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1 | I: Varieties

Remark 1.0.1: Some useful basic properties:

- Properties of V:
  - $\cap_{i\in I} V(\mathfrak{a}_i) = V\left(\sum_{i\in I} \mathfrak{a}_i\right)$.
    - E.g. $V(x) \cap V(y) = V(\langle x\rangle + \langle y\rangle) = V(x,y) = \{0\}$, the origin.
  - $\cup_{i\leq n} V(\mathfrak{a}_i) = V\left(\prod_{i\leq n} \mathfrak{a}_i\right)$.
    - E.g. $V(x) \cup V(y) = V(\langle x\rangle \langle y\rangle) = V(xy)$, the union of coordinate axes.
  - $V(\mathfrak{a})^c = \cup_{f\in\mathfrak{a}} D(f)$
  - $V(\mathfrak{a}_1) \subseteq V(\mathfrak{a}_2) \iff \sqrt{\mathfrak{a}_1} \supseteq \sqrt{\mathfrak{a}_2}$.
- Properties of I:
  - $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ and $V(I(Y)) = \mathrm{cl}_{\mathbf{A}^n}(Y)$. The containment correspondence is contravariant in both directions.
  - $I(\cup_i Y_i) = \cap_i I(Y_i)$.
- If F is a sheaf taking values in subsets of a giant ambient set, then $F(\cup U_i) = \cap F(U_i)$. For $\mathbf{A}^n/\mathbf{C}$, take $\mathbf{C}(x_1, \cdots, x_n)$, the field of rational functions, to be the ambient set.
- Distinguished open $D(f) \coloneqq \left\{p \in X \;\middle|\; f(p) \neq 0\right\}$:
  - $\mathcal{O}_X(D(f)) = A(X)\left[\frac{1}{f}\right] = \left\{\frac{g}{f^k} \;\middle|\; g \in A(X), k \geq 0\right\}$, and taking $f = 1$ shows $\mathcal{O}_X(X) = A(X)$, i.e. global regular functions are polynomial.
  - Generally $D(fg) = D(f) \cap D(g)$
  - For affines:

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R\left[\frac{1}{f}\right].$$

  - For $\mathbf{C}^n$,

$$\mathcal{O}_{\mathbf{C}^n}(D(f)) = k[x_1, \cdots, x_n][1/f] \implies \mathcal{O}_{\mathbf{C}^n}(V(\mathfrak{a})^c) = \cap_{f\in\mathfrak{a}} \mathcal{O}_{\mathbf{C}^n}(D(f)).$$

1.1 I.1: Affine Varieties ⋆

Remark 1.1.1: Summary:

- $\mathbf{A}^n_{/k} = \left\{[a_1, \cdots, a_n] \;\middle|\; a_i \in k\right\}$, and elements $f \in A \coloneqq k[x_1, \cdots, x_n]$ are functions on it.
- $Z(f) \coloneqq \left\{p \in \mathbf{A}^n \;\middle|\; f(p) = 0\right\}$, and for any $T \subseteq A$ we set $Z(T) \coloneqq \cap_{f\in T} Z(f)$.
  - Note that $Z(T) = Z(\langle T\rangle_A) = Z(\langle f_1, \cdots, f_r\rangle)$ for some generators f_i, using that A is a Noetherian ring. So every $Z(T)$ is the set of common zeros of finitely many polynomials, i.e. the intersection of finitely many hypersurfaces.

- **Algebraic**: $Y \subseteq \mathbf{A}^n$ is algebraic iff $Y = Z(T)$ for some $T \subsetneq A$.
- The Zariski topology is generated by open sets of the form $Z(T)^c$.
- $\mathbf{A}^1$ is a non-Hausdorff space with the cofinite topology.
- **Irreducible**: Y is reducible iff $Y = Y_1 \cup Y_2$ with Y_1, Y_2 proper subsets of Y which are closed in Y.
  - Nonempty open subsets of irreducible spaces are both irreducible and dense.
  - If $Y \subseteq X$ is irreducible then $\mathrm{cl}_X(Y) \subseteq X$ is again irreducible.
- **Affine (algebraic) varieties**: irreducible closed subsets of $\mathbf{A}^n$.
- **Quasi-affine varieties**: open subsets of affine varieties.
- The ideal of a subset: $I(Y) := \left\{ f \in A \;\middle|\; f(p) = 0 \ \forall p \in Y \right\}$.
- **Nullstellensatz**: if $k = \overline{k}$, $\mathfrak{a} \in \mathrm{Id}(k[x_1, \cdots, x_n])$, and $f \in k[x_1, \cdots, x_n]$ with $f(p) = 0$ for all $p \in V(\mathfrak{a})$, then $f^r \in \mathfrak{a}$ for some $r > 0$, so $f \in \sqrt{\mathfrak{a}}$. Thus there is a contravariant correspondence between radical ideals of $k[x_1, \cdots, x_n]$ and algebraic sets in $\mathbf{A}^n_{/k}$.
- **Irreducibility criterion**: Y is irreducible iff $I(Y) \in \operatorname{Spec} k[x_1, \cdots, x_n]$ (i.e. it is prime).
- **Affine curves**: if $f \in k[x, y]^{\mathrm{irr}}$ then $\langle f \rangle \in \operatorname{Spec} k[x, y]$ (since this is a UFD) so $Z(f)$ is irreducible and defines an affine curve of degree $d = \deg(f)$.
- **Affine surfaces**: $Z(f)$ for $f \in k[x_1, \cdots, x_n]^{\mathrm{irr}}$ defines a surface.
- **Coordinate rings**: $A(Y) := k[x_1, \cdots, x_n]/I(Y)$.
- **Noetherian spaces**: $X \in \mathsf{Top}$ is Noetherian iff the DCC on closed subsets holds.
- **Unique decomposition into irreducible components**: if $X \in \mathsf{Top}$ is Noetherian then every closed nonempty $Y \subseteq X$ is of the form $Y = \cup_{i=1}^r Y_i$ with Y_i a uniquely determined closed irreducible with $Y_i \not\subseteq Y_j$ for $i \neq j$, the *irreducible components* of Y.
- **Dimension**: for $X \in \mathsf{Top}$, the dimension is $\dim X := \sup \left\{ n \;\middle|\; \exists Z_0 \subset Z_1 \subset \cdots \subset Z_n \right\}$ with Z_i distinct irreducible closed subsets of X. Note that the dimension is the number of "links" here, not the number of subsets in the chain.
- **Height**: for $\mathfrak{p} \in \operatorname{Spec} A$ define $\mathrm{ht}(\mathfrak{p}) := \sup \left\{ n \;\middle|\; \exists \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p} \right\}$ with $\mathfrak{p}_i \in \operatorname{Spec} A$ distinct prime ideals.
- **Krull dimension**: define $\operatorname{krulldim} A := \sup_{\mathfrak{p} \in \operatorname{Spec} A} \mathrm{ht}(\mathfrak{p})$, the supremum of heights of prime ideals.

Exercise 1.1.2 (The Zariski topology)
Show that the class of algebraic sets form the closed sets of a topology, i.e. they are closed under finite unions, arbitrary intersections, etc.

Exercise 1.1.3 (The affine line)

- Show that $\mathbf{A}^1_{/k}$ has the cofinite topology when $k = \overline{k}$: the closed (algebraic) sets are finite sets and the whole space, so the opens are empty or complements of finite sets.[a]
- Show that this topology is not Hausdorff.
- Show that $\mathbf{A}^1$ is irreducible without using the Nullstellensatz.
- Show that $\mathbf{A}^n$ is irreducible.
- Show that maximal ideals $\mathfrak{m} \in \operatorname{mSpec} k[x_1, \cdots, x_n]$ correspond to minimal irreducible closed subsets $Y \subseteq \mathbf{A}^n$, which must be points.
- Show that $\operatorname{mSpec} k[x_1, \cdots, x_n] = \left\{ \langle x_1 - a_1, \cdots, x_n - a_n \rangle \;\middle|\; a_1, \cdots, a_n \in k \right\}$ for $k = \overline{k}$,

and that this fails for $k \neq \overline{k}$.

- Show that $\mathbf{A}^n$ is Noetherian.
- Show $\dim \mathbf{A}^1 = 1$.
- Show $\dim \mathbf{A}^n = n$.

[a]Hint: $k[x]$ is a PID and factor any $f(x)$ into linear factors using that $k = \overline{k}$ to write $Z(\mathfrak{a}) = Z(f) = \{a_1, \cdots, a_k\}$ for some k.

Exercise 1.1.4 (Commutative algebra)

- Show that if Y is affine then $A(Y)$ is an integral domain and in ${}_k\mathsf{Alg}^{\mathrm{fg}}$.
- Show that every $B \in {}_k\mathsf{Alg}^{\mathrm{fg}} \cap \mathsf{Domain}$ is of the form $B = A(Y)$ for some $Y \in \mathsf{AffVar}_{/k}$.
- Show that if Y is an affine algebraic set then $\dim Y = \operatorname{krulldim} A(Y)$.

Theorem 1.1.5 ***(Results from commutative algebra).***

- If $k \in \mathsf{Field}$, $B \in {}_k\mathsf{Alg}^{\mathrm{fg}} \cap \mathsf{Domain}$,
  - $\operatorname{krulldim} B = [K(B) : B]_{\mathrm{tr}}$ is the transcendence degree of the quotient field of B over B.
  - If $\mathfrak{p} \in \operatorname{Spec} B$ then $\operatorname{ht} \mathfrak{p} + \operatorname{krulldim}(B/\mathfrak{p}) = \operatorname{krulldim} B$.
- Krull's Hauptidealsatz:
  - If $A \in \mathsf{CRing}^{\mathrm{Noeth}}$ and $f \in A \setminus A^\times$ is not a zero divisor, then every minimal $\mathfrak{p} \in \operatorname{Spec} A$ with $\mathfrak{p} \ni f$ has height 1.
- If $A \in \mathsf{CRing}^{\mathrm{Noeth}} \cap \mathsf{Domain}$, then A is a UFD iff every $\mathfrak{p} \in \operatorname{Spec}(A)$ with $\operatorname{ht}(\mathfrak{p}) = 1$ is principal.

Exercise 1.1.6 (1.10)
Show that if Y is quasi-affine then

$$\dim Y = \dim \operatorname{cl}_{\mathbf{A}^n} Y.$$

Exercise 1.1.7 (1.13)
Show that if $Y \subseteq \mathbf{A}^n$ then $\operatorname{codim}_{\mathbf{A}^n}(Y) = 1 \iff Y = Z(f)$ for a single nonconstant $f \in k[x_1, \cdots, x_n]^{\mathrm{irr}}$.

Exercise 1.1.8 (?)
Show that if $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\operatorname{ht}(\mathfrak{p}) = 2$ then $\mathfrak{p}$ can not necessarily be generated by two elements.

1.2 I.2: Projective Varieties ⋆

Remark 1.2.1:

- **Projective space**: $\left\{ \mathbf{a} := [a_0, \cdots, a_n] \;\middle|\; a_i \in k \right\} / \sim$ where $\mathbf{a} \sim \lambda \mathbf{a}$ for all $\lambda \in k \setminus \{0\}$, i.e. lines in $\mathbf{A}^{n+1}$ passing through $\mathbf{0}$.

- **Graded rings**: a ring S with a decomposition $S = \oplus_{d \geq 0} S_d$ with each $S_d \in \mathsf{AbGrp}$ and $S_d S_e \subseteq S_{d+e}$; elements of S_d are **homogeneous of degree** d and any element in S is a finite sum of homogeneous elements of various degrees.

- **Homogeneous polynomials**: f is homogeneous of degree d if $f(\lambda x_0, \cdots, \lambda x_n) = \lambda^d f(x_0, \cdots, x_n)$.

- **Homogeneous ideals**: $\mathfrak{a} \subseteq S$ is homogeneous when it's of the form $\mathfrak{a} = \bigoplus_{d \geq 0} (\mathfrak{a} \cap S_d)$.

