JUST DO IT: A COLLECTION OF HARTSHORNE PROBLEMS

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Contents

1. I: Varieties

Remark 1.0.1. Some useful basic properties:

- Properties of V:
  - $\cap_{i\in I} V(\mathfrak{a}_i) = V\left(\sum_{i\in I} \mathfrak{a}_i\right)$.
    - ♢ E.g. $V(x) \cap V(y) = V(\langle x\rangle + \langle y\rangle) = V(x,y) = \{0\}$, the origin.
  - $\cup_{i\leq n} V(\mathfrak{a}_i) = V\left(\prod_{i\leq n} \mathfrak{a}_i\right)$.
    - ♢ E.g. $V(x) \cup V(y) = V(\langle x\rangle \langle y\rangle) = V(xy)$, the union of coordinate axes.
  - $V(\mathfrak{a})^c = \cup_{f\in\mathfrak{a}} D(f)$
  - $V(\mathfrak{a}_1) \subseteq V(\mathfrak{a}_2) \iff \sqrt{\mathfrak{a}_1} \supseteq \sqrt{\mathfrak{a}_2}$.
- Properties of I:
  - $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ and $V(I(Y)) = \mathrm{cl}_{\mathbf{A}^n}(Y)$. The containment correspondence is contravariant in both directions.
  - $I(\cup_i Y_i) = \cap_i I(Y_i)$.
- If F is a sheaf taking values in subsets of a giant ambient set, then $F(\cup U_i) = \cap F(U_i)$. For $\mathbf{A}^n/\mathbf{C}$, take $\mathbf{C}(x_1, \cdots, x_n)$, the field of rational functions, to be the ambient set.
- Distinguished open $D(f) \coloneqq \left\{p \in X \;\middle|\; f(p) \neq 0\right\}$:
  - $\mathcal{O}_X(D(f)) = A(X)\left[\frac{1}{f}\right] = \left\{\frac{g}{f^k} \;\middle|\; g \in A(X), k \geq 0\right\}$, and taking $f = 1$ shows $\mathcal{O}_X(X) = A(X)$, i.e. global regular functions are polynomial.
  - Generally $D(fg) = D(f) \cap D(g)$
  - For affines:
  - For $\mathbf{C}^n$,

1.1. I.1: Affine Varieties ⋆.

Remark 1.1.1. Summary:

- $\mathbf{A}^n_{/k} = \left\{[a_1, \cdots, a_n] \;\middle|\; a_i \in k\right\}$, and elements $f \in A \coloneqq k[x_1, \cdots, x_n]$ are functions on it.
- $Z(f) \coloneqq \left\{p \in \mathbf{A}^n \;\middle|\; f(p) = 0\right\}$, and for any $T \subseteq A$ we set $Z(T) \coloneqq \cap_{f\in T} Z(f)$.
  - Note that $Z(T) = Z(\langle T\rangle_A) = Z(\langle f_1, \cdots, f_r\rangle)$ for some generators f_i, using that A is a Noetherian ring. So every $Z(T)$ is the set of common zeros of finitely many polynomials, i.e. the intersection of finitely many hypersurfaces.
- **Algebraic**: $Y \subseteq \mathbf{A}^n$ is algebraic iff $Y = Z(T)$ for some $T \subseteq A$.
- The Zariski topology is generated by open sets of the form $Z(T)^c$.
- $\mathbf{A}^1$ is a non-Hausdorff space with the cofinite topology.
- **Irreducible**: Y is reducible iff $Y = Y_1 \cup Y_2$ with Y_1, Y_2 proper subsets of Y which are closed in Y.
  - Nonempty open subsets of irreducible spaces are both irreducible and dense.
  - If $Y \subseteq X$ is irreducible then $\mathrm{cl}_X(Y) \subseteq X$ is again irreducible.
- **Affine (algebraic) varieties**: irreducible closed subsets of $\mathbf{A}^n$.
- **Quasi-affine varieties**: open subsets of affine varieties.
- The ideal of a subset: $I(Y) \coloneqq \left\{f \in A \;\middle|\; f(p) = 0 \;\forall p \in Y\right\}$.
- **Nullstellensatz**: if $k = \overline{k}, \mathfrak{a} \in \mathrm{Id}(k[x_1, \cdots, x_n])$, and $f \in k[x_1, \cdots, x_n]$ with $f(p) = 0$ for all $p \in V(\mathfrak{a})$, then $f^r \in \mathfrak{a}$ for some $r > 0$, so $f \in \sqrt{\mathfrak{a}}$. Thus there is a contravariant correspondence between radical ideals of $k[x_1, \cdots, x_n]$ and algebraic sets in $\mathbf{A}^n_{/k}$.
- **Irreducibility criterion**: Y is irreducible iff $I(Y) \in \operatorname{Spec} k[x_1, \cdots, x_n]$ (i.e. it is prime).
- **Affine curves**: if $f \in k[x,y]^{\mathrm{irr}}$ then $\langle f\rangle \in \operatorname{Spec} k[x,y]$ (since this is a UFD) so $Z(f)$ is irreducible and defines an affine curve of degree $d = \deg(f)$.
- **Affine surfaces**: $Z(f)$ for $f \in k[x_1, \cdots, x_n]^{\mathrm{irr}}$ defines a surface.

- **Coordinate rings**: $A(Y) \coloneqq k[x_1, \cdots, x_n]/I(Y)$.
- **Noetherian spaces**: $X \in \mathsf{Top}$ is Noetherian iff the DCC on closed subsets holds.
- **Unique decomposition into irreducible components**: if $X \in \mathsf{Top}$ is Noetherian then every closed nonempty $Y \subseteq X$ is of the form $Y = \cup_{i=1}^{r} Y_i$ with Y_i a uniquely determined closed irreducible with $Y_i \not\subseteq Y_j$ for $i \neq j$, the *irreducible components* of Y.
- **Dimension**: for $X \in \mathsf{Top}$, the dimension is $\dim X \coloneqq \sup \left\{ n \;\middle|\; \exists Z_0 \subset Z_1 \subset \cdots \subset Z_n \right\}$ with Z_i distinct irreducible closed subsets of X. Note that the dimension is the number of "links" here, not the number of subsets in the chain.
- **Height**: for $\mathfrak{p} \in \operatorname{Spec} A$ define $\operatorname{ht}(\mathfrak{p}) \coloneqq \sup \left\{ n \;\middle|\; \exists \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p} \right\}$ with $\mathfrak{p}_i \in \operatorname{Spec} A$ distinct prime ideals.
- **Krull dimension**: define $\operatorname{krulldim} A \coloneqq \sup_{\mathfrak{p} \in \operatorname{Spec} A} \operatorname{ht}(\mathfrak{p})$, the supremum of heights of prime ideals.

Exercise 1.1.2 (The Zariski topology)**.** Show that the class of algebraic sets form the closed sets of a topology, i.e. they are closed under finite unions, arbitrary intersections, etc.

Exercise 1.1.3 (The affine line)**.**

- Show that $\mathbf{A}^1_{/k}$ has the cofinite topology when $k = \overline{k}$: the closed (algebraic) sets are finite sets and the whole space, so the opens are empty or complements of finite sets.[1]
- Show that this topology is not Hausdorff.
- Show that $\mathbf{A}^1$ is irreducible without using the Nullstellensatz.
- Show that $\mathbf{A}^n$ is irreducible.
- Show that maximal ideals $\mathfrak{m} \in \operatorname{mSpec} k[x_1, \cdots, x_n]$ correspond to minimal irreducible closed subsets $Y \subseteq \mathbf{A}^n$, which must be points.
- Show that $\operatorname{mSpec} k[x_1, \cdots, x_n] = \left\{ \langle x_1 - a_1, \cdots, x_n - a_n \rangle \;\middle|\; a_1, \cdots, a_n \in k \right\}$ for $k =$
$bark$, and that this fails for $k \neq$
$bark$.
- Show that $\mathbf{A}^n$ is Noetherian.
- Show $\dim \mathbf{A}^1 = 1$.
- Show $\dim \mathbf{A}^n = n$.

Exercise 1.1.4 (Commutative algebra)**.**

- Show that if Y is affine then $A(Y)$ is an integral domain and in ${}_k\mathsf{Alg}^{\mathrm{fg}}$.
- Show that every $B \in {}_k\mathsf{Alg}^{\mathrm{fg}} \cap \mathsf{Domain}$ is of the form $B = A(Y)$ for some $Y \in \mathsf{AffVar}_{/k}$.
- Show that if Y is an affine algebraic set then $\dim Y = \operatorname{krulldim} A(Y)$.

