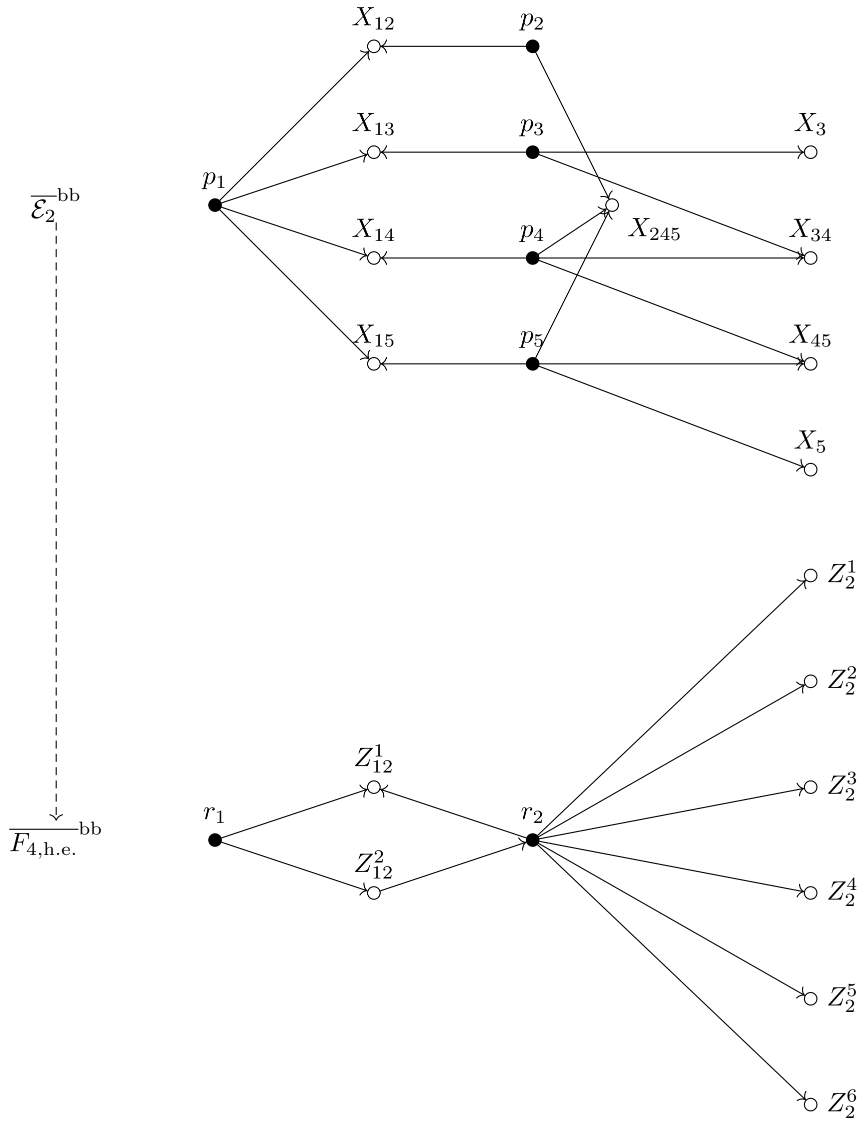



FIGURE 45. Goal 2

20F. **Classical and affine Dynkin diagrams. ??** is a table of (labeled) classical and affine Dynkin diagrams.