  - $\mathfrak{a}$ is homogeneous iff generated by homogeneous elements.
  - The class of homogeneous ideals is closed under sums, products, intersections, and radicals.
  - Primality of homogeneous ideals can be tested on homogeneous elements, i.e. it STS $fg \in \mathfrak{a} \implies f, g \in \mathfrak{a}$ for f, g homogeneous.

- $k[x_1, \cdots, x_n] = \bigoplus_{d \geq 0} k[x_1, \cdots, x_n]_d$ where the degree d part is generated by monomials of total weight d.

  - E.g.

$$\begin{aligned} k[x_1, \cdots, x_n]_1 &= \langle x_1, x_2, \cdots, x_n \rangle \\ k[x_1, \cdots, x_n]_2 &= \left\langle x_1^2, x_1 x^2, x_1 x_3, \cdots, x_2^2, x_2 x_3, x_2 x_4, \cdots, x_n^2 \right\rangle. \end{aligned}$$

  - Useful fact: by stars and bars, $\operatorname{rank}_k k[x_1, \cdots, x_n]_d = \binom{d+n}{n}$. E.g. for $(d, n) = (3, 2)$,

$$\begin{aligned} x_1^3 &\longleftrightarrow \star\star\star \mid \mid \\ x_1^2 x_2 &\longleftrightarrow \star\star \mid \star \mid \\ x_1^2 x_3 &\longleftrightarrow \star\star \mid \mid \star \\ x_1 x_2^2 &\longleftrightarrow \star \mid \star\star \mid \\ x_1 x_2 x_3 &\longleftrightarrow \star \mid \star \mid \star \\ x_1 x_3^2 &\longleftrightarrow \star \mid \mid \star\star \\ x_2^3 &\longleftrightarrow \mid \star\star\star \mid \\ x_2^2 x_3 &\longleftrightarrow \mid \star\star \mid \star \\ x_2 x_3^2 &\longleftrightarrow \mid \star \mid \star\star \\ x_3^3 &\longleftrightarrow \mid \mid \star\star\star \end{aligned}$$

- Arbitrary polynomials $f \in k[x_0, \cdots, x_n]$ do not define functions on $\mathbf{P}^n$ because of non-uniqueness of coordinates due to scaling, but homogeneous polynomials f being zero or not is well-defined and there is a function

$$\begin{aligned} \mathrm{ev}_f : \mathbf{P}^n &\to \{0, 1\} \\ p &\mapsto \begin{cases} 0 & f(p) = 0 \\ 1 & f(p) \neq 0. \end{cases}. \end{aligned}$$

  So $Z(f) := \left\{ p \in \mathbf{P}^n \;\middle|\; f(p) = 0 \right\}$ makes sense.

- **Projective algebraic varieties**: Y is projective iff it is an irreducible algebraic set in $\mathbf{P}^n$. Open subsets of $\mathbf{P}^n$ are **quasi-projective varieties**.

- **Homogeneous ideals of varieties**:

$$I(Y) := \left\{ f \in k[x_0, \cdots, x_n]^{\mathrm{homog}} \;\middle|\; f(p) = 0 \, \forall p \in Y \right\}.$$

- **Homogeneous coordinate rings**:

$$S(Y) := k[x_0, \cdots, x_n]/I(Y).$$

- $Z(f)$ for f a linear homogeneous polynomial defines a **hyperplane**.

Exercise 1.2.2 (Cor. 2.3)
Show $\mathbf{P}^n$ admits an open covering by copies of $\mathbf{A}^n$ by explicitly constructing open sets U_i and well-defined homeomorphisms $\varphi_i : U_i \to \mathbf{A}^n$.

1.3 I.3: Morphisms

1.4 I.4: Rational Maps

1.5 I.5: Nonsingular Varieties

1.6 I.6: Nonsingular Curves

1.7 I.7: Intersections in Projective Space

2 | II: Schemes

Note: there are many, many important notions tucked away in the exercises in this section.

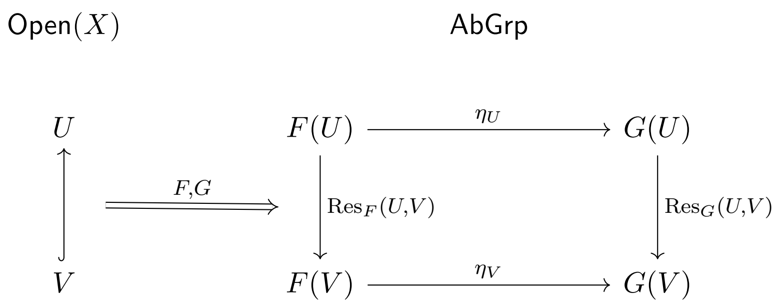
2.1 II.1: Sheaves ⋆

Remark 2.1.1:

- **Presheaves** F of abelian groups: contravariant functors $F \in \mathsf{Fun}(\mathsf{Open}(X), \mathsf{AbGrp})$.
  - Assigns every open $U \subseteq X$ some $F(U) \in \mathsf{AbGrp}$
  - For $\iota_{VU} : V \subseteq U$, restriction morphisms $\varphi_{UV} : F(U) \to F(V)$.
  - $F(\emptyset) = 0$, so $F(\emptyset^{\downarrow}) = 0_{\uparrow}$.
  - $\varphi_{UU} = \mathrm{id}_{F(U)}$
  - $W \subseteq V \subseteq U \implies \varphi_{UW} = \varphi_{VW} \circ \varphi_{UV}$.
- **Sections**: elements $s \in F(U)$ are sections of F over U. Also notation $\Gamma(U; F)$ and $H^0(U; F)$, and the restrictions are written $s|_V \coloneqq \varphi_{UV}(s)$ for $s \in F(U)$.
- **Sheaves**: presheaves F which are completely determined by local data. Additional requirements on open covers $\mathcal{V} \rightrightarrows U$:
  - If $s \in F(U)$ with $s|_{V_i} = 0$ for all i then $s \equiv 0 \in F(U)$.
  - Given $s_i \in F(V_i)$ where $s_i|_{V_{ij}} = s_j|_{V_{ij}} \in F(V_{ij})$ then $\exists s \in F(U)$ such that $s|_{V_i} = s_i$ for each i, which is unique by the previous condition.
- **Constant sheaf**: for $A \in \mathsf{AbGrp}$, define the constant sheaf

$$\underline{A}(U) \coloneqq \mathsf{Top}(U, A^{\mathrm{disc}}).$$

- **Stalks**: $F_p \coloneqq \varinjlim_{U \ni p} F(U)$ along the system of restriction maps.
  - These are represented by pairs (U, s) with $U \ni p$ an open neighborhood and $s \in F(U)$, modulo $(U, s) \sim (V, t)$ when $\exists W \subseteq U \cap V$ with $s|_w = t|_w$.
- **Germs**: a germ of a section of F at p is an elements of the stalk F_p.
- **Morphisms of presheaves**: natural transformations $\eta \in \mathrm{Mor}_{\mathsf{Fun}}(F, G)$, i.e. for every U, V, components η_U, η_V fitting into a diagram



Link to Diagram

- A morphism of sheaves is exactly a morphism of the underlying presheaves.
- Morphisms of sheaves $\eta : F \to G$ induce morphisms of rings on the stalks $\eta_p : F_p \to G_p$.
- Morphisms of sheaves are isomorphisms iff isomorphisms on all stalks, see exercise below.
- **Kernels, cokernels, images**: for $\varphi : F \to G$, sheafify the assignments to kernels/cokernels/images on open sets.
- **Sheafification**: for any $F \in \underset{\text{pre}}{\mathsf{Sh}}(X)$, there is a unique $F^+ \in \mathsf{Sh}(X)$ and a morphism $\theta : F \to F^+$ of presheaves such that any sheaf presheaf morphism $F \to G$ factors as $F \to F^+ \to G$.
  - The construction: $F^+(U) = \mathsf{Top}(U, \coprod_{p \in U} F_p)$ are all functions s into the union of stalks, subject to $s(p) \in F_p$ for all $p \in U$ and for each $p \in U$, there is a neighborhood $V \supseteq U \ni p$ and $t \in F(V)$ such that for all $q \in V$, the germ t_q is equal to $s(q)$.
  - Note that the stalks are the same: $(F^+)_p = F_p$, and if F is already a sheaf then θ is an isomorphism.
- **Subsheaves**: $F' \leq F$ iff $F'(U) \leq F(U)$ is a subgroup for every U and the restrictions on F' are induced by restrictions from F.
  - If $F' \leq F$ then $F'_p \leq F_p$.
  - **Injectivity**: $\varphi : F \to G$ is injective iff the sheaf kernel $\ker \varphi = 0$ as a subsheaf of F.
    - φ is injective iff injective on all sections.
  - $\operatorname{im} \varphi \leq G$ is a subsheaf.
  - **Surjectivity**: $\varphi : F \to G$ is surjective iff $\operatorname{im} \varphi = G$ as a subsheaf.
- **Exactness**: a sequence of sheaves $(F_i, \varphi_i : F_i \to F_{i+1})$ is exact iff $\ker \varphi_i = \operatorname{im} \varphi^{i-1}$ as subsheaves of F_i.
  - $\varphi : F \to G$ is injective iff $0 \to F \xrightarrow{\varphi} G$ is exact.
  - $\varphi : F \to G$ is surjective iff $F \xrightarrow{\varphi} G \to 0$ is exact.
  - Sequences of sheaves are exact iff exact on stalks.
- **Quotient sheaves**: F/F' is the sheafification of $U \mapsto F(U)/F'(U)$.
- **Cokernels**: for $\varphi : F \to G$, $\operatorname{coker} \varphi$ is sheafification of $U \mapsto \operatorname{coker}(F(U) \xrightarrow{\varphi(U)} G(U))$.
- **Direct images**: for $f \in \mathsf{Top}(X, Y)$, the sheaf defined on sections by $(f_* F)(V) \coloneqq F(f^{-1}(V))$ for any $V \subseteq Y$. Yields a functor $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$.
- **Inverse images**: denoted $f^{-1}G$, the sheafification of $U \mapsto \varinjlim_{V \supseteq f(U)} G(V)$, i.e. take the limit from above of all open sets V of Y containing the image $f(U)$. Yields a functor $f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$.

- **Restriction of a sheaf**: for $F \in \mathsf{Sh}(X)$ and $Z \subseteq X$ with $\iota : Z \hookrightarrow X$ the inclusion, define $i^{-1}F \in \mathsf{Sh}(Z)$ to be the restriction. Also denoted $F|_Z$. This has the same stalks: $(F|_Z)_p = F_p$.

- For any $U \subseteq X$, the global sections functor $\Gamma(U; -) : \mathsf{Sh}(X) \to \mathsf{AbGrp}$ is left-exact (proved in exercises).

- **Limits of sheaves**: for $\{F_i\}$ a direct system of sheaves, $\underrightarrow{\operatorname{colim}}_i F_i$ has underlying presheaf $U \mapsto \underrightarrow{\operatorname{colim}}_i F_i(U)$. If X is Noetherian, then this is already a sheaf, and commutes with sections: $\Gamma(X; \underrightarrow{\operatorname{colim}}_i F_i) = \underrightarrow{\operatorname{colim}}_i \Gamma(X; F_i)$.

  - Inverse limits exist and are defined similarly.

- **The espace étalé**: define $\text{Ét}(F) = \coprod_{p \in X} F_p$ and a projection $\pi : \text{Ét}(F) \to X$ by sending $s \in F_p$ to p. For each $U \subseteq X$ and $s \in F(U)$, there is a local section $\overline{s} : U \to \text{Ét}(F)$ where $p \mapsto s_p$, its germ at p; this satisfies $\pi \circ \overline{s} = \mathrm{id}_U$. Give $\text{Ét}(F)$ the strongest topology such that the $\overline{s}$ are all continuous. Then $F^+(U) \coloneqq \mathsf{Top}(U, \text{Ét}(F))$ is the set of continuous sections of $\text{Ét}(F)$ over U.

- **Support**: for $s \in F(U)$, $\operatorname{supp}(s) \coloneqq \left\{ p \in U \;\middle|\; s_p \neq 0 \right\}$ where s_p is the germ of s in F_p. This is closed.