Theorem 1.1.5 (Results from commutative algebra)**.**

- *If* $k \in \mathsf{Field}, B \in {}_k\mathsf{Alg}^{\mathrm{fg}} \cap \mathsf{Domain}$*,*
  - $\operatorname{krulldim} B = [K(B) : B]_{\mathrm{tr}}$ *is the transcendence degree of the quotient field of* B *over* B*.*
  - *If* $\mathfrak{p} \in \operatorname{Spec} B$ *then* $\operatorname{ht} \mathfrak{p} + \operatorname{krulldim}(B/\mathfrak{p}) = \operatorname{krulldim} B$*.*
- *Krull's Hauptidealsatz:*
  - *If* $A \in \mathsf{CRing}^{\mathrm{Noeth}}$ *and* $f \in A \setminus A^{\times}$ *is not a zero divisor, then every minimal* $\mathfrak{p} \in \operatorname{Spec} A$ *with* $\mathfrak{p} \ni f$ *has height 1.*
- *If* $A \in \mathsf{CRing}^{\mathrm{Noeth}} \cap \mathsf{Domain}$*, then* A *is a UFD iff every* $\mathfrak{p} \in \operatorname{Spec}(A)$ *with* $\operatorname{ht}(\mathfrak{p}) = 1$ *is principal.*

[1]Hint: $k[x]$ is a PID and factor any $f(x)$ into linear factors using that $k =$
$bark$ to write $Z(\mathfrak{a}) = Z(f) = \{a_1, \cdots, a_k\}$ for some k.

Exercise 1.1.6 (1.10)**.** Show that if Y is quasi-affine then

Exercise 1.1.7 (1.13)**.** Show that if $Y \subseteq \mathbf{A}^n$ then $\operatorname{codim}_{\mathbf{A}^n}(Y) = 1 \iff Y = Z(f)$ for a single nonconstant $f \in k[x_1, \cdots, x_n]^{\text{irr}}$.

Exercise 1.1.8 (?)**.** Show that if $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\operatorname{ht}(\mathfrak{p}) = 2$ then $\mathfrak{p}$ can not necessarily be generated by two elements.

1.2. **I.2: Projective Varieties** $\star$.

Remark 1.2.1.

- **Projective space**: $\left\{ \mathbf{a} := [a_0, \cdots, a_n] \;\middle|\; a_i \in k \right\} / \sim$ where $\mathbf{a} \sim \lambda \mathbf{a}$ for all $\lambda \in k \setminus \{0\}$, i.e. lines in $\mathbf{A}^{n+1}$ passing through $\mathbf{0}$.
- **Graded rings**: a ring S with a decomposition $S = \oplus_{d \geq 0} S_d$ with each $S_d \in \mathsf{AbGrp}$ and $S_d S_e \subseteq S_{d+e}$; elements of S_d are **homogeneous of degree** d and any element in S is a finite sum of homogeneous elements of various degrees.
- **Homogeneous polynomials**: f is homogeneous of degree d if $f(\lambda x_0, \cdots, \lambda x_n) = \lambda^d f(x_0, \cdots, x_n)$.
- **Homogeneous ideals**: $\mathfrak{a} \subseteq S$ is homogeneous when it's of the form $\mathfrak{a} = \bigoplus_{d \geq 0} (\mathfrak{a} \cap S_d)$.
  - $\mathfrak{a}$ is homogeneous iff generated by homogeneous elements.
  - The class of homogeneous ideals is closed under sums, products, intersections, and radicals.
  - Primality of homogeneous ideals can be tested on homogeneous elements, i.e. it STS $fg \in \mathfrak{a} \implies f, g \in \mathfrak{a}$ for f, g homogeneous.
- $k[x_1, \cdots, x_n] = \bigoplus_{d \geq 0} k[x_1, \cdots, x_n]_d$ where the degree d part is generated by monomials of total weight d.
  - E.g. $k[x_1, \cdots, x_n]_1 = \langle x_1, x_2, \cdots, x_n \rangle$, $k[x_1, \cdots, x_n]_2 = \langle x_1^2, x_1 x^2, x_1 x_3, \cdots, x_2^2, x_2 x_3, x_2 x_4, \cdots, x_n^2 \rangle$.
  - Useful fact: by stars and bars, $\operatorname{rank}_k k[x_1, \cdots, x_n]_d = \binom{d+n}{n}$. E.g. for $(d, n) = (3, 2)$,

$$
\begin{array}{rl}
x_1^3 & \longleftrightarrow \star\star\star \,|\,| \\
x_1^2 x_2 & \longleftrightarrow \star\star \,|\, \star \,| \\
x_1^2 x_3 & \longleftrightarrow \star\star \,|\,|\, \star \\
x_1 x_2^2 & \longleftrightarrow \star \,|\, \star\star \,| \\
x_1 x_2 x_3 & \longleftrightarrow \star \,|\, \star \,|\, \star \\
x_1 x_3^2 & \longleftrightarrow \star \,|\,|\, \star\star \\
x_2^3 & \longleftrightarrow \,|\, \star\star\star \,| \\
x_2^2 x_3 & \longleftrightarrow \,|\, \star\star \,|\, \star \\
x_2 x_3^2 & \longleftrightarrow \,|\, \star \,|\, \star\star \\
x_3^3 & \longleftrightarrow \,|\,|\, \star\star\star
\end{array}
$$

- Arbitrary polynomials $f \in k[x_0, \cdots, x_n]$ do not define functions on $\mathbf{P}^n$ because of non-uniqueness of coordinates due to scaling, but homogeneous polynomials f being zero or not is well-defined and there is a function So $Z(f) := \left\{ p \in \mathbf{P}^n \;\middle|\; f(p) = 0 \right\}$ makes sense.
- **Projective algebraic varieties**: Y is projective iff it is an irreducible algebraic set in $\mathbf{P}^n$. Open subsets of $\mathbf{P}^n$ are **quasi-projective varieties**.
- **Homogeneous ideals of varieties**:
- **Homogeneous coordinate rings**:
- $Z(f)$ for f a linear homogeneous polynomial defines a **hyperplane**.

Exercise 1.2.2 (Cor. 2.3)**.** Show $\mathbf{P}^n$ admits an open covering by copies of $\mathbf{A}^n$ by explicitly constructing open sets U_i and well-defined homeomorphisms $\varphi_i : U_i \to \mathbf{A}^n$.

1.3. **I.3: Morphisms.**

1.4. **I.4: Rational Maps.**

1.5. **I.5: Nonsingular Varieties.**

1.6. **I.6: Nonsingular Curves.**

1.7. **I.7: Intersections in Projective Space.**

2. II: Schemes

Note: there are many, many important notions tucked away in the exercises in this section.

2.1. II.1: Sheaves $\star$.

Remark 2.1.1.

- **Presheaves** F of abelian groups: contravariant functors $F \in \mathsf{Fun}(\mathsf{Open}(X), \mathsf{AbGrp})$.
  - Assigns every open $U \subseteq X$ some $F(U) \in \mathsf{AbGrp}$
  - For $\iota_{VU} : V \subseteq U$, restriction morphisms $\varphi_{UV} : F(U) \to F(V)$.
  - $F(\emptyset) = 0$, so $F(\emptyset^{\downarrow}) = {}_{\uparrow}$.
  - $\varphi UU = \mathrm{id}_{F(U)}$
  - $W \subseteq V \subseteq U \implies \varphi_{UW} = \varphi_{VW} \circ \varphi_{UV}$.
- **Sections**: elements $s \in F(U)$ are sections of F over U. Also notation $\Gamma(U; F)$ and $H^0(U; F)$, and the restrictions are written $s|_V \coloneqq \varphi_{UV}(s)$ for $s \in F(U)$.
- **Sheaves**: presheaves F which are completely determined by local data. Additional requirements on open covers $\mathcal{V} \rightrightarrows U$:
  - If $s \in F(U)$ with $s|_{V_i} = 0$ for all i then $s \equiv 0 \in F(U)$.
  - Given $s_i \in F(V_i)$ where $s_i|_{V_{ij}} = s_j|_{V_{ij}} \in F(V_{ij})$ then $\exists s \in F(U)$ such that $s|_{V_i} = s_i$ for each i, which is unique by the previous condition.
- **Constant sheaf**: for $A \in \mathsf{AbGrp}$, define the constant sheaf
- **Stalks**: $F_p \coloneqq \varinjlim_{U \ni p} F(U)$ along the system of restriction maps.
  - These are represented by pairs (U, s) with $U \ni p$ an open neighborhood and $s \in F(U)$, modulo $(U, s) \sim (V, t)$ when $\exists W \subseteq U \cap V$ with $s|_w = t|_w$.
- **Germs**: a germ of a section of F at p is an elements of the stalk F_p.
- **Morphisms of presheaves**: natural transformations $\eta \in \mathrm{Mor}_{\mathsf{Fun}}(F, G)$, i.e. for every U, V, components η_U, η_V fitting into a diagram