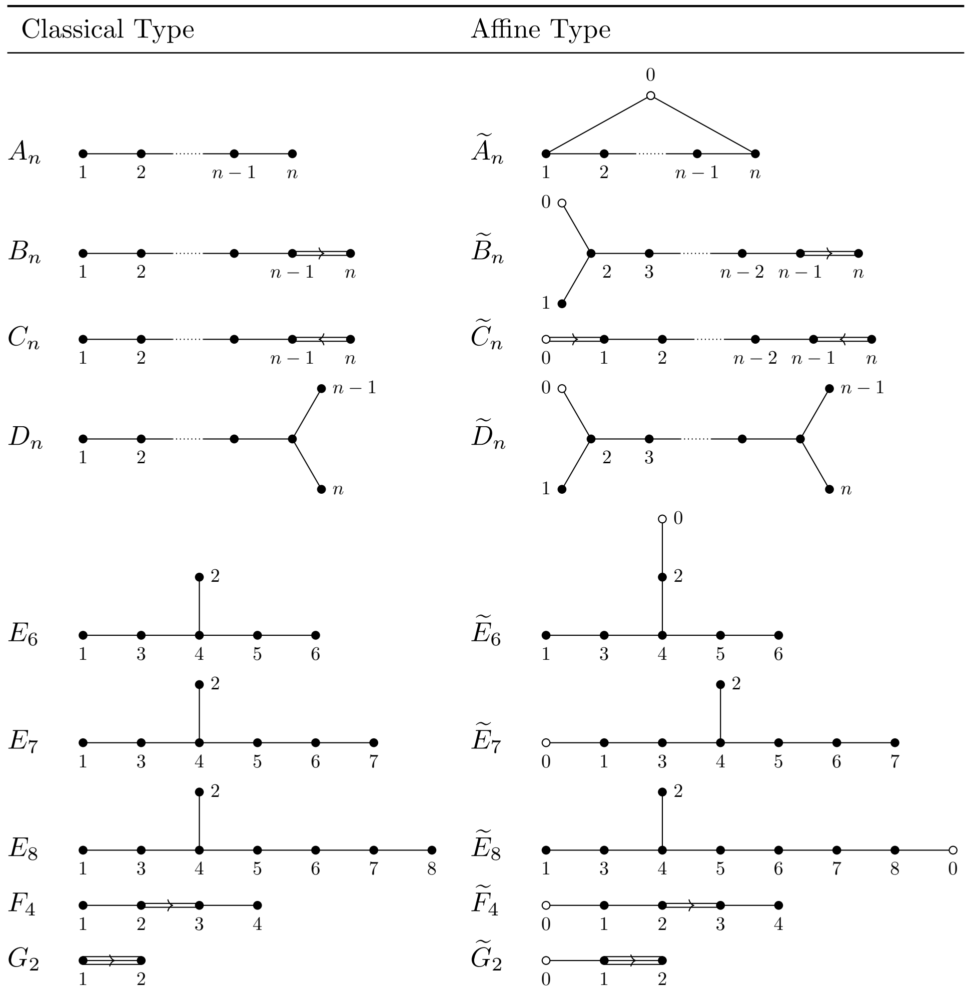



TABLE 11. Classical and Affine Dynkin Diagrams

**Remark 20.2.** The genera and number of cusps of the compactified modular curves $X_1(e)$ for various $e$ from Scattone:

| $e$ | = | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | $q > 3$ prime |
|---|---|---|---|---|---|---|---|---|---|---|
| genus | = | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 1 | $\frac{1}{24}(q-5)(q-7)$ |
| # cusps | = | 1 | 2 | 4 | 6 | 8 | 10 | 12 | 16 | $q-1$ |

**Remark 20.3.** Scattone conventions for cusp incidence diagrams:

- One vertex for each $\Gamma-$equivalence class of primitive isotropic sublattices of $L$.
- An edge

  $$\overset{[\mathbb{Z}v]}{\bullet}\longrightarrow\overset{[E]}{\bullet}$$

  if every sublattice $L' \leq L$ with $L \cong \mathbb{Z}v \mod \Gamma$ is contained in some sublattice $E'$ in the equivalence class $[E]$. We think of this as a weighted edge of weight 1 (and thus don't label the weight).
- A weighted edge

  $$\overset{[\mathbb{Z}v]}{\bullet}\overset{\mu}{\Longrightarrow}\overset{[E]}{\bullet}$$

  of weight $\mu \geq 2$ where $\mu$ is the number of $N_\Gamma(E)-$inequivalent sublattices $L' \leq L$ contained in $E$ where $L' \cong \mathbb{Z}v \mod \Gamma$.