  - This extends to $\operatorname{supp}(F) \coloneqq \left\{ p \in X \;\middle|\; F_p \neq 0 \right\}$, which need not be closed.

- **Sheaf hom**: $U \mapsto \operatorname{Hom}(F|_U, G|_U)$ forms a sheaf of local morphisms and is denoted $\mathcal{H}om(F, G)$.

- **Flasque sheaves**: a sheaf is flasque iff $V \hookrightarrow U \implies F(U) \twoheadrightarrow F(V)$.

- **Skyscraper sheaves**: for $A \in \mathsf{AbGrp}$ and $p \in X$, define

$$i_p(A)(U) = \begin{cases} A & p \in U \\ 0 & \text{otherwise.} \end{cases}.$$

  Also denoted $\iota_*(A)$ where $\iota : \operatorname{cl}_X(\{p\}) \hookrightarrow X$ is the inclusion.

  - The stalks are

$$(i_p(A))_q = \begin{cases} A & q \in \operatorname{cl}_X(\{p\}) \\ 0 & \text{otherwise.} \end{cases}.$$

- **Extension by zero**: if $\iota : Z \hookrightarrow X$ is the inclusion of a closed set and $U \coloneqq X \setminus Z$ with $j : U \to X$, then for $F \in \mathsf{Sh}(Z)$, the sheaf $\iota_* F \in \mathsf{Sh}(X)$ is the extension of F by zero outside of Z. The stalks are

$$(\iota_* F)_p = \begin{cases} F_p & p \in Z \\ 0 & \text{otherwise.} \end{cases}.$$

  - For the open U, extension by zero is $j_! F$ which has presheaf $V \mapsto F(V)$ if $V \subseteq U$ and 0 otherwise. The stalks are

$$(j_! F)_p = \begin{cases} F_p & p \in U \\ 0 & \text{otherwise.} \end{cases}.$$

- **Sheaf of ideals**: for $Y \subseteq X$ closed and $U \subseteq X$ open, $\mathcal{I}_Y(U)$ has presheaf $U \mapsto$ the ideal in $\mathcal{O}_X(U)$ of regular functions vanishing on all of $Y \cap U$. This is a subsheaf of $\mathcal{O}_X$.

- **Gluing sheaves**: given $\mathcal{U} \rightrightarrows X$ and sheaves $F_i \in \mathsf{Sh}(U_i)$, one can glue to a unique $F \in \mathsf{Sh}(X)$ if one is given morphisms $\varphi_{ij}\, F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ij}}$ where $\varphi_{ii} = \mathrm{id}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on U_{ijk}.

Warning 2.1.2
Some common mistakes:

- Kernel presheaves are already sheaves, but not cokernels or images. See exercise below.
- $\varphi : F \to G$ is injective iff injective on sections, but this is not true for surjectivity.
- The sheaves $f^{-1}G$ and $f^* G$ are different! See III.5 for the latter.
- Global sections need not be right-exact.

Exercise 2.1.3 (Regular functions on varieties form a sheaf)
For $X \in \mathsf{Var}_{/k}$, define the ring $\mathcal{O}_X(U)$ of literal regular functions $f_i : U \to k$ where restriction morphisms are induced by literal restrictions of functions. Show that $\mathcal{O}_X$ is a sheaf of rings on X.

Hint: Locally regular implies regular, and regular + locally zero implies zero.

Exercise 2.1.4 (?)
Show that for every connected open subset $U \subseteq X$, the constant sheaf satisfies $\underline{A}(U) = A$, and if U is open with open connected component so the $\underline{A}(U) = A^{\times \sharp \pi_0 U}$.

Exercise 2.1.5 (?)
Show that if $X \in \mathsf{Var}_{/k}$ and $\mathcal{O}_X$ is its sheaf of regular functions, then the stalk $\mathcal{O}_{X,p}$ is the *local ring of* p on X as defined in Ch. I.

Exercise 2.1.6 (Prop 1.1)
Let $\varphi : F \to G$ be a morphism in $\mathsf{Sh}(X)$ and show that φ is an isomorphism iff φ_p is an isomorphism on stalks for all $p \in X$. Show that this is false for presheaves.

Exercise 2.1.7 (?)
Show that for $\varphi \in \mathrm{Mor}_{\mathsf{Sh}(X)}(F, G)$, $\ker \varphi$ is a sheaf, but $\operatorname{coker} \varphi, \operatorname{im} \varphi$ are not in general.

Exercise 2.1.8 (?)
Show that if $\varphi : F \to G$ is surjective then the maps on sections $\varphi(U) : F(U) \to G(U)$ need not all be surjective.

2.2 II.2: Schemes

2.3 II.3: First Properties of Schemes

2.4 II.4: Separated and Proper Morphisms

2.5 II.5: Sheaves of Modules

2.6 II.6: Divisors

2.7 II.7: Projective Morphisms

2.8 II.8: Differentials

2.9 II.9: Formal Schemes

3 III: Cohomology

3.1 III.1: Derived Functors

3.2 III.2: Cohomology of Sheaves

3.3 III.3: Cohomology of a Noetherian Affine Scheme

3.4 III.4: Čech Cohomology

3.5 III.5: The Cohomology of Projective Space

3.6 III.6: Ext Groups and Sheaves

3.7 III.7: Serre Duality

3.8 III.8: Higher Direct Images of Sheaves

3.9 III.9: Flat Morphisms

3.10 III.10: Smooth Morphisms

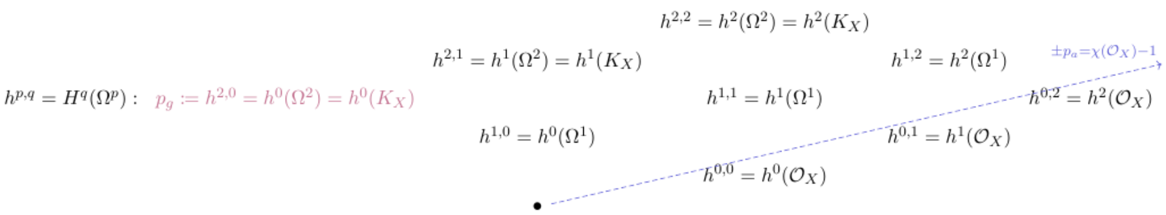
3.11 III.11: The Theorem on Formal Functions

3.12 III.12: The Semicontinuity Theorem

4 | IV: Curves ⋆

Remark 4.0.1: Summary of major results:

- $p_a(X) \coloneqq 1 - P_X(0) = (-1)^r(1 - \chi(\mathcal{O}_X))$.
  - Note: $P_X(\ell)$ is defined as the Hilbert polynomial of the homogeneous coordinate ring $S(Y)$, and then defined for graded S-modules M by setting $\varphi_M(\ell) = \dim_k M_\ell$ and showing $\exists! P_M(z) \in \mathbf{Q}[z]$ with $\varphi_M(\ell) = P_M(\ell)$ for $\ell \gg 0$.
- $p_g(X) \coloneqq h^0(\omega_X) = h^0(\mathcal{L}(K_X))$.
- Remembering these:



Link to Diagram

- For curves, $p_a(X) = p_g(X) = h^1(\mathcal{O}_X)$ by setting $D \coloneqq K_C$ in RR.
  - $\deg K_C = 2g - 2$.
- $D_1 \sim D_2 \iff D_1 - D_2 = (f)$ for $f \in K(X)$ rational, $|D| = \{D' \sim D\}$, and this bijects with points of $\frac{H^0(\mathcal{L}(D)) \setminus \{0\}}{\mathbf{G}_m}$.
  - Thus $\dim |D| = h^0(\mathcal{L}(D)) - 1 \coloneqq \ell(D) - 1$.
- X smooth $\implies \mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Pic}(X)$ via $D \mapsto \mathcal{L}(D)$.
- $h^0(\mathcal{L}(D)) > 0 \implies \deg(D) \geq 0$, and if $\deg D = 0$ then $D \sim 0$ and $\mathcal{L}(D) \cong \mathcal{O}_X$.
- RR:

$$\begin{aligned}\chi(\mathcal{L}(D) &= h^0(\mathcal{L}(D)) - h^1(\mathcal{L}(D)) \\ &= h^0(\mathcal{L}(D)) - h^0(\mathcal{L}(K - D)) \\ &= \deg(D) + (1 - g).\end{aligned}$$

  - How to remember: note $g = h^1(\mathcal{O}_X) = h^1(\mathcal{L}(0))$, and $H^0(\mathcal{O}_X) = k$ so $h^0(\mathcal{O}_X) = 1$, thus

$$\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) = 1 - g = \deg \mathcal{L}(0) + 1 - g.$$

  - For $C \subseteq \mathbf{P}^n, \deg(C) = d$ and $D = C \cap H$ a hyperplane section defining $\mathcal{L}(D) = \mathcal{O}_X(1)$,

$$\chi(\mathcal{L}(D)) = \deg(D) + (1 - g) = d + (1 - p_a(C))$$

- A curve is rational iff isomorphic to $\mathbf{P}^1$ iff $g = 0$.
- $K \sim 0$ on an elliptic curve since $\deg K = 2g - 2 = 0$ and $\deg D = 0 \implies D \sim 0$.
- For X elliptic, $\operatorname{Pic}^0(X) := \left\{ D \in \operatorname{Div}(X) \;\middle|\; \deg D = 0 \right\}$ and $|X| \xrightarrow{\sim} \left|\operatorname{Pic}^0(X)\right|$ via $p \mapsto \mathcal{L}(p - p_0)$ for any fixed $p_0 \in X$, inducing its group structure. (This is proved with RR.)

Remark 4.0.2: Comments from preface:

- The statement of Riemann-Roch is important; less so its proof.
- Representing curves:
  - A branched covering of $\mathbf{P}^1$,
  - More generally a branched covering of another curve,
  - Nonsingular projective curves: admit embeddings into $\mathbf{P}^3$, maps to $\mathbf{P}^2$ birationally such that the image is at worst a nodal curve.
- The central result regarding representing curves: Hurwitz's theorem which compares K_X, K_Y for a cover $Y \to X$ of curves.
- Curves of genus 1: elliptic curves.
- Later sections: the canonical embedding of a curve.

4.1 IV.1: Riemann-Roch

Definition 4.1.1 (Curves)
A **curve** over $k = \overline{k}$ is a scheme over $\operatorname{Spec} k$ which is

- Integral
- Dimension 1
- Proper over k
- With regular local rings

In particular, a curve is smooth, complete, and necessary projective. A **point** on a curve is a closed point.

Definition 4.1.2 (Arithmetic genus)
The **arithmetic genus** of a projective curve X is

$$p_a(X) := 1 - P_X(0)$$

where $P_X(t)$ is the **Hilbert polynomial** of X.

Definition 4.1.3 (Geometric genus)
The **geometric genus** of a curve is

$$p + g(X) := \dim_k H^0(X; \omega_X)$$

where ω_X is the canonical sheaf.

Exercise 4.1.4 (?)
Show that if X is a curve, there is a single well-defined **genus**

$$g \coloneqq p_A(X) = p_G(X) = \dim_k H^1(X; \mathcal{O}_X).$$

Hint: see Ch. III Ex. 5.3, and use Serre duality for p_g.

Exercise 4.1.5 (?)
Show that for any $g \geq 0$ there exists a curve of genus g.

Hint: take a divisor of type $(g+1, 2)$ on a smooth quadric which is irreducible and smooth with $p_a = g$.