  Link to Diagram
- A morphism of sheaves is exactly a morphism of the underlying presheaves.
- Morphisms of sheaves $\eta : F \to G$ induce morphisms of rings on the stalks $\eta_p : F_p \to G_p$.
- Morphisms of sheaves are isomorphisms iff isomorphisms on all stalks, see exercise below.
- **Kernels, cokernels, images**: for $\varphi : F \to G$, sheafify the assignments to kernels/cokernels/images on open sets.
- **Sheafification**: for any $F \in \underset{\mathsf{pre}}{\mathsf{Sh}}(X)$, there is a unique $F^+ \in \mathsf{Sh}(X)$ and a morphism $\theta : F \to F^+$ of presheaves such that any sheaf presheaf morphism $F \to G$ factors as $F \to F^+ \to G$.
  - The construction: $F^+(U) = \mathsf{Top}(U, \coprod_{p \in U} F_p)$ are all functions s into the union of stalks, subject to $s(p) \in F_p$ for all $p \in U$ and for each $p \in U$, there is a neighborhood $V \supseteq U \ni p$ and $t \in F(V)$ such that for all $q \in V$, the germ t_q is equal to $s(q)$.
  - Note that the stalks are the same: $(F^+)_p = F_p$, and if F is already a sheaf then θ is an isomorphism.
- **Subsheaves**: $F' \leq F$ iff $F'(U) \leq F(U)$ is a subgroup for every U and the restrictions on F' are induced by restrictions from F.
  - If $F' \leq F$ then $F'_p \leq F_p$.
  - **Injectivity**: $\varphi : F \to G$ is injective iff the sheaf kernel $\ker \varphi = 0$ as a subsheaf of F.
    - $\Diamond$ φ is injective iff injective on all sections.
  - $\operatorname{im} \varphi \leq G$ is a subsheaf.
  - **Surjectivity**: $\varphi : F \to G$ is surjective iff $\operatorname{im} \varphi = G$ as a subsheaf.

- **Exactness**: a sequence of sheaves $(F_i, \varphi_i : F_i \to F_{i+1})$ is exact iff $\ker \varphi_i = \operatorname{im} \varphi^{i-1}$ as subsheaves of F_i.
  - $\varphi : F \to G$ is injective iff $0 \to F \xrightarrow{\varphi} G$ is exact.
  - $\varphi : F \to G$ is surjective iff $F \xrightarrow{\varphi} G \to 0$ is exact.
  - Sequences of sheaves are exact iff exact on stalks.
- **Quotient sheaves**: F/F' is the sheafification of $U \mapsto F(U)/F'(U)$.
- **Cokernels**: for $\varphi : F \to G$, $\operatorname{coker} \varphi$ is sheafification of $U \mapsto \operatorname{coker}(F(U) \xrightarrow{\varphi(U)} G(U))$.
- **Direct images**: for $f \in \mathsf{Top}(X, Y)$, the sheaf defined on sections by $(f_* F)(V) := F(f^{-1}(V))$ for any $V \subseteq Y$. Yields a functor $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$.
- **Inverse images**: denoted $f^{-1}G$, the sheafification of $U \mapsto \varinjlim_{V \supseteq f(U)} G(V)$, i.e. take the limit from above of all open sets V of Y containing the image $f(U)$. Yields a functor $f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$.
- **Restriction of a sheaf**: for $F \in \mathsf{Sh}(X)$ and $Z \subseteq X$ with $\iota : Z \hookrightarrow X$ the inclusion, define $i^{-1}F \in \mathsf{Sh}(Z)$ to be the restriction. Also denoted $F|_Z$. This has the same stalks: $(F|_Z)_p = F_p$.
- For any $U \subseteq X$, the global sections functor $\Gamma(U; -) : \mathsf{Sh}(X) \to \mathsf{AbGrp}$ is left-exact (proved in exercises).
- **Limits of sheaves**: for $\{F_i\}$ a direct system of sheaves, $\varinjlim_i F_i$ has underlying presser $U \mapsto \varinjlim_i F_i(U)$. If X is Noetherian, then this is already a sheaf, and commutes with sections: $\Gamma(X; \varinjlim_i F_i) = \varinjlim_i \Gamma(X; F_i)$.
  - Inverse limits exist and are defined similarly.
- **The espace étalé**: define $\text{Ét}(F) = \coprod_{p \in X} F_p$ and a projection $\pi : \text{Ét}(F) \to X$ by sending $s \in F_p$ to p. For each $U \subseteq X$ and $s \in F(U)$, there is a local section $\overline{s} : U \to \text{Ét}(F)$ where $p \mapsto s_p$, its germ at p; this satisfies $\pi \circ \overline{s} = \mathrm{id}_U$. Give $\text{Ét}(F)$ the strongest topology such that the $\overline{s}$ are all continuous. Then $F^+(U) := \mathsf{Top}(U, \text{Ét}(F))$ is the set of continuous sections of $\text{Ét}(F)$ over U.
- **Support**: for $s \in F(U)$, $\operatorname{supp}(s) := \left\{ p \in U \;\middle|\; s_p \neq 0 \right\}$ where s_p is the germ of s in F_p. This is closed.
  - This extends to $\operatorname{supp}(F) := \left\{ p \in X \;\middle|\; F_p \neq 0 \right\}$, which need not be closed.
- **Sheaf hom**: $U \mapsto \operatorname{Hom}(F|_U, G|_U)$ forms a sheaf of local morphisms and is denoted $\mathcal{H}om(F, G)$.
- **Flasque sheaves**: a sheaf is flasque iff $V \hookrightarrow U \implies F(U) \twoheadrightarrow F(V)$.
- **Skyscraper sheaves**: for $A \in \mathsf{AbGrp}$ and $p \in X$, define $i_p(A)$ by $U \mapsto A$ if $p \in U$ and 0 otherwise. Also denoted $\iota_*(A)$ where $\iota : \operatorname{cl}_X(\{p\}) \hookrightarrow X$ is the inclusion.
  - The stalks are $(i_p(A))_q = A$ if $q \in \operatorname{cl}_X(\{p\})$ and 0 otherwise.
- **Extension by zero**: if $\iota : Z \hookrightarrow X$ is the inclusion of a closed set and $U := X \setminus Z$ with $j : U \to X$, then for $F \in \mathsf{Sh}(Z)$, the sheaf $\iota_* F \in \mathsf{Sh}(X)$ is the extension of F by zero outside of Z. The stalks $(\iota_* F)_p$ are F_p is $p \in Z$ and 0 otherwise.
  - For the open U, extension by zero is $j_! F$ which has presheaf $V \mapsto F(V)$ if $V \subseteq U$ and 0 otherwise. The stalks $(j_! F)_p$ are F_p if $p \in U$ and 0 otherwise.
- **Sheaf of ideals**: for $Y \subseteq X$ closed and $U \subseteq X$ open, $\mathcal{I}_Y(U)$ has presheaf $U \mapsto$ the ideal in $\mathcal{O}_X(U)$ of regular functions vanishing on all of $Y \cap U$. This is a subsheaf of $\mathcal{O}_X$.
- **Gluing sheaves**: given $\mathcal{U} \rightrightarrows X$ and sheaves $F_i \in \mathsf{Sh}(U_i)$, one can glue to a unique $F \in \mathsf{Sh}(X)$ if one is given morphisms $\varphi_{ij} \, F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ij}}$ where $\varphi_{ii} = \mathrm{id}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on U_{ijk}.

Warning 2.1.2. Some common mistakes:

- Kernel presheaves are already sheaves, but not cokernels or images. See exercise below.

- $\varphi : F \to G$ is injective iff injective on sections, but this is not true for surjectivity.
- The sheaves $f^{-1}G$ and f^*G are different! See III.5 for the latter.
- Global sections need not be right-exact.

Exercise 2.1.3 (Regular functions on varieties form a sheaf)**.** For $X \in \mathsf{Var}_{/k}$, define the ring $\mathcal{O}_X(U)$ of literal regular functions $f_i : U \to k$ where restriction morphisms are induced by literal restrictions of functions. Show that $\mathcal{O}_X$ is a sheaf of rings on X.