This describes $\partial\overline{\Gamma\backslash D}$, where an edge of the form

$$\overset{a}{\bullet}\longrightarrow\overset{b}{\bullet}$$

represents a point $p$ corresponding to $a$ and a curve $C$ corresponding to $b$ with $p \in \overline{C}$, where $C$ has a self-intersection at $p$ if $\mu \geq 2$. Note that

$$N_\Gamma(E) \coloneqq \operatorname{im}\left(\operatorname{Stab}_\Gamma(E) \xrightarrow{r_\Gamma} \mathrm{GL}(E)\right) \subseteq \mathrm{GL}(E).$$

Define $I_n(L)$ for the isotropic rank $n$ sublattice of $L$; then $[E] \in \Gamma\backslash I_2(L)$ are BB boundary curves $C_E$ realized as quotients of disks in $\mathbb{P}(E_\mathbb{C})$ by $N_\Gamma(E)$. In many reasonable examples, $N_\Gamma(E) \leq \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup, i.e. it contains a principal congruence subgroup of the form $\Gamma(n)$ for some $n$.

**Remark 20.4.** Scattone's treatment of $F_{2k}$ for $k = 1, 2$: we first consider $\overline{\Gamma_k\backslash\Omega_k}^{\mathrm{bb}}$. These are isomorphic to $\overline{\mathrm{O}_-(L_k)\backslash\Omega_k}^{\mathrm{bb}}$ when $k = 1, 2$ by Corollary 5.6.10. By results 4.0.1 and 5.0.2, $\partial\overline{\mathrm{O}_-(L_k)\backslash\Omega_k}^{\mathrm{bb}}$ has one point $p(0)$ and $h(k)$ modular curves isomorphic to a fixed curve $C$, where $h(k)$ is the genus of $\langle -2k\rangle \oplus {E_8}^{\oplus^2}$. Scattone computes $h(1) = 4$ and $h(2) = 9$ and continues to describe how to compute $h(k)$ in general for $F_{2k}$. This seems to be shown in Figure 5.5.7 in the case $N = 1$, in which case $C \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. For $k = 1$, in section 6.2.1 he picks another lattice in the same genus, $A_1 \oplus {E_8}^{\oplus^2}$ and shows $h(1) = 4$ by producing 4 types distinguished by their root systems:

$$\widetilde{A}_1 \oplus {\widetilde{E}_8}^{\oplus^2}, \widetilde{A}_1 \oplus \widetilde{D}_{16}, \widetilde{E}_7 \oplus \widetilde{D}_{10}, \widetilde{A}_{17}.$$

Why this bijection works: let $G(1)$ be the iso classes of lattices of genus 1. Note $L_1 \coloneqq \langle -2\rangle \oplus H^{\oplus^2} \oplus {E_8}^{\oplus^2}$ is the polarized K3 lattice for $F_2$; each isotropic plane

$E \leq L_1$ determines a class $E^{\perp}/E \in G(1)$ and a boundary curve $C_E \in \overline{\Gamma_1 \backslash \Omega_1}^{\text{bb}}$. We then set up a bijection between $G(1)$ and boundary curves: send $E^{\perp}/E \in G(1)$ to $E \mod \Gamma_1$, then send $E \mod \Gamma_1$ to $C_E$. This geometrically corresponds to having 4 distinct possibilities for geometric configurations of the polarizing class.

Alternatively, one can compute $h(2) = 4$ using Vinberg's method. Letting $L \coloneqq \langle -2 \rangle \oplus H \oplus {E_8}^{\oplus 2}$, so $\operatorname{sgn} L = (1, 18)$, we have a hyperbolic lattice and a corresponding Coxeter polytope $P$ with Coxeter diagram $D$. We get 24 vertices in $D$ corresponding to 24 hyperplanes bounding $P$, which is diagram $(19, 1, 1)$ in the new notation:

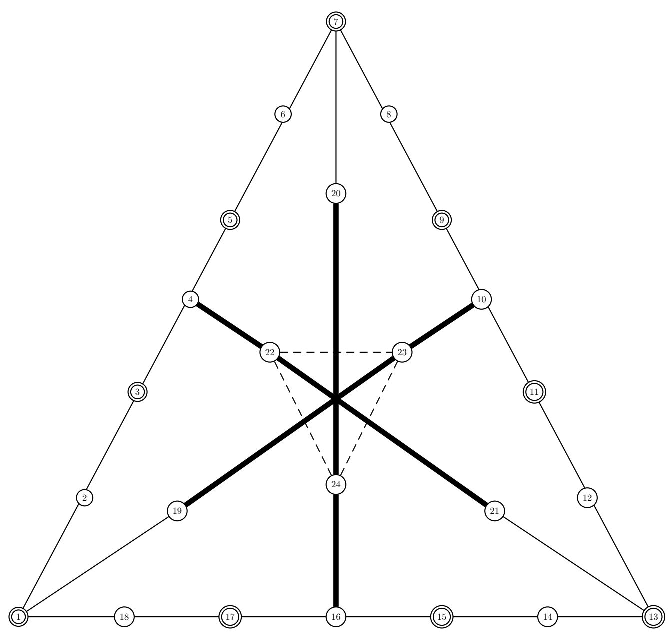
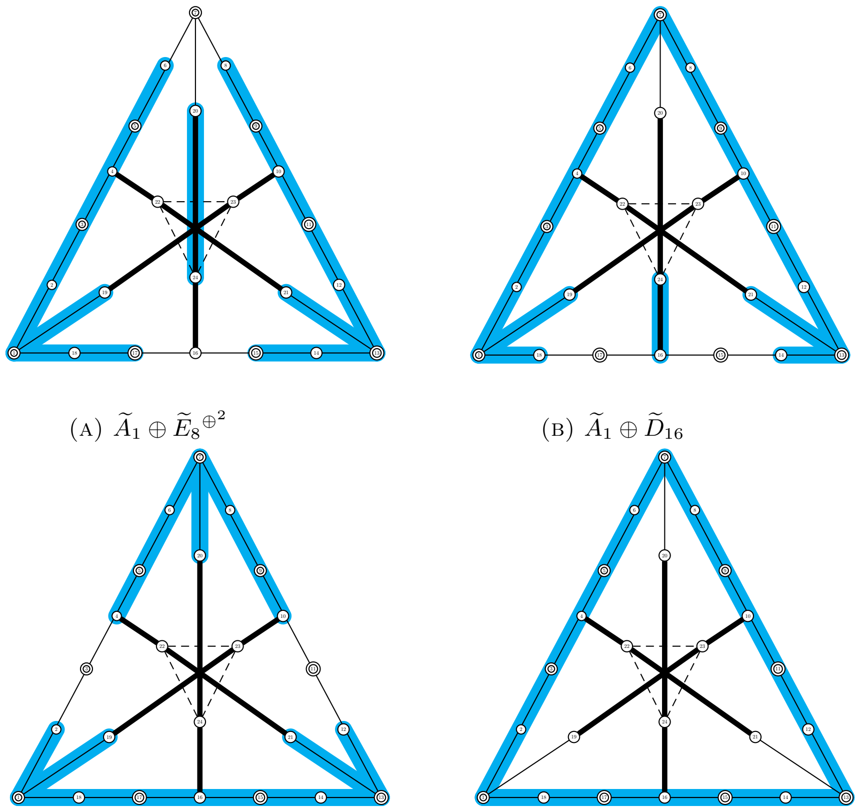



FIGURE 46. Coxeter diagram for $(19, 1, 1)$.

One then finds the 4 root systems by finding all parabolic subdiagrams:



(A) $\widetilde{A}_1 \oplus \widetilde{E}_8{}^{\oplus^2}$ (B) $\widetilde{A}_1 \oplus \widetilde{D}_{16}$

(C) $\widetilde{E}_7 \oplus \widetilde{D}_{10}$ (D) $\widetilde{A}_{17}$

FIGURE 47. 4 parabolic subdiagrams of $(19, 1, 1)$ corresponding to 4 lattices in the genus $G(1)$.