Definition 4.1.6 (Divisors on a curve)
Reviewing divisors:

- The **divisor group**: $\operatorname{Div}(X) = \mathbf{Z}[X_{\mathrm{cl}}]$
- **Degrees**: $\deg(\sum n_i D_i) \coloneqq \sum n_i$, and
- **Linear equivalence**: $D_1 \sim D_2 \iff D_1 - D_1 = \operatorname{Div}(f)$ for some $f \in k(X)$ a rational function.
- D is **effective** if $n_i \geq 0$ for all i.
- $|D| \coloneqq \left\{ D' \in \operatorname{Div}(X) \;\middle|\; D' \sim D \right\}$ is the **complete linear system** of D.
- $|D| \cong \mathbf{P} H^0(X; \mathcal{L}(D))$
- **Dimensions of linear systems**: $\ell(D) \coloneqq \dim_k H^0(X; \mathcal{L}(D))$ and $\dim |D| \coloneqq \ell(D) - 1$.
- **Relative differentials**: $\Omega_X \coloneqq \Omega_{X_{/k}}$ is the sheaf of relative differentials on X.
  - The technical definition: $\Omega_{X_{/S}} \coloneqq \Delta^*_{X_{/Y}}(\mathcal{I}/\mathcal{I}^2)$ where $\mathcal{I}$ is the sheaf of ideals defining the locally closed subscheme $\operatorname{im}(\Delta_{X_{/Y}}) \subseteq X \mathrm{fp} Y X$.
  - On affine schemes: on the ring side, $\Omega_{B_{/A}} \in {}_B\mathsf{Mod}$ equipped with a differential $d : B \to \Omega B_{/A}$, defined as $\left\langle db \;\middle|\; b \in B \right\rangle_B / \langle d(b_1 + b_2) = db_1 + db_2, d(b_1 b_2) = d(b_1) b_2 + b_1 d(b_2), da = 0 \,\forall a \in A \rangle_B$.
  - On curves, $\Omega_{X_{/Y}}$ measures the “difference” between K_X and K_Y.
- **Canonical sheaf**: $\dim X = 1, \Omega_{X_{/k}} \cong \omega_X$.
- **Canonical divisor**: K_X 2is any divisor in the linear equivalence class corresponding to ω_X
- D is **special** iff its **index of speciality** $\ell(K - D) > 0$, otherwise D is **nonspecial**.

Exercise 4.1.7 (?)
Show that $D_1 \sim D_2 \implies \deg(D_1) = \deg(D_2)$.

Exercise 4.1.8 (?)
Show that

$$|D| \rightleftharpoons \mathbf{P}H^0(X; \mathcal{L}(D)),$$

so $|D|$ has the structure of the closed points of some projective space.

Exercise 4.1.9 (Lemma 1.2)
Show that if $D \in \operatorname{Div}(X)$ for X a curve and $\ell(D) \neq 0$, then $\deg(D) \geq 0$.
Show that is $\ell(D) \neq 0$ and $\deg D = 0$ then $D \sim 0$ and $\mathcal{L}(D) \cong \mathcal{O}_X$.

Theorem 4.1.10 ***(Riemann-Roch).***

$$\ell(D) - \ell(K - D) = \deg(D) + (1 - g).$$

Exercise 4.1.11 (Ingredients for proof of RR)
Show the following:

- The divisor $K - D$ corresponds to $\omega_X \otimes \mathcal{L}(D)^\vee \in \operatorname{Pic}(X)$.
- $H^1(X; \mathcal{L}(D))^\vee \cong H^0(X; \omega_X \otimes \mathcal{L}(D)^\vee)$.
- If X is any projective variety,

$$H^0(X; \mathcal{O}_X) = k.$$

Exercise 4.1.12 (?)
Show that if $X \subseteq \mathbf{P}^n$ is a curve with $\deg X = d$ and $D = X \cap H$ is a hyperplane section, then $\mathcal{L}(D) = \mathcal{O}_X(1)$ and $\chi(\mathcal{L}(D)) = d + 1 - p_a$.

Exercise 4.1.13 (?)
Show that if $g(X) = g$ then $\deg K_X = 2g - 2$.

Hint: set $D = K$ and use $\ell(K) = p_g = g$ and $\ell(0) = 1$.

Remark 4.1.14: More definitions:

- X is **rational** iff birational to $\mathbf{P}^1$.
- X is **elliptic** if $g = 1$.

Exercise 4.1.15 (?)
Show that

1. If $\deg D > 2g-2$ then D is nonspecial.
2. $p_a(\mathbf{P}^1) = 0$.
3. A complete nonsingular curve is rational iff $X \cong \mathbf{P}^1$ iff $g(X) = 0$.
4. If X is elliptic then $K \sim 0$

Hint: for (3) apply RR to $D = p - q$ for points $p \neq q$, and use $\deg(K - D) = -2$ and $\deg(D) = 0 \implies D \sim 0 \implies p \sim q$. For (4), show $\ell(K) = p_g = 1$.

Exercise 4.1.16 (?)
If X is elliptic and $p \in X$, then there is a bijection

$$\begin{aligned} m_p : X &\xrightarrow{\sim} \operatorname{Pic}(X) \\ x &\mapsto \mathcal{L}(x-p), \end{aligned}$$

so $\operatorname{Pic}(X) \in \mathsf{Grp}$.

Hint: show that if $\deg(D) = 0$ then there is some $x \in X$ such that $D \sim x - p$ and apply RR to $D + p$.

4.2 IV.2: Hurwitz ⋆

Remark 4.2.1: Summary of results:

- For curves, complete = projective.
- Riemann-Hurwitz: for $f : X \to Y$ finite separable,

$$K_X \sim f^* K_Y + R \implies \deg(K_X) = \deg(f^* K_Y) + \deg(R) \implies$$

$$\chi(X) = \deg(f) \cdot \chi(Y) + \deg R, \qquad \deg R = \sum_{p \in X} (e_p - 1).$$

- $\deg f := [K(X) : K(Y)]$ for finite morphisms of curves.
- $e_p := v_p(f_*^\sharp t)$ where t is uniformizer in $\mathcal{O}_{f(p)}$ and $f^\sharp : \mathcal{O}_{Y, f(p)} \to \mathcal{O}_{X,p}$ for $f : X \to Y$.
  - $e_p > 1 \implies$ ramification.
  - Unramified everywhere implies etale (since automatically flat).
  - $p \mid e_{x_0} \implies$ wild ramification, otherwise tame.
- $\exists f^* : \operatorname{Div}(Y) \to \operatorname{Div}(X)$ where $q \mapsto \sum_{p \mapsto q} e_p p$.

- Pullback commutes with forming line bundles:

$$f^*\mathcal{L}(D) \cong \mathcal{L}(f^*D)$$

  where the LHS $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$.

- The fundamental SES for relative differentials: if $f : X \to Y$ is finite separable,

$$f^*\Omega_Y \hookrightarrow \Omega_X \twoheadrightarrow \Omega_{X/Y}.$$

- $\frac{\partial t}{\partial u}$ for t a uniformizer at $f(p)$ and u a uniformizer at p is defined by noting $\Omega Y, f(p) = \langle\, dt\rangle\,, \Omega_{X,p} = \langle\, du\rangle$, and there is some $g \in \mathcal{O}_{X,p}$ such that $f^*\,dt = g\,du$; set $g \coloneqq \frac{\partial t}{\partial u}$.

- For finite separable morphisms of curves $f : X \to Y$,

  - $\operatorname{supp}\Omega_{X/Y} = \mathrm{Ram}(f)$ is the ramification locus, and $\Omega_{X/Y}$ is torsion so $\mathrm{Ram}(f)$ is finite.
  - $\mathrm{length}(\Omega_{X,Y})_p = v_p\left(\frac{\partial t}{\partial u}\right)$ for any $p \in X$
  - Tamely ramified $\implies$ $\mathrm{length}(\Omega_{X/Y})_p = e_p - 1$, and wild ramification increases this length. Recall that length is the largest size of chains of submodules.

- The ramification divisor:

$$R \coloneqq \sum_{p \in X} \mathrm{length}(\Omega_{X/Y})_p p.$$

- $K_X \sim f^*K_Y + R$

- $\mathbf{P}^1$ can't admit an unramified cover: for $n \geq 1$,

$$\chi(X) = n\chi(\mathbf{P}^1) + \deg R \implies \chi(X) = -2n + \deg R \implies \chi(X) = -2n \leq -2,$$

  which forces $g(X) = 0, n = 1, X = \mathbf{P}^1, f = \mathrm{id}$.

- The Frobenius morphism on schemes is defined by taking $f^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ to be the pth power map; pullback yields a definition of X_p, the Frobenius twist of X.

  - $F : X_p \to X$ is finite, $\deg F = p$, and corresponds to $K(X) \hookrightarrow K(X)^{\frac{1}{p}}$

- If $f : X \to Y$ induces a purely inseparable extension $K(X)/K(Y)$, then $X \xrightarrow{\sim} Y$ as schemes, $g(X) = g(Y)$, and f is a composition of Frobenii.

- Everywhere ramified extensions: $f : Y_p \to Y$, where $e_q = p$ for every $q \in X$. Induces $\Omega_{X/Y} \cong \Omega_X$.

- $\deg R$ is always even.

- Finite implies proper: finite implies separated, of finite type, closed by "going up" and universally closed by since finiteness is preserved under base change.

- $\mathbf{P}^1$ no nontrivial etale covers.

- If $f : X \to Y$ then $g(X) \geq g(Y)$.

  - Thus $\exists \mathbf{P}^1 \to Y$ finite $\implies g(Y) = 0$.

Remark 4.2.2: Preface:

- **Degree**: for a finite morphism of curves $X \xrightarrow{f} Y$, set $\det(f) := [k(X) : k(Y)]$, the degree of the extension of function fields.
- **Ramification indices** e_p: for $p \in X$, let $q = f(p)$ and $t \in \mathcal{O}_q$ a local coordinate. Pull back to $t \in \mathcal{O}_p$ via $f^\sharp$ and define $e_p := v_p(t)$ using the valuation v_p for the DVR $\mathcal{O}_p$.
- **Ramified**: $e_p > 1$, and **unramified** if $e_p = 1$.
- **Branch points** any $q = f(p)$ where f is ramified.
- **Tame ramification**: for $\operatorname{ch}(k) = p$, tame if $p \nmid e_P$.
- **Wild ramification**: when $p \mid e_P$.
- Pullback maps on divisor groups:

$$
\begin{aligned}
f^* : \operatorname{Div}(Y) &\to \operatorname{Div}(X) \\
Q &\mapsto \sum_{P \xrightarrow{f} q} e_P [P].
\end{aligned}
$$

  - This commutes with taking line bundles (exercise), so induces a well-defined map $f^* : \operatorname{Pic}(X) \to \operatorname{Pic}(Y)$.

- f is **separable** if $k(X)/k(Y)$ is a separable field extension.

Exercise 4.2.3 (?)
Misc:

- Show that if f is everywhere unramified then it is an étale morphism.
- Show that $f^* \mathcal{L}(D) = \mathcal{L}(f^* D)$

Exercise 4.2.4 (Prop 2.1)
Show that if $X \xrightarrow{f} Y$ is a finite separable morphism of curves, there is a SES

$$
f^* \Omega_Y \hookrightarrow \Omega_X \twoheadrightarrow \Omega_{X/Y}.
$$

Remark 4.2.5: Definitions:

- **Derivatives**: for $f : X \to Y$, let t be a parameter at $Q = f(P)$ and u at P. Then $\Omega_{Y,Q} = \langle dt \rangle_{\mathcal{O}_Q}$ and $\mathcal{O}_{X,P} = \langle du \rangle_{\mathcal{O}_P}$ and $\exists! g \in \mathcal{O}_P$ such that $f^* dt = du$ so we write $\frac{\partial t}{\partial u} := g$.

- **Ramification divisor**: $R \coloneqq \sum_{P \in X} \operatorname{length}(\Omega_{X/Y})_P [P] \in \operatorname{Div}(X)$

Exercise 4.2.6 (Prop 2.2)
For $X \xrightarrow{f} Y$ a finite separable morphism of curves,

a. $\Omega_{X/Y}$ is a torsion sheaf on X with support equal to the ramification locus of f. Thus f is ramified at finitely many points.
b. The stalks $(\Omega_{X/Y})_P$ are principal $\mathcal{O}_P$-modules of finite length equal to $v_p\left(\frac{\partial t}{\partial u}\right)$
c.

$$\operatorname{length}(\Omega_{X/Y})_P \begin{cases} = e_p - 1 & f \text{ is tamely ramified at } P \\ > e_p - 1 & f \text{ is wildly ramified at } P. \end{cases}.$$

Exercise 4.2.7 (Prop 2.3)
If $X \xrightarrow{f} Y$ is a finite separable morphism of curves, then

$$K_X \sim f^* K_Y + R,$$

where R is the ramification divisor of f.