Hint: Locally regular implies regular, and regular + locally zero implies zero.

Exercise 2.1.4 (?)**.** Show that for every connected open subset $U \subseteq X$, the constant sheaf satisfies $\underline{A}(U) = A$, and if U is open with open connected component so the $\underline{A}(U) = A^{\times^{\sharp \pi_0 U}}$.

Exercise 2.1.5 (?)**.** Show that if $X \in \mathsf{Var}_{/k}$ and $\mathcal{O}_X$ is its sheaf of regular functions, then the stalk $\mathcal{O}_{X,p}$ is the *local ring of* p on X as defined in Ch. I.

Exercise 2.1.6 (Prop 1.1)**.** Let $\varphi : F \to G$ be a morphism in $\mathsf{Sh}(X)$ and show that φ is an isomorphism iff φ_p is an isomorphism on stalks for all $p \in X$. Show that this is false for presheaves.

Exercise 2.1.7 (?)**.** Show that for $\varphi \in \mathrm{Mor}_{\mathsf{Sh}(X)}(F, G)$, $\ker \varphi$ is a sheaf, but $\operatorname{coker} \varphi, \operatorname{im} \varphi$ are not in general.

Exercise 2.1.8 (?)**.** Show that if $\varphi : F \to G$ is surjective then the maps on sections $\varphi(U) : F(U) \to G(U)$ need not all be surjective.

2.2. **II.2: Schemes.**

2.3. **II.3: First Properties of Schemes.**

2.4. **II.4: Separated and Proper Morphisms.**

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2.6. **II.6: Divisors.**

2.7. **II.7: Projective Morphisms.**

2.8. **II.8: Differentials.**

2.9. **II.9: Formal Schemes.**

3. III: Cohomology

3.1. III.1: Derived Functors.

3.2. III.2: Cohomology of Sheaves.

3.3. III.3: Cohomology of a Noetherian Affine Scheme.

3.4. III.4: Čech Cohomology.

3.5. III.5: The Cohomology of Projective Space.

3.6. III.6: Ext Groups and Sheaves.

3.7. III.7: Serre Duality.

3.8. III.8: Higher Direct Images of Sheaves.

3.9. III.9: Flat Morphisms.

3.10. III.10: Smooth Morphisms.

3.11. III.11: The Theorem on Formal Functions.

3.12. III.12: The Semicontinuity Theorem.

4. IV: Curves ⋆

Remark 4.0.1. Summary of major results:

- $p_a(X) \coloneqq 1 - P_X(0)$
- $p_g(X) \coloneqq h^0(\omega_X) = h^0(\mathcal{L}(K_X))$
- For curves, $p_a(X) = p_g(X) = h^1(\mathcal{O}_X)$ by setting $D \coloneqq K_C$ in RR.
  - $\deg K_C = 2g - 2$.
- $D_1 \sim D_2 \iff D_1 - D_2 = (f)$ for $f \in K(X)$ rational, $|D| = \{D' \sim D\}$, and this bijects with points of $\frac{H^0(\mathcal{L}(D))\setminus\{0\}}{\mathbf{G}_m}$.
  - Thus $\dim |D| = h^0(\mathcal{L}(D)) - 1 \coloneqq \ell(D) - 1$.
- X smooth $\implies \mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Pic}(X)$ via $D \mapsto \mathcal{L}(D)$.
- $h^0(\mathcal{L}(D)) > 0 \implies \deg(D) \geq 0$, and if $\deg D = 0$ then $D \sim 0$ and $\mathcal{L}(D) \cong \mathcal{O}_X$.
- RR:
  - How to remember: note $g = h^1(\mathcal{O}_X) = h^1(\mathcal{L}(0))$, and $H^0(\mathcal{O}_X) = k$ so $h^0(\mathcal{O}_X) = 1$, thus
  - For $C \subseteq \mathbf{P}^n$, $\deg(C) = d$ and $D = C \cap H$ a hyperplane section defining $\mathcal{L}(D) = \mathcal{O}_X(1)$,
- A curve is rational iff isomorphic to $\mathbf{P}^1$ iff $g = 0$.
- $K \sim 0$ on an elliptic curve since $\deg K = 2g - 2 = 0$ and $\deg D = 0 \implies D \sim 0$.
- For X elliptic, $\mathrm{Pic}^0(X) \coloneqq \left\{ D \in \mathrm{Div}(X) \;\middle|\; \deg D = 0 \right\}$ and $|X| \xrightarrow{\sim} \left|\mathrm{Pic}^0(X)\right|$ via $p \mapsto \mathcal{L}(p - p_0)$ for any fixed $p_0 \in X$, inducing its group structure. (This is proved with RR.)

Remark 4.0.2. Comments from preface:

- The statement of Riemann-Roch is important; less so its proof.
- Representing curves:
  - A branched covering of $\mathbf{P}^1$,
  - More generally a branched covering of another curve,
  - Nonsingular projective curves: admit embeddings into $\mathbf{P}^3$, maps to $\mathbf{P}^2$ birationally such that the image is at worst a nodal curve.
- The central result regarding representing curves: Hurwitz's theorem which compares K_X, K_Y for a cover $Y \to X$ of curves.
- Curves of genus 1: elliptic curves.
- Later sections: the canonical embedding of a curve.

4.1. IV.1: Riemann-Roch.

Definition 4.1.1 (Curves)**.** A **curve** over $k = \overline{k}$ is a scheme over $\operatorname{Spec} k$ which is

- Integral
- Dimension 1
- Proper over k
- With regular local rings

In particular, a curve is smooth, complete, and necessary projective. A **point** on a curve is a closed point.

Definition 4.1.2 (Arithmetic genus)**.** The **arithmetic genus** of a projective curve X is where $P_X(t)$ is the **Hilbert polynomial** of X.

Definition 4.1.3 (Geometric genus)**.** The **geometric genus** of a curve is where ω_X is the canonical sheaf.

Exercise 4.1.4 (?)**.** Show that if X is a curve, there is a single well-defined **genus**

Hint: see Ch. III Ex. 5.3, and use Serre duality for p_g.

Exercise 4.1.5 (?)**.** Show that for any $g \geq 0$ there exists a curve of genus g.

Hint: take a divisor of type $(g+1,2)$ on a smooth quadric which is irreducible and smooth with $p_a = g$.

Definition 4.1.6 (Divisors on a curve)**.** Reviewing divisors:

- The **divisor group**: $\operatorname{Div}(X) = \mathbf{Z}\left[X_{\mathrm{cl}}\right]$
- **Degrees**: $\deg(\sum n_i D_i) \coloneqq \sum n_i$, and
- **Linear equivalence**: $D_1 \sim D_2 \iff D_1 - D_1 = \operatorname{Div}(f)$ for some $f \in k(X)$ a rational function.
- D is **effective** if $n_i \geq 0$ for all i.
- $|D| \coloneqq \left\{ D' \in \operatorname{Div}(X) \;\middle|\; D' \sim D \right\}$ is the **complete linear system** of D.
- $|D| \cong \mathbf{P}H^0(X; \mathcal{L}(D))$
- **Dimensions of linear systems**: $\ell(D) \coloneqq \dim_k H^0(X; \mathcal{L}(D))$ and $\dim |D| \coloneqq \ell(D) - 1$.
- **Relative differentials**: $\Omega_X \coloneqq \Omega_{X_{/k}}$ is the sheaf of relative differentials on X.
  - The technical definition: $\Omega_{X_{/S}} \coloneqq \Delta^*_{X_{/Y}}(\mathcal{I}/\mathcal{I}^2)$ where $\mathcal{I}$ is the sheaf of ideals defining the locally closed subscheme $\operatorname{im}(\Delta_{X_{/Y}}) \subseteq X \mathrm{fp} Y X$.
  - On affine schemes: on the ring side, $\Omega_{B_{/A}} \in {}_B\mathsf{Mod}$ equipped with a differential $d: B \to \Omega B_{/A}$, defined as $\left\langle db \;\middle|\; b \in B \right\rangle_B / \left\langle d(b_1 + b_2) = db_1 + db_2, d(b_1 b_2) = d(b_1) b_2 + b_1 d(b_2), da = 0 \, \forall a \in A \right\rangle_B$
  - On curves, $\Omega_{X_{/Y}}$ measures the "difference" between K_X and K_Y.
- **Canonical sheaf**: $\dim X = 1, \Omega_{X_{/k}} \cong \omega_X$.
- **Canonical divisor**: K_X 2is any divisor in the linear equivalence class corresponding to ω_X
- D is **special** iff its **index of speciality** $\ell(K - D) > 0$, otherwise D is **nonspecial**.