**Remark 20.5.** We can now carry out the same analysis for $F_4$ where $k = 2$. In this case $N \cong D_7$. The only lattices $L \in ?^{24}$ that contain $D_7$ are

$$\widetilde{E}_8{}^{\oplus^3}, \widetilde{E}_8 \oplus \widetilde{D}_{16}, \widetilde{E}_7{}^{\oplus^2} \oplus \widetilde{D}_{10}, \widetilde{D}_{12}{}^{\oplus^2}, \widetilde{D}_8{}^{\oplus^3}, \widetilde{D}_9 \oplus \widetilde{A}_{15}, \widetilde{E}_6 \oplus \widetilde{D}_7 \oplus \widetilde{A}_{11}$$

which have orthogonal complements

$$\langle -4\rangle \oplus \widetilde{E}_8{}^{\oplus^2}, \langle -4\rangle \oplus \widetilde{D}_{16}, \widetilde{E}_8 \oplus \widetilde{D}_9, \widetilde{E}_7{}^{\oplus^2} \oplus \widetilde{A}_3, \widetilde{D}_{17}, \widetilde{D}_{12} \oplus \widetilde{D}_5, \langle -4\rangle \oplus \widetilde{D}_8{}^{\oplus^2}, \widetilde{A}_1{}^{\oplus^2} \oplus \widetilde{A}_{15}, \widetilde{E}_6 \oplus \widetilde{A}_{11}$$

so there are 9 distinct possibilities determined by the geometry. One could recover these root systems by applying Vinberg's algorithm to $L := \langle -4\rangle \oplus U \oplus E_8{}^{\oplus^2}$. One obtains the following Coxeter diagram as in [**?**, Figure 6.3.1]:

Note: this is slightly different to $(18, 2, 0)$ in AET19, very mysterious!

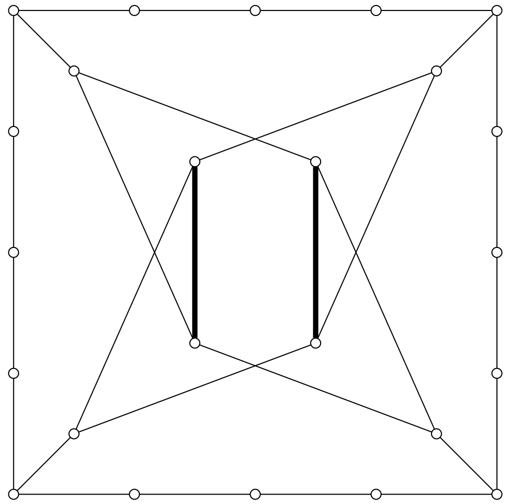


FIGURE 48. Scattone's diagram

20G. **Singularities.**

todo

**Definition 20.6** ($\mathrm{I}_n$ singularities)**.**

**Remark 20.7.** We record the classification of singular fibres of elliptic fibrations due to Kodaira. The type is listed along with $A$ the local monodromy matrix and the Cartan intersection matrix of irreducible components in such a fiber.

- $I_0$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, type $\widetilde{A}_0$.
- $I_n$, $A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, type $\widetilde{A}_{n-1}$.
- $I_0^*$, $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, type $\widetilde{D}_4$.
- $I_n^*$, $A = \begin{bmatrix} -1 & -n \\ 0 & -1 \end{bmatrix}$, type $\widetilde{D}_{4+n}$.
- $II$, $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, type $\widetilde{A}_0$.
- $II^*$, $A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, type $\widetilde{E}_8$.
- $III$, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, type $\widetilde{A}_1$.
- $III^*$, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, type $\widetilde{E}_7$
- $IV$, $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, type $\widetilde{A}_2$.
- $IV^*$, $A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$, type $\widetilde{E}_6$.

Of particular importance to us:

$$I_1 : \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \widetilde{A}_0 \qquad I_n : \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, \widetilde{A}_{n-1}.$$

Note $I_0$ is a smooth elliptic curve, $I_1$ is a nodal curve (ordinary double point), $I_2$ is two curves intersecting at two points, and $I_n$ is a cycle of curves.