Theorem 4.2.8 ***(Hurwitz).***
If $X \xrightarrow{f} Y$ is a finite separable morphism of curves, then

$$2g(X) - 2 = \deg(f)(2g(Y) - 2) + \deg(R),$$

and if f has only tame ramification then $\deg(R) = \sum_{P \in X}(e_P - 1)$.

Remark 4.2.9 ***(proof of Hurwitz):*** Take degrees of the divisor equation:

$$\begin{aligned}
\deg(K_X) &= \deg(f^* K_Y + R) \\
\implies \chi_{\mathsf{Top}}(X) &= \deg(f^* K_Y) + \deg(R) \\
\implies 2g(X) - 2 &= \deg(f)\deg(K_Y) + \deg(R) \\
\implies 2g(X) - 2 &= \deg(f)\chi_{\mathsf{Top}}(Y) + \deg(R) \\
\implies 2g(X) - 2 &= \deg(f)(2g(Y) - 2) + \deg(R) \\
\implies 2g(X) - 2 &= \deg(f)(2g(Y) - 2) + \sum_{P \in X}(e_P - 1)
\end{aligned}$$

,

using tame ramification in the last step which implies $\operatorname{length}(\Omega_{X/Y})_P = (e_p - 1)$.

Remark 4.2.10: Consider the purely inseparable case.

- **Frobenius morphism**: for $X \in \mathsf{Sch}$ where $\mathcal{O}_P \supseteq \mathbf{Z}/p\mathbf{Z}$ for all P, define $\operatorname{Frob} : X \to X$ by $F(|X|) = |X|$ on spaces and $F^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ is $f \mapsto f^p$. This is a morphism since $F^\sharp$ induces a morphism on all local rings, which are all characteristic p.

- **The k-linear Frobenius morphism**: define X_p to be X with the structure morphism $F \circ \pi$, so $k \curvearrowright \mathcal{O}_{X_p}$ by pth powers and F becomes a k-linear morphism $F' : X_p \to X$.
  - Why this is necessary: F as before is not a morphism in $\mathsf{Sch}_{/k}$, and instead forms a commuting square involving $F : \operatorname{Spec} k \to \operatorname{Spec} k$ and the structure maps $X \xrightarrow{\pi} \operatorname{Spec} k$.

Exercise 4.2.11 (?)
Find examples where

- $X_p \cong X \in \mathsf{Sch}_{/k}$, and
- $X_p \not\cong X \in \mathsf{Sch}_{/k}$.

Hint: consider $X = \operatorname{Spec} k[t]$ for k perfect.

Exercise 4.2.12 (?)
Show that if $X \xrightarrow{f} Y$ is separable then $\deg(R)$ is always even.

Skipped some stuff around Example 2.4.2, I don't necessarily need characteristic p things right now.

Remark 4.2.13: Definitions:

- **Étale covers**: $X \xrightarrow{f} Y$ is an étale cover if f is a finite étale morphism,, i.e. f is flat and $\Omega^1_{X/Y} = 0$.
- Y is a **trivial** cover if $X \cong \coprod_{i \in I} Y$ a finite disjoint union of copies of Y,
- Y is **simply connected** if there are no nontrivial étale covers.

Exercise 4.2.14 (?)

- Show that a connected regular curve is irreducible.
- Show that if f is etale then X is smooth over k.
- Show that if f is finite, X must be a curve.
- Show that if f is étale, then f must be separable.
- Show that $\pi_1^{\text{ét}}(\mathbf{P}^1) = 0$.

Hint: use Hurwitz and that when f is unramified, $R = 0$.

Exercise 4.2.15 (?)

- Show that the genus of a curve doesn't change under purely inseparable extensions.
- Show that if $f : X \to Y$ is a finite morphism of curves then $g(X) \geq g(Y)$.

Exercise 4.2.16 (Lüroth)
Show that if L is a subfield of a purely transcendental extension $k(t)/k$ where $k = \overline{k}$, then L is also purely transcendental.[a]

Hint: Assume $[L : k]_{\mathrm{tr}} = 1$, so $L = k(X)$ for Y a curve and $L \subseteq k(t)$ corresponds to a finite morphism $f : \mathbf{P}^1 \to Y$. Conclude $g(Y) = 0$ so $Y \cong \mathbf{P}^1$ and $L \cong k(u)$ for some u.

[a] This is true over any field k in dimension 1, over $k = \overline{k}$ in dimension 2, and false in dimension 3 by the existence of nonrational unirational threefolds.

4.3 IV.3: Embeddings in Projective Space ⋆

Remark 4.3.1: A summary of major results:

- For $D \in \mathrm{Div}(C)$ with $g = g(C)$,
  - D is ample iff $\deg D > 0$.
  - D is BPF iff $\deg D \geq 2g$.
  - D is very ample iff $\deg D \geq 2g + 1$.
- Being very ample is equivalent to being a hyperplane section under a projective embedding.
- Divisors $D \in \mathrm{Div}(\mathbf{P}^n)$ are ample iff very ample iff $\deg D \geq 1$.
  - E.g. if E is elliptic then D is very ample if $\deg D \geq 3$, and for hyperelliptic, very ample if $\deg D \geq 5$.
- If D is very ample then $\deg \varphi(X) = \deg D$.
- Curves $C \subseteq \mathbf{P}^n$ for $n \geq 4$ can be projected away from a point $p \notin X$ to get a closed immersion into $\mathbf{P}^m$ for some $m \leq n - 1$. So any curve is birational to a nodal curve in $\mathbf{P}^2$.
- Genus of normalizations of nodal curves: $g = \frac{1}{2}(d-1)(d-2) - \sharp\{\text{nodes}\}$.
- Any curve embeds into $\mathbf{P}^3$, and maps into $\mathbf{P}^2$ with at worst nodal singularities.

Remark 4.3.2: Main result: any curve can be embedded in $\mathbf{P}^3$, and is birational to a nodal curve in $\mathbf{P}^2$. Some recollections:

- **Very ample line bundles**: $\mathcal{L} \in \mathrm{Pic}(X)$ is very ample if $\mathcal{L} \cong \mathcal{O}_X(1)$ for some immersion of $f : X \hookrightarrow \mathbf{P}^N$.
- **Ample**: $\mathcal{L}$ is ample when $\forall \mathcal{F} \in \mathsf{Coh}(X)$, the twist $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated for $n \gg 0$.
- **(Very) ample divisors**: $D \in \mathrm{Div}(X)$ is (very) ample iff $\mathcal{L}(D) \in \mathrm{Pic}(X)$ is (very) ample.
- **Linear systems**: a linear system is any set $S \leq |D|$ of effective divisors yielding a linear subspace.
- **Base points**: P is a base point of S iff $P \in \operatorname{supp} D$ for all $D \in S$.
- **Secant lines**: the secant line of $P, Q \in X$ is the line in $\mathbf{P}^N$ joining them.
- **Tangent lines**: at $P \in X$, the unique line $L \subseteq \mathbf{P}^N$ passing through p such that $\mathbf{T}_P(L) = \mathbf{T}_P(X) \subseteq \mathbf{T}_P(\mathbf{P}^N)$.

- **Nodes**: a singularity of multiplicity 2.
  - $y^2 = x^3 + x^2$ is a **node**.
  - $y^2 = x^3$ is a **cusp**.
  - $y^2 = x^4$ is a **tacnode**.
- **Multisecant**: for $X \subseteq \mathbf{P}^3$, a line meeting X in 3 or more distinct points.
- A **secant with coplanar tangent lines** is a secant through P, Q whose tangent lines L_P, L_Q lie in a common plane, or equivalently L_P intersects L_Q.

Exercise 4.3.3 (II.8.20.2)
Show that by Bertini's theorem there are irreducible smooth curves of degree d in $\mathbf{P}^2$ for any d.

Exercise 4.3.4 (?)
Show that

- $\mathcal{L}$ is ample iff $\mathcal{L}^n$ is very ample for $b \gg 0$.
- $|D|$ is basepoint free iff $\mathcal{L}(D)$ is globally generated.
- If D is very ample, then $|D|$ is basepoint free.
- If D is ample, $nD \sim H$ a hyperplane section for a projective embedding for some n.
- If $g(X) = 0$ then D is ample iff very ample iff $\deg D > 0$.
- If D is very ample and corresponds to a closed immersion $\varphi : X \hookrightarrow \mathbf{P}^n$ then $\deg \varphi(X) = \deg D$.
- If XS is elliptic, any D with $\deg D = 3$ is very ample and $\dim |D| = 2$, and so can be embedded into $\mathbf{P}^2$ as a cubic curve.
- Show that if $g(X) = 1$ then D is very ample iff $\deg D \geq 3$.
- Show that if $g(X) = 2$ and $\deg D = 5$ then D is very ample, so any genus 2 curve embeds in $\mathbf{P}^3$ as a curve of degree 5.

Exercise 4.3.5 (Prop 3.1: when a linear system yields a closed immersion into $\mathbf{P}^N$)
Let $D \in \operatorname{Div}(X)$ for X a curve and show

- $|D|$ is basepoint free iff $\dim |D - P| = \dim |D| - 1$ for all points $p \in X$.
- D is very ample iff $\dim |D - P - Q| = \dim |D| - 2$ for all points $P, Q \in X$.

Hint: use the SES $\mathcal{L}(D - P) \hookrightarrow \mathcal{L}(D) \twoheadrightarrow k(P)$ where $k(P)$ is the skyscraper sheaf at P.

Exercise 4.3.6 (Cor 3.2)
Let $D \in \operatorname{Div}(X)$.

- If $\deg D \geq 2g(X)$ then $|D|$ is basepoint free.
- If $\deg D \geq 2g(X) + 1$ then D is very ample.
- D is ample iff $\deg D > 0$
- This bounds is not sharp.

Hint: apply RR. For the bound, consider a plane curve X of degree 4 and $D = X.H$.

Remark 4.3.7: Idea behind embedding in $\mathbf{P}^3$: embed into $\mathbf{P}^n$ and project away from a point in the complement.

Exercise 4.3.8 (3.4, 3.5, 3.6)
Let $X \subseteq \mathbf{P}^N$ be a curve and $O \notin X$, let $\varphi : X \to \mathbf{P}^{n-1}$ be projection away from O. Then φ is a closed immersion iff

- O is not on any secant line of X, and
- O is not on any tangent line of X.

Show that if $N \geq 4$ then there exists such a point O yielding a closed immersion into $\mathbf{P}^{N-1}$. Conclude that any curve can be embedded into $\mathbf{P}^3$.

Hint: $\dim \operatorname{Sec}(X) \leq 3$ *and* $\dim \operatorname{Tan}(X) \leq 2$.

Proposition 4.3.9 ***(3.7).***
Let $X \subseteq \mathbf{P}^3$, $O \notin X$, and $\varphi : X \to \mathbf{P}^2$ be the projection from O. Then $X \overset{\sim}{\dashrightarrow} \varphi(X)$ iff $\varphi(X)$ is nodal iff the following hold:

- O is only on finitely many secants of X,
- O is on no tangents,
- O is on no multisecant,
- O is on no secant with coplanar tangent lines.

Skipped things around Prop 3.8. The hard part: showing not every secant is a multisecant, and not every secant has coplanar tangent lines. Skipped strange curves.

Remark 4.3.10: Classifying all curves: any curve is birational to a nodal plane curve, so study the family $\mathcal{F}_{d,r}$ of plane curves of degree d and r nodes. The family $\mathcal{F}_d$ of all plane curves is a linear system of dimension

$$\dim |\mathcal{F}_d| = \frac{d(d+3)}{2}.$$

For any such curve X, consider its normalization $\nu(X)$, then

$$g(\nu(X)) = \frac{(d-1)(d-2)}{2} - r.$$

Thus for $\mathcal{F}_{d,r}$ to be nonempty, one needs

$$0 \leq r \leq \frac{(d-1)(d-2)}{2}.$$

Both extremes can occur: $r = 0$ follows from Bertini, and $r = \frac{(d-1)(d-2)}{2}$ by embedding $\mathbf{P}^1 \hookrightarrow \mathbf{P}^d$ as a curve of degree d and projecting down to a nodal curve in $\mathbf{P}^2$ of genus zero. Severi states and Harris proves that for every r in this range $\mathcal{F}_{d,r}$ is irreducible, nonempty, and $\dim \mathcal{F}_{d,r} = \frac{d(d+3)}{2} - r$.