Exercise 4.1.7 (?)**.** Show that $D_1 \sim D_2 \implies \deg(D_1) = \deg(D_2)$.

Exercise 4.1.8 (?)**.** Show that so $|D|$ has the structure of the closed points of some projective space.

Exercise 4.1.9 (Lemma 1.2)**.** Show that if $D \in \operatorname{Div}(X)$ for X a curve and $\ell(D) \neq 0$, then $\deg(D) \geq 0$.

Show that is $\ell(D) \neq 0$ and $\deg D = 0$ then $D \sim 0$ and $\mathcal{L}(D) \cong \mathcal{O}_X$.

Theorem 4.1.10 (Riemann-Roch)**.**

Exercise 4.1.11 (Ingredients for proof of RR)**.** Show the following:

- The divisor $K - D$ corresponds to $\omega_X \otimes \mathcal{L}(D)^\vee \in \operatorname{Pic}(X)$.
- $H^1(X; \mathcal{L}(D))^\vee \cong H^0(X; \omega_X \otimes \mathcal{L}(D)^\vee)$.
- If X is any projective variety,

Exercise 4.1.12 (?)**.** Show that if $X \subseteq \mathbf{P}^n$ is a curve with $\deg X = d$ and $D = X \cap H$ is a hyperplane section, then $\mathcal{L}(D) = \mathcal{O}_X(1)$ and $\chi(\mathcal{L}(D)) = d + 1 - p_a$.

Exercise 4.1.13 (?)**.** Show that if $g(X) = g$ then $\deg K_X = 2g - 2$.

Hint: set $D = K$ and use $\ell(K) = p_g = g$ and $\ell(0) = 1$.

Remark 4.1.14. More definitions:

- X is **rational** iff birational to $\mathbf{P}^1$.
- X is **elliptic** if $g = 1$.

Exercise 4.1.15 (?)**.** Show that

1. If $\deg D > 2g - 2$ then D is nonspecial.
2. $p_a(\mathbf{P}^1) = 0$.
3. A complete nonsingular curve is rational iff $X \cong \mathbf{P}^1$ iff $g(X) = 0$.

4. If X is elliptic then $K \sim 0$

   Hint: for (3) apply RR to $D = p - q$ for points $p \neq q$, and use $\deg(K - D) = -2$ and $\deg(D) = 0 \implies D \sim 0 \implies p \sim q$. For (4), show $\ell(K) = p_g = 1$.

Exercise 4.1.16 (?)**.** If X is elliptic and $p \in X$, then there is a bijection so $\operatorname{Pic}(X) \in \mathsf{Grp}$.

Hint: show that if $\deg(D) = 0$ then there is some $x \in X$ such that $D \sim x - p$ and apply RR to $D + p$.

4.2. IV.2: Hurwitz $\star$.

Remark 4.2.1. Summary of results:

- For curves, complete = projective.
- Riemann-Hurwitz: for $f : X \to Y$ finite separable,
- $\deg f := [K(X) : K(Y)]$ for finite morphisms of curves.
- $e_p := v_p(f^\sharp_* t)$ where t is uniformizer in $\mathcal{O}_{f(p)}$ and $f^\sharp : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ for $f : X \to Y$.
  - $e_p > 1 \implies$ ramification.
  - Unramified everywhere implies etale (since automatically flat).
  - $p \mid e_{x_0} \implies$ wild ramification, otherwise tame.
- $\exists f^* : \operatorname{Div}(Y) \to \operatorname{Div}(X)$ where $q \mapsto \sum_{p \mapsto q} e_p p$.
- Pullback commutes with forming line bundles: where the LHS $f^* : \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$.
- The fundamental SES for relative differentials: if $f : X \to Y$ is finite separable,
- $\frac{\partial t}{\partial u}$ for t a uniformizer at $f(p)$ and u a uniformizer at p is defined by noting $\Omega Y, f(p) = \langle\, dt \rangle, \Omega_{X,p} = \langle\, du \rangle$, and there is some $g \in \mathcal{O}_{X,p}$ such that $f^* \, dt = g \, du$; set $g := \frac{\partial t}{\partial u}$.
- For finite separable morphisms of curves $f : X \to Y$,
  - $\operatorname{supp} \Omega_{X/Y} = \operatorname{Ram}(f)$ is the ramification locus, and $\Omega_{X/Y}$ is torsion so $\operatorname{Ram}(f)$ is finite.
  - $\operatorname{length}(\Omega_{X,Y})_p = v_p\left(\frac{\partial t}{\partial u}\right)$ for any $p \in X$
  - Tamely ramified $\implies$ $\operatorname{length}(\Omega_{X/Y})_p = e_p - 1$, and wild ramification increases this length. Recall that length is the largest size of chains of submodules.
- The ramification divisor:
- $K_X \sim f^* K_Y + R$
- $\mathbf{P}^1$ can't admit an unramified cover: for $n \geq 1$, which forces $g(X) = 0, n = 1, X = \mathbf{P}^1, f = \mathrm{id}$.
- The Frobenius morphism on schemes is defined by taking $f^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ to be the pth power map; pullback yields a definition of X_p, the Frobenius twist of X.
  - $F : X_p \to X$ is finite, $\deg F = p$, and corresponds to $K(X) \hookrightarrow K(X)^{\frac{1}{p}}$
- If $f : X \to Y$ induces a purely inseparable extension $K(X)/K(Y)$, then $X \xrightarrow{\sim} Y$ as schemes, $g(X) = g(Y)$, and f is a composition of Frobenii.
- Everywhere ramified extensions: $f : Y_p \to Y$, where $e_q = p$ for every $q \in X$. Induces $\Omega_{X/Y} \cong \Omega_X$.
- $\deg R$ is always even.
- Finite implies proper: finite implies separated, of finite type, closed by "going up" and universally closed by since finiteness is preserved under base change.
- $\mathbf{P}^1$ no nontrivial etale covers.
- If $f : X \to Y$ then $g(X) \geq g(Y)$.
  - Thus $\exists \mathbf{P}^1 \to Y$ finite $\implies g(Y) = 0$.

Remark 4.2.2. Preface:

- **Degree**: for a finite morphism of curves $X \xrightarrow{f} Y$, set $\det(f) := [k(X) : k(Y)]$, the degree of the extension of function fields.

- **Ramification indices** e_p: for $p \in X$, let $q = f(p)$ and $t \in \mathcal{O}_q$ a local coordinate. Pull back to $t \in \mathcal{O}_p$ via $f^\sharp$ and define $e_p \coloneqq v_p(t)$ using the valuation v_p for the DVR $\mathcal{O}_p$.
- **Ramified**: $e_p > 1$, and **unramified** if $e_p = 1$.
- **Branch points** any $q = f(p)$ where f is ramified.
- **Tame ramification**: for $\mathrm{ch}(k) = p$, tame if $p \nmid e_P$.
- **Wild ramification**: when $p \mid e_P$.
- Pullback maps on divisor groups:
  - This commutes with taking line bundles (exercise), so induces a well-defined map $f^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$.
- f is **separable** if $k(X)/k(Y)$ is a separable field extension.

Exercise 4.2.3 (?). Misc:

- Show that if f is everywhere unramified then it is an étale morphism.
- Show that $f^*\mathcal{L}(D) = \mathcal{L}(f^*D)$

Exercise 4.2.4 (Prop 2.1). Show that if $X \xrightarrow{f} Y$ is a finite separable morphism of curves, there is a SES

Remark 4.2.5. Definitions:

- **Derivatives**: for $f : X \to Y$, let t be a parameter at $Q = f(P)$ and u at P. Then $\Omega_{Y,Q} = \langle dt \rangle_{\mathcal{O}_Q}$ and $\mathcal{O}_{X,P} = \langle du \rangle_{\mathcal{O}_P}$ and $\exists! g \in \mathcal{O}_P$ such that $f^*dt = du$ so we write $\frac{\partial t}{\partial u} \coloneqq g$.
- **Ramification divisor**: $R \coloneqq \sum_{P \in X} \mathrm{length}(\Omega_{X/Y})_P [P] \in \mathrm{Div}(X)$

Exercise 4.2.6 (Prop 2.2). For $X \xrightarrow{f} Y$ a finite separable morphism of curves,

a. $\Omega_{X/Y}$ is a torsion sheaf on X with support equal to the ramification locus of f. Thus f is ramified at finitely many points.
b. The stalks $(\Omega_{X/Y})_P$ are principal $\mathcal{O}_P$-modules of finite length equal to $v_p\left(\frac{\partial t}{\partial u}\right)$
c.