## 21. Unsorted

### 21A. The main conversion processes/bijections we use.

**Remark 21.1.** Several conversion processes to spell out:

- $d-$semistable Type III $\mathcal{X}_0 \rightsquigarrow$ triangulated IAS$^2$. See [**?**, §6.12].
- Singular triangulated IAS$^2$ with $\sum_i Q(p_i) = 24 \rightsquigarrow$ Type III $\mathcal{X}_0$. See [**?**, §6.12].
- Certain Lagrangian torus fibrations $(X, \omega) \to B \rightsquigarrow$ IAS$^2$ (structure on $B$). See [**?**, §6D].
- $B =$ IAS$^2$ with $\sum Q(p_i) = 24$ and only $\mathrm{I}_n$ singularities $\rightsquigarrow (X, \omega) \to B$ with $X$ a K3. See [**?**, §6D].
- Type III KPP model $\rightleftharpoons$ IAS$^2$: unknown.
- Vinberg diagram at 0-cusp $\rightleftharpoons$ Type III KPP model: unknown.

### 21B. Nikulin involutions.

**Remark 21.2.** A **Nikulin involution** on a K3 surface is a symplectic involution, i.e., one leaving non-zero holomorphic two forms invariant. See Morrison, D.: On K3 surfaces with large Picard number. Every Nikulin involution has eight fixed points leading to a quotient surface with eight nodes; the blow-up of these nodes yields a K3 surface.

### 21C. Importing work from AET.

**Remark 21.3.** Making explicit [**?**, Thm. 7.4] in our case: the general setup is

- $\widehat{S}$ a lattice appearing as the target of a mirror move $S \to \widehat{S} := \bar{T}$, which is of the form

$$\widehat{S} = (r, a, \delta) = (10 + k - (g - 1), 10 - k - (g - 1), \delta)$$

  where
- $\widehat{\mathcal{X}} \to \Delta$ is a 1-parameter degeneration with central fiber $\mathcal{X}_0 = \widehat{\mathcal{Y}} \coprod_D \widehat{\mathcal{Y}}$, two copies of a surface $\widehat{\mathcal{Y}}$ glued along a divisor $D$.
- $\widehat{\mathcal{X}}_t$ is a surface with $\widehat{S} = (\operatorname{Pic} \widehat{\mathcal{X}}_t)^+$,
- $L \in \widehat{S} \otimes_{\mathbb{Z}} \mathbb{Q}$ an ample $\mathbb{Q}-$line bundle on $\widehat{\mathcal{X}}_t$,
- The involution-equivariant Lagrangian torus fibration $\mu : (\widehat{\mathcal{X}}_t, \omega) \to B$ with $\omega = [L]$ and base given by $B = P \coprod_{\partial P} P^{\mathrm{op}}$, a pushout of Symington polytopes glued along their boundaries by the identity,
- **For the Halphen case:**
  -

$$(r, a, \delta) = (10, 10, 1), \qquad (g, k, \delta) = (1, 0, 1)$$

    yielding the following Coxeter diagram for $\widehat{S}$:

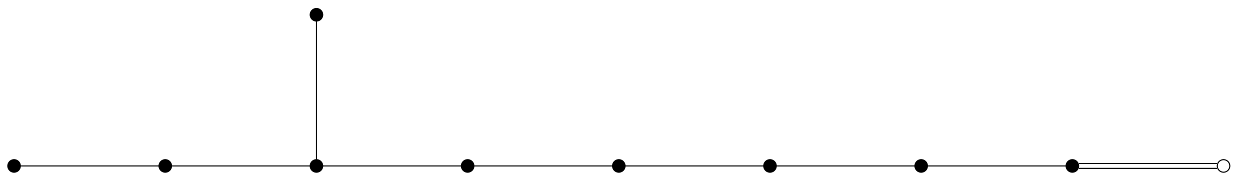


FIGURE 49. Coxeter diagram $\widetilde{E}_{9'}$ for the Halphen case $\widehat{S} = (10, 10, 1)$.