4.4 IV.4: Elliptic Curves ⋆

Remark 4.4.1: Curves E with $g(E) = 1$; we'll assume $\operatorname{ch} k \neq 2$ throughout. Outline:

- Define the j-invariant, classifies isomorphism classes of elliptic curves.
- Group structure on the curve.
- $E = \operatorname{Jac}(E)$.
- Results about elliptic functions over $\mathbf{C}$.
- The Hasse invariant of $E/\mathbf{F}_q$ in characteristic p.
- $E(\mathbf{Q})$.

4.4.1 The j-invariant

Remark 4.4.2: The j-invariant:

- $j(E) \in k$, so $\mathbf{A}^1_{/k}$ is a coarse moduli space for elliptic curves over K.
- Defining $j(E)$:
  - Let $p_0 \in X$, consider the linear system $L := |2p_0|$.
  - Nonspecial, so RR shows $\dim(L) = 1$.
  - BPF, otherwise E is rational.
  - Defines a morphism $\varphi_L : E \to \mathbf{P}^1_{/k}$ with $\deg \varphi_L = 2$.
  - Up to change of coordinates, $f(p_0) = \infty$.
  - By Hurwitz, f is ramified at 4 branch points a, b, c, p_0.
  - Move $a \mapsto 0, b \mapsto 1$ by a Mobius transformation fixing ∞, so f is branched over $0, 1, \lambda, \infty$ where $\lambda \in k \setminus \{0, 1\}$.
  - Use λ to define the invariant:

$$j(E) = j(\lambda) = 2^8 \left(\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2} \right).$$

- Theorem 4.1:
  - j depends only on the curve E and not λ.
  - $E \cong E' \iff j(E) = j(E')$.
  - Every element of k occurs as $j(E)$ for some E.
  - So this yields a bijection

$$\left\{ \text{Elliptic curves over } k \right\}_{/\sim} \rightleftharpoons \mathbf{A}^1_{/k}$$
$$E \mapsto j(E).$$

- Some facts that go into proving this:
  - $\forall p, q \in X\ \exists \sigma \in \mathrm{Aut}(X)$ such that $\sigma^2 = 1, \sigma(p) = q$, for any $r \in X$, one has $r + \sigma(r) \sim p + q$.
  - $\mathrm{Aut}(X) \curvearrowright X$ transitively.
  - Any two degree two maps $f_1, f_2 : X \to \mathbf{P}^1$ fit into a commuting square.
  - Under $S_3 \curvearrowright \mathbf{A}^1_{/k} \setminus \{0,1\}$, the orbit of λ is

$$S_3.\lambda = \left\{\lambda, \lambda^{-1}, s_1 = 1 - \lambda, s_1^{-1} = (1-\lambda)^{-1}, s_2 = \lambda(\lambda - 1)^{-1}, s_3 = \lambda^{-1}(\lambda - 1)\right\}.$$

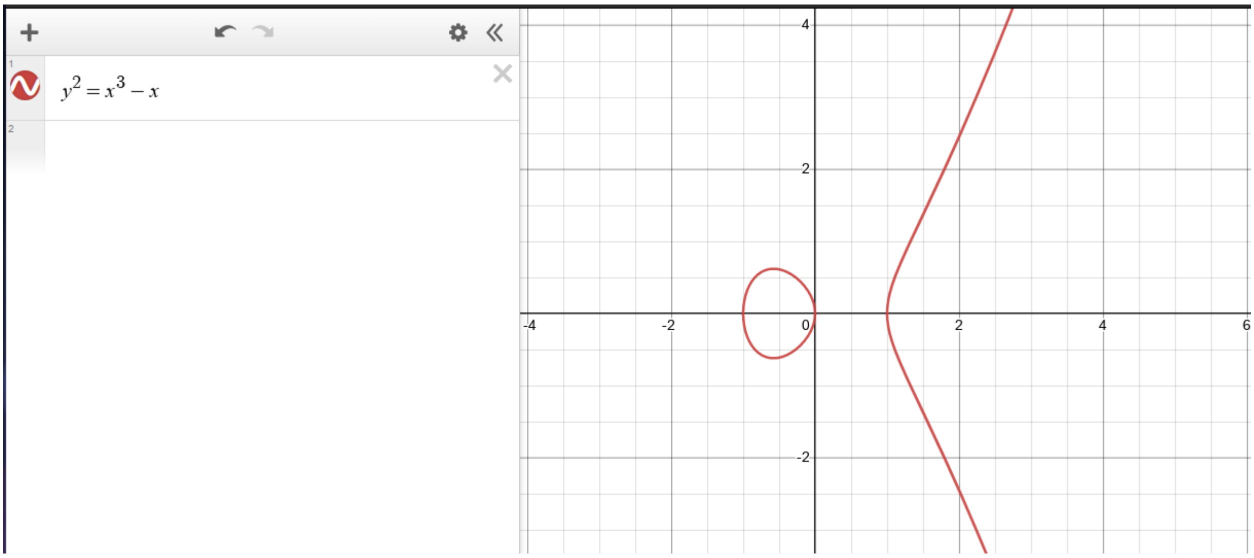
  - Fixing $p \in X$, there is a closed immersion $X \to \mathbf{P}^2$ whose image is $y^2 = x(x-1)(x-\lambda)$ where $p \mapsto \infty = [0:1:0]$ and this λ is either the λ from above or one of $s_1^{\pm 1}, s_2^{\pm 1}$.
    - ♢ Idea of proof: embed $X \hookrightarrow \mathbf{P}^2$ by $L := |3p|$, use RR to compute $h^0(\mathcal{O}(np)) = n$ so $h^0(\mathcal{O}(6p)) = 6$.
    - ♢ So $\{1, x, y, x^2, xy, y^2, x^3\}$ has a linear dependence where x^3, y^2 have nonzero coefficients since they have poles at p.
    - ♢ Rescale x^3, y^2 to coefficient 1 to get

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

  - Do a change of variable to put in the desired form: complete the square on the LHS, factor as $y^2 = (x-a)(x-b)(x-c)$, send $a \to 0, b \to 1$ by a Mobius transformation.
- Note that one can project from p to the x-axis to get a finite degree 2 morphism ramified at $0, 1, \lambda, \infty$.

Example 4.4.3 *(?)*: An elliptic curve that is smooth over every field of non-2 characteristic:

$$E : y^2 = x^3 - x, \qquad \lambda = -1,\ j(E) = 2^6 \cdot 3^3 = 1728.$$



One that is smooth over every k with $\operatorname{ch} k \neq 3$: the Fermat curve

$$E : x^3 + y^3 = z^3, \qquad \lambda = \pm \zeta_3^k,\ j(E) = 0.$$

Theorem 4.4.4 ***(Orders of automorphism groups of elliptic curves).***

$$\sharp\operatorname{Aut}(X,p)=\begin{cases}2 & j(E)\neq 0,1728\\ 4 & j(E)=1728,\operatorname{ch}k\neq 3\\ 6 & j(E)=0,\operatorname{ch}k\neq 3\\ 12 & j(E)=0,1728,\operatorname{ch}k=3\end{cases}.$$

Remark 4.4.5 ***(Proof idea):*** Idea: take the degree 2 morphism $f:X\to\mathbf{P}^1$ with $f(p)=\infty$ branched over $\{0,1,\lambda,\infty\}$. Produce two elements in G: for $\sigma\in G$, find $\tau\in\operatorname{Aut}(\mathbf{P}^1)$ so $f\sigma=\tau f$; then either $\tau\neq\mathrm{id}$, so $\{\sigma,\tau\}\subseteq G$, or $\tau=id$ and either $\sigma=\mathrm{id}$ or σ exchanges the sheets of f.

If $\tau\neq\mathrm{id}$, it permutes $\{0,1,\lambda\}$ and sends $\lambda\mapsto\lambda^{-1},s_1^{\pm1},s_2^{\pm1}$ from above. Cases:

1. $j=1728$: If $\lambda=-1,1/2,2,\operatorname{ch}k\neq 3$, then λ coincides with *one* other element of $S_3.\lambda$, so $\sharp G=4$.
2. $j=0$: If $\lambda=-\zeta_3,-\zeta_3^2,\operatorname{ch}k\neq 3$ then λ coincides with *two* elements in $S_3.\lambda$ so $\sharp G=6$.
3. $j=0=1728$: If $\lambda=-1,\operatorname{ch}k=3$ then $S_3.\lambda=\{\lambda\}$ and $\sharp G=12$.

4.4.2 The group structure

Remark 4.4.6: The group structure:

- Fixing $p_o\in E$, the map $p\mapsto\mathcal{L}(p-p_0)$ induces a bijection $E\xrightarrow{\sim}\operatorname{Pic}^0(E)$, so the group structure on E is the pullback along this with $p_0=\mathrm{id}$ and $p+q=r\iff p+q\sim r+p_0\in\operatorname{Div}(E)$.
- Under the embedding of $|3p_0|$, points p,q,r are collinear iff $p+q+r\sim 3p_0$, so $p+q+r=0$ in the group structure.
- E is a group variety, since $p\mapsto -p$ and $(p,q)\mapsto p+q$ are morphisms. Thus there is a morphism $[n]:E\to E$, multiplication by n, which is a finite morphism of degree n^2 with kernel $\ker[n]=C_n^2$ if $(n,\operatorname{ch}k)=1$.and $\ker[n]=C_p,0$ if $n=\operatorname{ch}k$, depending on the Hasse invariant.
- If $f:E_1\to E_2$ is a morphism of curves with $f(p_1)=p_2$ then f induces a group morphism.
- $\operatorname{End}(E,p_0)$ forms a ring under $f+g=\mu\circ(f\times g)$ and $f\cdot g:=f\circ g$.
- The map $n\mapsto([n]:E\to E)$ defines a finite ring morphism $\mathbf{Z}\to\operatorname{End}(E,p_0)$ for $n\neq 0$.
- $R:=\operatorname{End}(E,p_0)^\times=\operatorname{Aut}(E)$, and if $j=0,1728$ then R contains $\{\pm1\}$ and is thus bigger than $\mathbf{Z}$.

Remark 4.4.7: The Jacobian: a variety that generalizes to make sense for any curve, a moduli space of degree zero divisor classes.

- For X/k a curve and $T\in\mathsf{Sch}_{/k}$, define

$$\operatorname{Pic}^0(X\times T):=\left\{\mathcal{F}\in\operatorname{Pic}(X\times T)\;\middle|\;\deg\mathcal{F}|_{X_t}=0\,\forall t\in T\right\},\qquad\operatorname{Pic}(X/T):=\operatorname{Pic}^0(X\times T)/p^*\operatorname{Pic}(T)$$

where $p : X \times T \to T$ is the second projection. Regard this as *families of sheaves of degree 0 on X parameterized by T*.

- The Jacobian variety of a curve X: $\mathrm{Jac}(X) \in \mathsf{Sch}^{\mathrm{ft}}_{/k}$ along with $\mathcal{L} \in \mathrm{Pic}^0(X/\mathrm{Jac}(X))$ such that for any $T \in \mathsf{Sch}^{\mathrm{ft}}_{/k}$ and any $\mathcal{M} \in \mathrm{Pic}^0(X/T)$, $\exists!\, f : T \to \mathrm{Jac}(X)$ such that $f^*\mathcal{L} = \mathcal{M}$. Thus J represents the functor $\mathrm{Pic}^0(X/-)$.

- For E elliptic, $E = \mathrm{Jac}(E)$.