Exercise 4.2.7 (Prop 2.3). If $X \xrightarrow{f} Y$ is a finite separable morphism of curves, then where R is the ramification divisor of f.

Theorem 4.2.8 (Hurwitz). *If $X \xrightarrow{f} Y$ is a finite separable morphism of curves, then and if f has only tame ramification then* $\deg(R) = \sum_{P \in X}(e_P - 1)$.

Remark 4.2.9 (proof of Hurwitz). Take degrees of the divisor equation: using tame ramification in the last step which implies $\mathrm{length}(\Omega_{X/Y})_P = (e_p - 1)$.

Remark 4.2.10. Consider the purely inseparable case.

- **Frobenius morphism**: for $X \in \mathsf{Sch}$ where $\mathcal{O}_P \supseteq \mathbf{Z}/p\mathbf{Z}$ for all P, define $\mathrm{Frob} : X \to X$ by $F(|X|) = |X|$ on spaces and $F^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ is $f \mapsto f^p$. This is a morphism since $F^\sharp$ induces a morphism on all local rings, which are all characteristic p.
- **The k-linear Frobenius morphism**: define X_p to be X with the structure morphism $F \circ \pi$, so $k \curvearrowright \mathcal{O}_{X_p}$ by pth powers and F becomes a k-linear morphism $F' : X_p \to X$.
  - Why this is necessary: F as before is not a morphism in $\mathsf{Sch}_{/k}$, and instead forms a commuting square involving $F : \operatorname{Spec} k \to \operatorname{Spec} k$ and the structure maps $X \xrightarrow{\pi} \operatorname{Spec} k$.

Exercise 4.2.11 (?). Find examples where

- $X_p \cong X \in \mathsf{Sch}_{/k}$, and
- $X_p \not\cong X \in \mathsf{Sch}_{/k}$.

  Hint: consider $X = \operatorname{Spec} k[t]$ for k perfect.

Exercise 4.2.12 (?)**.** Show that if $X \xrightarrow{f} Y$ is separable then $\deg(R)$ is always even.

> Skipped some stuff around Example 2.4.2, I don't necessarily need characteristic p things right now.

Remark 4.2.13. Definitions:

- **Étale covers**: $X \xrightarrow{f} Y$ is an étale cover if f is a finite étale morphism,, i.e. f is flat and $\Omega^1_{X/Y} = 0$.
- Y is a **trivial** cover if $X \cong \coprod_{i \in I} Y$ a finite disjoint union of copies of Y,
- Y is **simply connected** if there are no nontrivial étale covers.

Exercise 4.2.14 (?)**.**

- Show that a connected regular curve is irreducible.
- Show that if f is etale then X is smooth over k.
- Show that if f is finite, X must be a curve.
- Show that if f is étale, then f must be separable.
- Show that $\pi_1^{\text{ét}}(\mathbf{P}^1) = 0$.

  Hint: use Hurwitz and that when f is unramified, $R = 0$.

Exercise 4.2.15 (?)**.**

- Show that the genus of a curve doesn't change under purely inseparable extensions.
- Show that if $f : X \to Y$ is a finite morphism of curves then $g(X) \geq g(Y)$.

Exercise 4.2.16 (Lüroth)**.** Show that if L is a subfield of a purely transcendental extension $k(t)/k$ where $k = \overline{k}$, then L is also purely transcendental.[2]

> Hint: Assume $[L : k]_{\text{tr}} = 1$, so $L = k(X)$ for Y a curve and $L \subseteq k(t)$ corresponds to a finite morphism $f : \mathbf{P}^1 \to Y$. Conclude $g(Y) = 0$ so $Y \cong \mathbf{P}^1$ and $L \cong k(u)$ for some u.

4.3. IV.3: Embeddings in Projective Space ⋆.

Remark 4.3.1. A summary of major results:

- For $D \in \operatorname{Div}(C)$ with $g = g(C)$,
  - D is ample iff $\deg D > 0$.
  - D is BPF iff $\deg D \geq 2g$.
  - D is very ample iff $\deg D \geq 2g + 1$.
- Being very ample is equivalent to being a hyperplane section under a projective embedding.
- Divisors $D \in \operatorname{Div}(\mathbf{P}^n)$ are ample iff very ample iff $\deg D \geq 1$.
  - E.g. if E is elliptic then D is very ample if $\deg D \geq 3$, and for hyperelliptic, very ample if $\deg D \geq 5$.
- If D is very ample then $\deg \varphi(X) = \deg D$.
- Curves $C \subseteq \mathbf{P}^n$ for $n \geq 4$ can be projected away from a point $p \notin X$ to get a closed immersion into $\mathbf{P}^m$ for some $m \leq n - 1$. So any curve is birational to a nodal curve in $\mathbf{P}^2$.
- Genus of normalizations of nodal curves: $g = \frac{1}{2}(d-1)(d-2) - \sharp\{\text{nodes}\}$.
- Any curve embeds into $\mathbf{P}^3$, and maps into $\mathbf{P}^2$ with at worst nodal singularities.

Remark 4.3.2. Main result: any curve can be embedded in $\mathbf{P}^3$, and is birational to a nodal curve in $\mathbf{P}^2$. Some recollections:

- **Very ample line bundles**: $\mathcal{L} \in \operatorname{Pic}(X)$ is very ample if $\mathcal{L} \cong \mathcal{O}_X(1)$ for some immersion of $f : X \hookrightarrow \mathbf{P}^N$.

[2]This is true over any field k in dimension 1, over $k = \overline{k}$ in dimension 2, and false in dimension 3 by the existence of nonrational unirational threefolds.

- **Ample**: $\mathcal{L}$ is ample when $\forall \mathcal{F} \in \mathsf{Coh}(X)$, the twist $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated for $n \gg 0$.
- **(Very) ample divisors**: $D \in \operatorname{Div}(X)$ is (very) ample iff $\mathcal{L}(D) \in \operatorname{Pic}(X)$ is (very) ample.
- **Linear systems**: a linear system is any set $S \leq |D|$ of effective divisors yielding a linear subspace.
- **Base points**: P is a base point of S iff $P \in \operatorname{supp} D$ for all $D \in S$.
- **Secant lines**: the secant line of $P, Q \in X$ is the line in $\mathbf{P}^N$ joining them.
- **Tangent lines**: at $P \in X$, the unique line $L \subseteq \mathbf{P}^N$ passing through p such that $\mathbf{T}_P(L) = \mathbf{T}_P(X) \subseteq \mathbf{T}_P(\mathbf{P}^N)$.
- **Nodes**: a singularity of multiplicity 2.
  - $y^2 = x^3 + x^2$ is a **node**.
  - $y^2 = x^3$ is a **cusp**.
  - $y^2 = x^4$ is a **tacnode**.
- **Multisecant**: for $X \subseteq \mathbf{P}^3$, a line meeting X in 3 or more distinct points.
- A **secant with coplanar tangent lines** is a secant through P, Q whose tangent lines L_P, L_Q lie in a common plane, or equivalently L_P intersects L_Q.

Exercise 4.3.3 (II.8.20.2)**.** Show that by Bertini's theorem there are irreducible smooth curves of degree d in $\mathbf{P}^2$ for any d.

Exercise 4.3.4 (?)**.**

Show that

- $\mathcal{L}$ is ample iff $\mathcal{L}^n$ is very ample for $b \gg 0$.
- $|D|$ is basepoint free iff $\mathcal{L}(D)$ is globally generated.
- If D is very ample, then $|D|$ is basepoint free.
- If D is ample, $nD \sim H$ a hyperplane section for a projective embedding for some n.
- If $g(X) = 0$ then D is ample iff very ample iff $\deg D > 0$.
- If D is very ample and corresponds to a closed immersion $\varphi : X \hookrightarrow \mathbf{P}^n$ then $\deg \varphi(X) = \deg D$.
- If XS is elliptic, any D with $\deg D = 3$ is very ample and $\dim |D| = 2$, and so can be embedded into $\mathbf{P}^2$ as a cubic curve.
- Show that if $g(X) = 1$ then D is very ample iff $\deg D \geq 3$.
- Show that if $g(X) = 2$ and $\deg D = 5$ then D is very ample, so any genus 2 curve embeds in $\mathbf{P}^3$ as a curve of degree 5.