- $\widehat{\mathcal{Y}}_t = \widehat{\mathcal{X}}_t/\widehat{\iota}$
- $B$ is an IAS$^2$ with 12 $\mathrm{I}_1$ singularities in each hemisphere with locations determined by length parameters $\ell_i$ for the Symington polytope $P$ of $\widehat{\mathcal{Y}}$,
- $P$ is a Symington polytope for $\widehat{\mathcal{Y}}_t$:
- The degeneration $\mathcal{X}$ is given by colliding $F_0$ a smooth elliptic fiber and $F$ a special fiber of $X$ double covering a fiber in $Y$,
- $\widehat{\mathcal{Y}}$ is an elliptic fibration as described in [**?**, Lem. 7.1].
- The surfaces $V_j$ corresponding to equatorial points in $B$ are all toric and of type E1 as described in [**?**, Lem. 8.2] corresponding to "odd equatorial behavior": if $(V_j, D_j)$ is one such torus pair, there is a generically $\mathbb{P}^1$ fibration $\pi : V \to \mathbb{P}^1$ with sections $s_1, s_2$ where the fixed locus $V_j^\iota$ is a fiber of $\pi$ plus two points, one on each section $s_i$.

- **For the Enriques case:**
  -

$$(r, a, \delta) = (10, 10, 0), \qquad (g, k, \delta) = (1, 0, 0)$$

yielding the following Coxeter diagram for $\widehat{S}$:

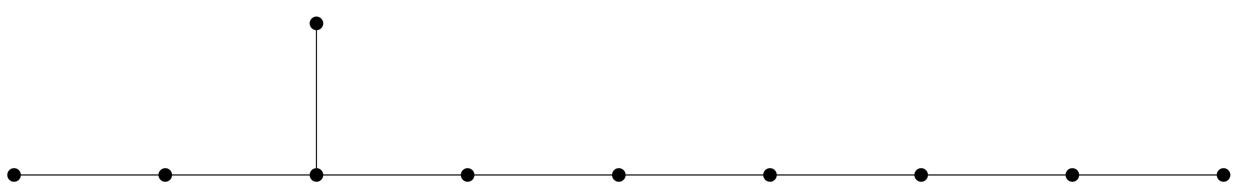


FIGURE 50. Coxeter diagram $E_{10}$ for the Enriques case $\widehat{S} = (10, 10, 0)$.

- The $E_{10}$ Coxeter-Dynkin diagram is associated with the root lattice $Q(E_{10}) = \mathrm{II}_{1,9}$ and has the following Cartan matrix which defines an infinite-dimensional Kac-Moody algebra of hyperbolic type:

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

- The degeneration $\mathcal{X}$ is given by colliding $F$ and $F_0$, the two double fibers of the elliptic fibration, as described in [**?**, Lem. 7.1].
- $\widehat{\mathcal{X}_0} = \widehat{\mathcal{Y}} \coprod_D \widehat{\mathcal{Y}}$ where $\widehat{\mathcal{Y}}$ is the surface from the Halphen case above, but the involution on $\mathcal{X}_0$ is basepoint-free.
- $P$ is the Symington polytope from the Halphen case above,
- $B = P \coprod_\tau P^{\mathrm{op}}$ is two copies of $P$ from the Halphen case, glued along a half-twist $\tau$ of the boundaries.
- $B/\iota \cong \mathbb{RP}^2$

21D. **Describing surfaces in degenerations using Coxeter diagrams.** Interpreting this geometrically: consider the cycle of $2\bar{k}$ white vertices cycling between plain and double-circled:

- See [**?**]
- Each edge on the outer cycle corresponds to $\mathbb{P}^2$
- Single circle vertices (with odd $i$) corresponds to a line in $\mathbb{P}^2$
- Double-circled vertices (with even $i$) correspond to conics on the $\mathbb{P}^2$
- Explicit example worked out in [**?**, §5].
- It seems like that from the Coxeter diagram, you draw the fan of a toric surface, you compute the charge, and then, if this is not 24, you fix it by blowing up some non-torus fixed points along the toric boundary.

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