  - In general, $|\mathrm{Jac}(X)| \cong \left|\mathrm{Pic}^0(X)\right|$ on points, since points of $\mathrm{Jac}(X)$ are morphisms $\operatorname{Spec} k \to \mathrm{Jac}(X)$, which correspond to elements in $\mathrm{Pic}^0(X/k) = \mathrm{Pic}^0(X)$.

- $\mathrm{Jac}(X) \in \mathsf{GrpSch}_{/k}$ where $e : \operatorname{Spec} k \to \mathrm{Jac}(X)$ corresponds to $0 \in \mathrm{Pic}^0(X/k)$, $\rho : \mathrm{Jac}(X) \to \mathrm{Jac}(X)$ is $\mathcal{L} \mapsto \mathcal{L}^{-1} \in \mathrm{Pic}^0(X/\mathrm{Jac}(X))$, and $\mu : \mathrm{Jac}(X)^{\times^2} \to \mathrm{Jac}(X)$ is $\mathcal{L} \mapsto p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \in \mathrm{Pic}^0(X/\mathrm{Pic}(X)^{\times^2})$.

- $\mathbf{T}_0 \mathrm{Jac}(X) \cong H^1(X; \mathcal{O}_X)$: giving an element of $\mathbf{T}_p X$ is the same as a morphism $T := \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \to X$ sending $\operatorname{Spec} k \to p$. So $\mathbf{T}_0 \mathrm{Jac}(X)$, this means giving $\mathcal{M} \in \mathrm{Pic}^0(X/T)$ whose restriction to $\mathrm{Pic}^0(X/k)$ is zero. Use the SES $H^1(X; \mathcal{O}_X) \hookrightarrow \operatorname{Pic} X[\varepsilon] \to \mathrm{Pic}(X)$.

- $\mathrm{Jac}(X)$ is proper over k by the valuative criterion. Just show that an invertible sheaf $\mathcal{M}$ on $X \times \operatorname{Spec} K$ lifts unique to $\tilde{\mathcal{M}}$ on $X \times \operatorname{Spec} R$, but $X \times \operatorname{Spec} R$ is regular, so apply II.6.5.

- For any n there is a morphism
$$\begin{aligned} \varphi^n : X^{\times^n} &\to \mathrm{Jac}(X) \\ (p_1, \cdots, p_n) &\mapsto \mathcal{L}(\textstyle\sum p_i - np_0). \end{aligned}$$
This is surjective for $n \geq g(X)$ by RR since every divisor class of degree $d \geq g$ has an effective representative. The fibers of φ^n are all tuples $(p_1, \cdots, p_n)$ such that $D = \sum p_i$ forms a complete linear system.

  - Most fibers are finite, so $\mathrm{Jac}(X)$ is irreducible of dimension g.
  - Smoothness: $\dim \mathbf{T}_0 \mathrm{Jac}(X) = \dim H^1(X; \mathcal{O}_X) = g$, so smooth at zero, and group schemes are homogeneous so smooth everywhere.

4.4.3 Elliptic functions

Stopped at elliptic functions.

4.5 IV.5: The Canonical Embedding

4.6 IV.6: Classification of Curves in $\mathbb{P}^3$

5 V: Surfaces

5.1 V.1: Geometry on a Surface

5.2 V.2: Ruled Surfaces

5.3 V.3: Monoidal Transformations

5.4 V.4: The Cubic Surface in $\mathbb{P}^3$

5.5 V.5: Birational Transformations

5.6 V.6: Classification of Surfaces

6 Toric Varieties

6.1 Summaries

6.1.1 Quick Criteria

Remark 6.1.1: Quick criteria:

- **Normal** $\iff$ **Saturated**: For affines, $X = \operatorname{Spec} \mathbf{C}[S]$ where $S \subseteq M$ is a **saturated** semigroup. This is true for $S = S_\sigma = \sigma^\vee \cap M$ where σ is any SCRPC.
- **Complete/proper** $\iff$ **Full support**: X_Σ is complete iff $\operatorname{supp} \Sigma = N_{\mathbf{R}}$.
- **Smooth** $\iff$ **Lattice basis**:
  - For a **cone** $\sigma = \operatorname{Cone}(S)$ is smooth iff $\det S = \pm 1$, the volume of the standard lattice $\mathbf{Z}^n$.
    - ♢ Consequences of smoothness:

  - ♢ $\mathrm{CDiv}(X) = \mathrm{Div}(X)$
  - ♢ $\mathrm{Cl}(X) = \mathrm{Pic}(X)$
- – Smooth implies simplicial, so non-simplicial cones are singular.
- – For p_σ the T-fixed point corresponding to σ, $T_p X \cong H$ where H is a Hilbert basis for S_σ.

- **Simplicial** $\iff$ **Euclidean basis**: For $\sigma = \mathrm{Cone}(S)$, σ is simplicial iff $\det(S) \neq 0$.

- **Orbifold singularities** $\iff$ **Simplicial**: X_Σ has at worst finite quotient singularities iff Σ is simplicial.

- **Projectivity** $\iff$ **Admits a strictly upper convex support function**: For h a support function and D_h its associated divisor, the linear system $|D_h|$ defines an embedding $X(\Delta) \hookrightarrow \mathbf{P}^N$ iff h is strictly upper convex.

  – Alternatively, X_Σ is projective iff Σ arises as the normal fan of a polytope.

- **Globally generated/basepoint free** $\iff$ **Upper convex support function**: $\mathcal{O}(D)$ is globally generated iff ψ_D is upper convex.

- **Ample** $\iff$ **Strictly upper convex support function**:
  $D \in \mathrm{CDiv}_T(X)$ is ample iff ψ_D is strictly upper convex.

- **Very ample** $\iff$ **ample and semigroup generation**: for Σ complete, D is very ample iff ψ_D is strictly upper convex **and** S_σ is generated by $\left\{ u - u(\sigma) \;\middle|\; u \in P_D \cap M \right\}$, or equivalently the semigroup $\left\{ u - u' \;\middle|\; u' \in P \cap M \right\}$ is saturated in M.

  – For $\mathbf{P}^n$: $D = \sum a_i D_i$ is globally generated iff $\sum a_i \geq 0$ and ample $\iff$ $\sum a_i > 0$.
  – For $\mathbf{F}_m$: $D = \sum a_i D_i$ is globally generated iff $a_2 + a_4 \geq 0$, $a_1 + a_3 \geq m a_1$, $\mathrm{Pic}(\mathbf{F}_n) = \langle D_1, D_4 \rangle$, and $D = aD_1 + bD_4$ is ample iff $a, b > 0$.
  – For $\dim X_\Sigma = 2$ and X complete: ample $\iff$ very ample.

- **Q-factorial** $\iff$ **simplicial**: iff every cone is simplicial.

- **Fundamental groups**:

  – For U_σ affine, $U_\sigma \cong \mathbf{A}^k \times \mathbf{G}_m^{n-k}$ so $\pi_1 U_\sigma \cong \mathbf{Z}^{n-k}$ since $\mathbf{G}_m^{n-k} \simeq (S^1)^{n-k}$.
  – Can write $\pi_1 U_\sigma = N/N_\sigma$ where N_σ is the sublattice generated by σ.
  – By a Van Kampen argument, $\pi_1 X_\Sigma = N/N'$ where $N' = \left\langle \sigma \cap N \;\middle|\; \sigma \in \Sigma \right\rangle$:

$$\pi_1 X_\Sigma = \pi_1 \cup U_\sigma = \varinjlim \pi_1 U_\sigma = \varinjlim N/N_\sigma = N/\sum N_\sigma = N/N'.$$

- **Euler characteristic**: $\chi X_\Sigma = \sharp\Sigma(n)$.

– Why: $H^i(U_\sigma;\mathbf{Z}) = \bigwedge^i M(\sigma)$ where $M(\sigma) \coloneqq \sigma^\vee \cap M$, so one gets a spectral sequence

$$E_1^{p,q} = \bigoplus_{I^p = i_0 < \cdots < i_p} H^q(U_{\sigma_{I^p}};\mathbf{Z}) \Rightarrow H^{p+q}(X_\Sigma;\mathbf{Z}), \qquad \sigma_{I^p} = \sigma_{i_0} \cap \cdots \sigma_{i_p}, \sigma_{i_j} \in \Sigma(n)$$

$$\rightsquigarrow E_1^{p,q} = \bigoplus_{I^p} \bigwedge^q M(\sigma_{I^p}) \Rightarrow H^{p+q}(X_\Sigma;\mathbf{Z})$$

$$\implies \chi X_\Sigma = \sum (-1)^{p+q} \operatorname{rank}_{\mathbf{Z}} E_1^{p,q} = \sharp\Sigma(n),$$

using that

$$\sum (-1)^q \operatorname{rank}_{\mathbf{Z}}^q M(\tau) = \begin{cases} 0 & \dim\tau < n \\ 1 & \dim\tau = n. \end{cases}.$$

- **Higher homology**:
  – If all maximal cones of Σ are n-dimensional, $H^2(X_\Sigma;\mathbf{Z}) \cong \operatorname{Pic}(X_\Sigma)$.
- **Global sections**: for $D \in \operatorname{Div}_T(X)$, P_D its associated polyhedron,

$$H^0(X;\mathcal{O}_X(D)) = \bigoplus_{m \in P_D \cap M} \mathbf{C}\,\chi^m.$$

- **Betti numbers**:

$$\beta_{2k} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} \sharp\Sigma(n-i).$$

- **Canonical bundles/divisors**: $\omega_{X_\Sigma} \coloneqq \det \Omega_{X_\Sigma/k} = \mathcal{O}(K_{X_\Sigma})$ where $K_{X_\Sigma} = -\sum_{\rho_i} D_i$.
  – For a smooth complete surface with $D_i^2 = -d_i$,

$$K^2 = \sum D_i^2 + 2d = -\sum d_i + 2d = -(3d-12) + 2d = 12 - d.$$

- **Degree** $= n! \cdot (P)$ (for X_P projective)

Remark 6.1.2: Some common counterexamples:

- An ample divisor that is not very ample: $P \coloneqq *([0,0,0],[0,1,1],[1,0,1],[1,1,0])$; then take D_P. X_P is a double cover of $\mathbf{P}^3$ branched along the 4 boundary divisors.
- A Weil divisor that is not Cartier: ????
- A complete variety that is not projective: ???

6.1.2 Cones and Lattices

Remark 6.1.3:

- **Characters**: for groups G, a map $\chi \in \mathsf{Grp}(G, \mathbf{C}^\times)$. For $G = T = (\mathbf{C}^\times)^n$, there is an isomorphism

$$\mathbf{Z}^n \xrightarrow{\sim} \mathsf{Grp}(T, \mathbf{C}^\times)$$
$$m = [m_1, \cdots, m_n] \mapsto \chi_m : [t_1, \cdots, t_n] \mapsto \prod t_i^{m_i}.$$

  Generally set $M \coloneqq \mathsf{Grp}(T, \mathbf{C}^\times)$, the character lattice.

  - M is a lattice, $M_{\mathbf{R}} \coloneqq M \otimes_{\mathbf{Z}} \mathbf{R}$ is its associated Euclidean space.

- **Cocharacters / one-parameter subgroups**: for groups G, a map $\lambda \in \mathsf{Grp}(\mathbf{C}^\times, G)$. For $G = T = \mathbf{C}^\times$, there is again an isomorphism

$$\mathbf{Z}^n \mapsto \mathsf{Grp}(\mathbf{C}^\times, T)$$
$$u = [u_1, \cdots, u_n] \mapsto \lambda^u : t \mapsto [t^{u_1}, \cdots, t^{u_n}].$$

  Define $N \coloneqq \mathsf{Grp}(\mathbf{C}^\times, T)$ the cocharacter lattice.

  - N is a lattice, $N_{\mathbf{R}} \coloneqq N \otimes_{\mathbf{Z}} \mathbf{R}$ its associated euclidean space.

- There is a perfect pairing

$$\langle -,\ - \rangle : M \times N \to \mathbf{Z}$$
$$,$$

  defined using the fact that if $m \in M, n \in N$ then $\chi^m \circ \lambda^n \in \mathsf{Grp}(\mathbf{C}^\times, \mathbf{C}^\times)$ is of the form $t \mapsto t^\ell$, so set $\langle m,\ n \rangle \coloneqq \ell$.