Exercise 4.3.5 (Prop 3.1: when a linear system yields a closed immersion into $\mathbf{P}^N$)**.** Let $D \in \operatorname{Div}(X)$ for X a curve and show

- $|D|$ is basepoint free iff $\dim |D - P| = \dim |D| - 1$ for all points $p \in X$.
- D is very ample iff $\dim |D - P - Q| = \dim |D| - 2$ for all points $P, Q \in X$.

  Hint: use the SES $\mathcal{L}(D - P) \hookrightarrow \mathcal{L}(D) \twoheadrightarrow k(P)$ where $k(P)$ is the skyscraper sheaf at P.

Exercise 4.3.6 (Cor 3.2)**.** Let $D \in \operatorname{Div}(X)$.

- If $\deg D \geq 2g(X)$ then $|D|$ is basepoint free.
- If $\deg D \geq 2g(X) + 1$ then D is very ample.
- D is ample iff $\deg D > 0$
- This bounds is not sharp.

  Hint: apply RR. For the bound, consider a plane curve X of degree 4 and $D = X.H$.

Remark 4.3.7. Idea behind embedding in $\mathbf{P}^3$: embed into $\mathbf{P}^n$ and project away from a point in the complement.

Exercise 4.3.8 (3.4, 3.5, 3.6)**.** Let $X \subseteq \mathbf{P}^N$ be a curve and $O \notin X$, let $\varphi : X \to \mathbf{P}^{n-1}$ be projection away from O. Then φ is a closed immersion iff

- O is not on any secant line of X, and
- O is not on any tangent line of X.

Show that if $N \geq 4$ then there exists such a point O yielding a closed immersion into $\mathbf{P}^{N-1}$. Conclude that any curve can be embedded into $\mathbf{P}^3$.

Hint: $\dim \operatorname{Sec}(X) \leq 3$ and $\dim \operatorname{Tan}(X) \leq 2$.

Proposition 4.3.9 (3.7). *Let $X \subseteq \mathbf{P}^3$, $O \notin X$, and $\varphi : X \to \mathbf{P}^2$ be the projection from O. Then $X \overset{\sim}{\dashrightarrow} \varphi(X)$ iff $\varphi(X)$ is nodal iff the following hold:*

- *O is only on finitely many secants of X,*
- *O is on no tangents,*
- *O is on no multisecant,*
- *O is on no secant with coplanar tangent lines.*

Skipped things around Prop 3.8. The hard part: showing not every secant is a multisecant, and not every secant has coplanar tangent lines. Skipped strange curves.

Remark 4.3.10. Classifying all curves: any curve is birational to a nodal plane curve, so study the family $\mathcal{F}_{d,r}$ of plane curves of degree d and r nodes. The family $\mathcal{F}_d$ of all plane curves is a linear system of dimension For any such curve X, consider its normalization $\nu(X)$, then Thus for $\mathcal{F}_{d,r}$ to be nonempty, one needs Both extremes can occur: $r = 0$ follows from Bertini, and $r = \frac{(d-1)(d-2)}{2}$ by embedding $\mathbf{P}^1 \hookrightarrow \mathbf{P}^d$ as a curve of degree d and projecting down to a nodal curve in $\mathbf{P}^2$ of genus zero. Severi states and Harris proves that for every r in this range $\mathcal{F}_{d,r}$ is irreducible, nonempty, and $\dim \mathcal{F}_{d,r} = \frac{d(d+3)}{2} - r$.

4.4. IV.4: Elliptic Curves $\star$.

Remark 4.4.1. Curves E with $g(E) = 1$; we'll assume $\operatorname{ch} k \neq 2$ throughout. Outline:

- Define the j-invariant, classifies isomorphism classes of elliptic curves.
- Group structure on the curve.
- $E = \operatorname{Jac}(E)$.
- Results about elliptic functions over $\mathbf{C}$.
- The Hasse invariant of $E/\mathbf{F}_q$ in characteristic p.
- $E(\mathbf{Q})$.

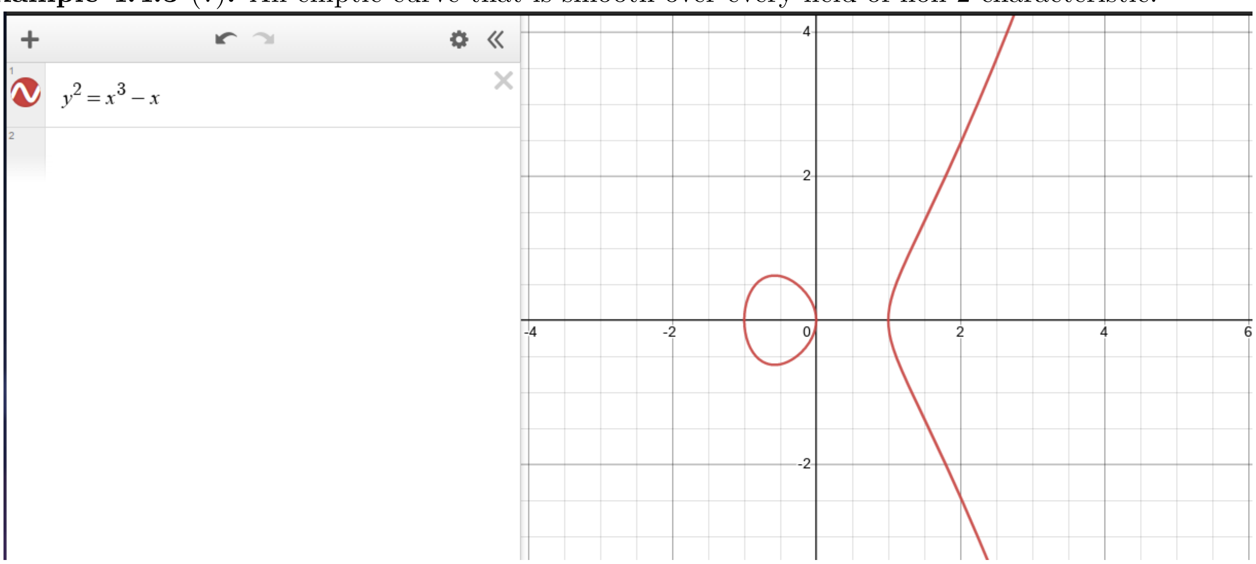
4.4.1. *The j-invariant.*

Remark 4.4.2. The j-invariant:

- $j(E) \in k$, so $\mathbf{A}^1_{/k}$ is a coarse moduli space for elliptic curves over K.
- Defining $j(E)$:
  - Let $p_0 \in X$, consider the linear system $L := |2p_0|$.
  - Nonspecial, so RR shows $\dim(L) = 1$.
  - BPF, otherwise E is rational.
  - Defines a morphism $\varphi_L : E \to \mathbf{P}^1_{/k}$ with $\deg \varphi_L = 2$.
  - Up to change of coordinates, $f(p_0) = \infty$.
  - By Hurwitz, f is ramified at 4 branch points a, b, c, p_0.
  - Move $a \mapsto 0, b \mapsto 1$ by a Mobius transformation fixing ∞, so f is branched over $0, 1, \lambda, \infty$ where $\lambda \in k \setminus \{0, 1\}$.
  - Use λ to define the invariant:
- Theorem 4.1:
  - j depends only on the curve E and not λ.
  - $E \cong E' \iff j(E) = j(E')$.
  - Every element of k occurs as $j(E)$ for some E.

  - So this yields a bijection
- Some facts that go into proving this:
  - $\forall p, q \in X\ \exists \sigma \in \operatorname{Aut}(X)$ such that $\sigma^2 = 1, \sigma(p) = q$, for any $r \in X$, one has $r + \sigma(r) \sim p + q$.
  - $\operatorname{Aut}(X) \curvearrowright X$ transitively.
  - Any two degree two maps $f_1, f_2 : X \to \mathbf{P}^1$ fit into a commuting square.
  - Under $S_3 \curvearrowright \mathbf{A}^1_{/k} \setminus \{0, 1\}$, the orbit of λ is
  - Fixing $p \in X$, there is a closed immersion $X \to \mathbf{P}^2$ whose image is $y^2 = x(x-1)(x-\lambda)$ where $p \mapsto \infty = [0 : 1 : 0]$ and this λ is either the λ from above or one of $s_1^{\pm 1}, s_2^{\pm 1}$.
    - Idea of proof: embed $X \hookrightarrow \mathbf{P}^2$ by $L := |3p|$, use RR to compute $h^0(\mathcal{O}(np)) = n$ so $h^0(\mathcal{O}(6p)) = 6$.
    - So $\{1, x, y, x^2, xy, y^2, x^3\}$ has a linear dependence where x^3, y^2 have nonzero coefficients since they have poles at p.
    - Rescale x^3, y^2 to coefficient 1 to get
  - Do a change of variable to put in the desired form: complete the square on the LHS, factor as $y^2 = (x-a)(x-b)(x-c)$, send $a \to 0, b \to 1$ by a Mobius transformation.
- Note that one can project from p to the x-axis to get a finite degree 2 morphism ramified at $0, 1, \lambda, \infty$.

Example 4.4.3 (?)**.** An elliptic curve that is smooth over every field of non-2 characteristic:



One that is smooth over every k with $\operatorname{ch} k \neq 3$: the Fermat curve

Theorem 4.4.4 (Orders of automorphism groups of elliptic curves)**.**

Remark 4.4.5 (Proof idea)**.** Idea: take the degree 2 morphism $f : X \to \mathbf{P}^1$ with $f(p) = \infty$ branched over $\{0, 1, \lambda, \infty\}$. Produce two elements in G: for $\sigma \in G$, find $\tau \in \operatorname{Aut}(\mathbf{P}^1)$ so $f\sigma = \tau f$; then either $\tau \neq \mathrm{id}$, so $\{\sigma, \tau\} \subseteq G$, or $\tau = id$ and either $\sigma = \mathrm{id}$ or σ exchanges the sheets of f.

If $\tau \neq \mathrm{id}$, it permutes $\{0, 1, \lambda\}$ and sends $\lambda \mapsto \lambda^{-1}, s_1^{\pm 1}, s_2^{\pm 1}$ from above. Cases:

1. $j = 1728$: If $\lambda = -1, 1/2, 2, \operatorname{ch} k \neq 3$, then λ coincides with *one* other element of $S_3.\lambda$, so $\sharp G = 4$.
2. $j = 0$: If $\lambda = -\zeta_3, -\zeta_3^2, \operatorname{ch} k \neq 3$ then λ coincides with *two* elements in $S_3.\lambda$ so $\sharp G = 6$.
3. $j = 0 = 1728$: If $\lambda = -1, \operatorname{ch} k = 3$ then $S_3.\lambda = \{\lambda\}$ and $\sharp G = 12$.

4.4.2. *The group structure.*

Remark 4.4.6. The group structure:

- Fixing $p_o \in E$, the map $p \mapsto \mathcal{L}(p - p_0)$ induces a bijection $E \xrightarrow{\sim} \mathrm{Pic}^0(E)$, so the group structure on E is the pullback along this with $p_0 = \mathrm{id}$ and $p + q = r \iff p + q \sim r + p_0 \in \mathrm{Div}(E)$.
- Under the embedding of $|3p_0|$, points p, q, r are collinear iff $p + q + r \sim 3p_0$, so $p + q + r = 0$ in the group structure.
- E is a group variety, since $p \mapsto -p$ and $(p, q) \mapsto p + q$ are morphisms. Thus there is a morphism $[n] : E \to E$, multiplication by n, which is a finite morphism of degree n^2 with kernel $\ker[n] = C_n^2$ if $(n, \mathrm{ch}\, k) = 1$.and $\ker[n] = C_p, 0$ if $n = \mathrm{ch}\, k$, depending on the Hasse invariant.
- If $f : E_1 \to E_2$ is a morphism of curves with $f(p_1) = p_2$ then f induces a group morphism.
- $\mathrm{End}(E, p_0)$ forms a ring under $f + g = \mu \circ (f \times g)$ and $f \cdot g \coloneqq f \circ g$.
- The map $n \mapsto ([n] : E \to E)$ defines a finite ring morphism $\mathbf{Z} \to \mathrm{End}(E, p_0)$ for $n \neq 0$.
- $R \coloneqq \mathrm{End}(E, p_0)^\times = \mathrm{Aut}(E)$, and if $j = 0, 1728$ then R contains $\{\pm 1\}$ and is thus bigger than $\mathbf{Z}$.

Remark 4.4.7. The Jacobian: a variety that generalizes to make sense for any curve, a moduli space of degree zero divisor classes.

- For X/k a curve and $T \in \mathsf{Sch}_{/k}$, define where $p : X \times T \to T$ is the second projection. Regard this as *families of sheaves of degree 0 on X parameterized by T.*
- The Jacobian variety of a curve X: $\mathrm{Jac}(X) \in \mathsf{Sch}^{\mathrm{ft}}_{/k}$ along with $\mathcal{L} \in \mathrm{Pic}^0(X/\mathrm{Jac}(X))$ such that for any $T \in \mathsf{Sch}^{\mathrm{ft}}_{/k}$ and any $\mathcal{M} \in \mathrm{Pic}^0(X/T)$, $\exists!\, f : T \to \mathrm{Jac}(X)$ such that $f^*\mathcal{L} = \mathcal{M}$. Thus J represents the functor $\mathrm{Pic}^0(X/-)$.
- For E elliptic, $E = \mathrm{Jac}(E)$.
  - In general, $|\mathrm{Jac}(X)| \cong \left|\mathrm{Pic}^0(X)\right|$ on points, since points of $\mathrm{Jac}(X)$ are morphisms $\mathrm{Spec}\, k \to \mathrm{Jac}(X)$, which correspond to elements in $\mathrm{Pic}^0(X/k) = \mathrm{Pic}^0(X)$.
- $\mathrm{Jac}(X) \in \mathsf{GrpSch}_{/k}$ where $e : \mathrm{Spec}\, k \to \mathrm{Jac}(X)$ corresponds to $0 \in \mathrm{Pic}^0(X/k)$, $\rho : \mathrm{Jac}(X) \to \mathrm{Jac}(X)$ is $\mathcal{L} \mapsto \mathcal{L}^{-1} \in \mathrm{Pic}^0(X/\mathrm{Jac}(X))$, and $\mu : \mathrm{Jac}(X)^{\times^2} \to \mathrm{Jac}(X)$ is $\mathcal{L} \mapsto p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \in \mathrm{Pic}^0(X/\mathrm{Pic}(X)^{\times^2})$.
- $\mathbf{T}_0\, \mathrm{Jac}(X) \cong H^1(X; \mathcal{O}_X)$: giving an element of $\mathbf{T}_p X$ is the same as a morphism $T \coloneqq \mathrm{Spec}\, k[\varepsilon]/\varepsilon^2 \to X$ sending $\mathrm{Spec}\, k \to p$. So $\mathbf{T}_0\, \mathrm{Jac}(X)$, this means giving $\mathcal{M} \in \mathrm{Pic}^0(X/T)$ whose restriction to $\mathrm{Pic}^0(X/k)$ is zero. Use the SES $H^1(X; \mathcal{O}_X) \hookrightarrow \mathrm{Pic}\, X[\varepsilon] \to \mathrm{Pic}(X)$.
- $\mathrm{Jac}(X)$ is proper over k by the valuative criterion. Just show that an invertible sheaf $\mathcal{M}$ on $X \times \mathrm{Spec}\, K$ lifts unique to $\tilde{\mathcal{M}}$ on $X \times \mathrm{Spec}\, R$, but $X \times \mathrm{Spec}\, R$ is regular, so apply II.6.5.
- For any n there is a morphism This is surjective for $n \geq g(X)$ by RR since every divisor class of degree $d \geq g$ has an effective representative. The fibers of φ^n are all tuples $(p_1, \cdots, p_n)$ such that $D = \sum p_i$ forms a complete linear system.
  - Most fibers are finite, so $\mathrm{Jac}(X)$ is irreducible of dimension g.
  - Smoothness: $\dim \mathbf{T}_0\, \mathrm{Jac}(X) = \dim H^1(X; \mathcal{O}_X) = g$, so smooth at zero, and group schemes are homogeneous so smooth everywhere.

4.4.3. *Elliptic functions.*

Stopped at elliptic functions.

4.5. IV.5: The Canonical Embedding.

4.6. IV.6: Classification of Curves in $\mathbf{P}^3$.

5. V: Surfaces

5.1. **V.1: Geometry on a Surface.**

5.2. **V.2: Ruled Surfaces.**

5.3. **V.3: Monoidal Transformations.**

5.4. **V.4: The Cubic Surface in $\mathbf{P}^3$.**

5.5. **V.5: Birational Transformations.**

5.6. **V.6: Classification of Surfaces.**