  - Thus $M = \mathsf{Grp}(M, \mathbf{Z})$ and $N = \mathsf{Grp}(N, \mathbf{Z})$.
  - How to recover the torus:

$$N \otimes_{\mathbf{Z}} \mathbf{C}^\times \to T$$
$$u \otimes t \mapsto \lambda^u(t).$$

- Δ is a **fan**, a collection of **strongly convex rational polyhedral cones**:

  - **Cone**: $0 \in \sigma$ and $\mathbf{R}_{\geq 0}\sigma \subseteq \sigma$.
  - **Strongly convex**: contains no nonzero subspace, i.e. no line through $\mathbf{0} \in N_{\mathbf{R}}$. Equivalently, $\dim \sigma^\vee = n$.
  - **Rational**: generated by $\{v_i\} \subseteq N$, i.e. of the form $\mathrm{Cone}(S)$ for $S \subseteq N$.

- **Dual cones**:

$$\sigma^\vee \coloneqq \left\{ u \in M \;\middle|\; \langle u,\ v \rangle \geq 0 \ \forall v \in M_{\mathbf{R}} \right\}.$$

  - If $\sigma^\vee = \bigcap_{i=1}^s H_{m_i}^+$ for $m_i \subseteq \sigma^\vee$ then $\sigma^\vee = \mathrm{Cone}(m_1, \cdots, m_s)$.

- **Hyperplanes** and **closed half-spaces**:

$$H_m \coloneqq \left\{ u \in N_{\mathbf{R}} \;\middle|\; \langle m,\ u \rangle = 0 \right\} \subseteq N_{\mathbf{R}}$$
$$H_m^+ \coloneqq \left\{ u \in N_{\mathbf{R}} \;\middle|\; \langle m,\ u \rangle \geq 0 \right\} \subseteq N_{\mathbf{R}}.$$

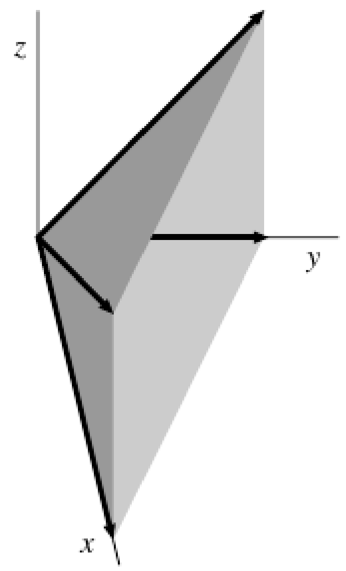
- **Face**: $\tau \leq \sigma$ is a face iff τ is of the form $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee \subseteq M_{\mathbf{R}}$.
- **Facet**: codimension one faces, $\Sigma(n-1)$ where $n \coloneqq \dim N$.
- **Ray**: dimension 1 faces, $\Sigma(1)$.
- The **semigroup** of a cone:

$$S_\sigma \coloneqq \sigma^\vee \cap M = \left\{ u \in M \;\middle|\; \langle u,\ v \rangle \geq 0\ \forall v \in \sigma \right\}.$$

- The **semigroup algebra** of a semigroup:

$$\mathbf{C}[S] \coloneqq \left\{ \sum_{s \in S} c_s \chi^s \;\middle|\; c_s \in \mathbf{C}, c_s = 0 \text{a.e.} \right\}, \qquad \chi^{m_1} \cdot \chi^{m_2} \coloneqq \chi^{m_1 + m_2}.$$

- **Simplicial**: the generators can be extended to an $\mathbf{R}$-basis of $N_{\mathbf{R}}$. E.g. not simplicial:



- **Smooth**: the minimal generators can be extended to a $\mathbf{Z}$-basis of N.

  - Checking T_pX: m is **decomposable** in S_σ iff $m = m_1 + m_2$ with $m_i \in S_\sigma$; the maximal ideal at p corresponding to σ is $\mathfrak{m}_p = \left\{ \chi^m \;\middle|\; m \in S_\sigma \right\}$, and $\mathfrak{m}_p/\mathfrak{m}_p^2 = \left\{ \chi^m \;\middle|\; m \text{ is indecomposable in } S_\sigma \right\}$. This exactly corresponds to a Hilbert basis.

- **Facet**: face of codimension 1.

- **Edge**: face of dimension 1. Note that facets = edges in $\dim N = 2$.

- **Saturated**: S is saturated if for all $k \in \mathbb{N} \setminus \{0\}$ and all $m \in M$, $km \in S \implies m \in S$. Any SCRPC is saturated.

  - E.g. $S = \{(4,0),(3,1),(1,3),(0,4)\}$ is not saturated since $2 \cdot (2,2) = (4,4) \in \mathbb{N}S$ but $(2,2) \notin S$.

- **Normalization**: in the affine case, write $X = \operatorname{Spec} \mathbf{C}[S]$ with torus character lattice $M = \mathbf{Z}S$, take a finite generating set S', and set $\sigma = \operatorname{Cone}(S')^\vee$. Then $\operatorname{Spec} \mathbf{C}[\sigma^\vee \cap M] \to X$ is the normalization.

- **Distinguished points**: each strongly convex $\sigma \rightsquigarrow \gamma_\sigma \in U_\sigma$ a unique point corresponding to the semigroup morphism $m \mapsto \mathbb{1}\,[(]\, m \in \sigma^\vee \cap M)$, which is T-fixed iff σ is full-dimensional.

- **Orbits**: $\operatorname{Orb}(\sigma) = T.\gamma_\sigma$, and $V(\sigma) \coloneqq \operatorname{clOrb}(\sigma)$.

- **Orbit-Cone correspondence**: there is a correspondence

  $\{\text{Cones } \sigma \in \Sigma\} \rightleftharpoons \{T\text{-orbits in } X_\Sigma\}$

  $$\sigma \mapsto \operatorname{Orb}(\sigma) \coloneqq T.\gamma_\sigma = \left\{ \gamma : S_\sigma \to \mathbf{C} \;\middle|\; \gamma(m) \neq 0 \iff m \in \sigma^\vee \cap M \right\} \cong \mathsf{Grp}(\sigma \cap M, \mathbf{C}^\times),$$

  where $\dim \operatorname{Orb}(\sigma) = \operatorname{codim}_{N_\mathbf{R}} \sigma$, and $\tau \leq \sigma \implies \operatorname{clOrb}(\tau) \supseteq \operatorname{clOrb}(\sigma)$ and in fact $\operatorname{clOrb}(\sigma) = \coprod_{\tau \leq \sigma} \operatorname{clOrb}(\tau)$.

- **Star**: define $N_\tau \coloneqq \mathbf{Z}\langle \tau \cap N \rangle$ and $N(\tau)\mathbf{R} \coloneqq N_\mathbf{R}/(N_\tau)_\mathbf{R}$ and $\overline{\sigma}$ for the image of σ under the quotient map, then

  $$\operatorname{Star}(\tau) \coloneqq \left\{ \overline{\sigma} \subseteq N(\tau)_\mathbf{R} \;\middle|\; \sigma \leq \tau \right\} \subseteq N(\tau)_\mathbf{R}.$$

  This is always a fan, and $V(\tau) = X_{\operatorname{Star}(\tau)}$.

- **Star subdivision**: for $\sigma = \operatorname{Cone}(S)$ for $S \coloneqq \{u_1, \cdots, u_n\}$, set $u_0 \coloneqq \sum u_i$ and take $\Sigma'(\sigma)$ defined as the cones generated by subsets of $\{u_0, u_1, \cdots, u_n\}$ not containing S. The star subdivision of Σ along σ is $\Sigma^\star(\sigma) \coloneqq (\Sigma \setminus \{\sigma\}) \cup \Sigma'(\sigma)$.

- **Blowups**: $\varphi : X_{\Sigma^\star(\sigma)} \to X_\Sigma$ is the blowup at γ_σ.

6.1.3 Divisors

Remark 6.1.4:

- **(Weil) divisor**: $\mathrm{Div}(X) = \left\{ \sum n_i V_i \;\middle|\; V_i \subseteq X, \operatorname{codim} V_i = 1 \right\}$.
  - $\mathcal{O}_X(D)$: the (coherent) sheaf associated to a Weil divisor D.
- **Cartier divisor**: $\mathrm{CDiv}(X) = H^0(X; \mathcal{K}_X^\times / \mathcal{O}_X^\times)$, the quotient of rational functions by regular functions. For X normal, equivalently locally principal (Weil) divisors, so $D \rightsquigarrow \{(U_i, f_i)\}$ where $D|_{U_i} = \mathrm{Div}(f_i)$.
  - **Q-Cartier divisor**: A **Q**-divisor $D = \sum n_i D_i$ with $n_i \in \mathbf{Q}$ is **Q**-Cartier when mD is Cartier for some $m \in \mathbf{Z}_{\geq 0}$.
  - **Q-factorial**: every prime divisor is **Q**-Cartier.
- **Ray divisors**: every $\rho \in \Sigma(1)$ defines a divisor $D_\rho := V(\rho) := \mathrm{clOrb}(\rho)$.
- **Very Ample**: $\mathcal{L}$ which defines a morphism into $\mathbf{P}H^0(X; \mathcal{L}) \cong \mathbf{P}^N$.
- **Ample**: $\mathcal{L}$ is basepoint free and some power $\mathcal{L}^n$ is very ample.
  - D is (very) ample iff $\mathcal{O}_X(D)$ is (very) ample, i.e. D is ample iff nD is very ample for some n.
- **Upper convex**: $f(n_1 + n_2) \leq f(n_1) + f(n_2)$.
  - **Strictly upper convex**: $\sigma_1 \neq \sigma_2 \implies f_{\sigma_1} \neq f_{\sigma_2}$.
- **Linearly equivalent divisors**: $D_1 \sim D_2 \iff D_1 - D_2 = \mathrm{Div}(f)$ for some f.
- **Complete linear systems**: $|D| = \left\{ D' \in \mathrm{Div}(X) \;\middle|\; D' \sim D \right\}$.
- **Support function**: $\varphi : \operatorname{supp} \Sigma \to \mathbf{R}$ where $\varphi|_\sigma$ is linear for each cone σ.
  - **Integral** with respect to N iff $\varphi(\operatorname{supp} \Sigma \cap N) \subseteq \mathbf{Z}$. Defines a set of integral support functions $\mathrm{SF}(\Sigma, N)$.
- The class group complement exact sequence: for $D_1, \cdots, D_n \in \mathrm{Div}(X)$ distinct,

$$\begin{aligned} \mathbf{Z}^n &\to \mathrm{Cl}(X) \twoheadrightarrow \mathrm{Cl}(X \setminus \cup D_i) \\ e_1 &\mapsto [D_i]. \end{aligned}$$

- $\mathcal{O}_X(D)$ is the sheaf

$$U \mapsto \left\{ f \in \mathcal{K}(X)^\times(U) \;\middle|\; \mathrm{Div}(f) + D|_U \geq 0 \in \mathrm{Cl}(U) \right\}.$$

  Then $D \in \mathrm{CDiv}(X) \iff \mathcal{O}_X(D) \in \mathrm{Pic}(X)$.

- The toric class group exact sequence:

$$\begin{aligned} M &\to \mathrm{Div}_T(X) \twoheadrightarrow \mathrm{Cl}(X) \\ m &\mapsto \mathrm{Div}(\chi^m) = \sum_\rho \langle m,\ u_\rho\rangle [D_\rho] \end{aligned}$$

  where u_ρ are minimal ray generators.

6.1.4 Polytopes

Remark 6.1.5:

- **Supporting hyperplanes**: the positive side of an affine hyperplane

$$\begin{aligned} H_{u,b} &:= \left\{ m \in M_{\mathbf{R}} \;\middle|\; \langle m,\ u\rangle = b \right\} \\ H_{u,b}^+ &:= \left\{ m \in M_{\mathbf{R}} \;\middle|\; \langle m,\ u\rangle \geq b \right\}. \end{aligned}$$

  - If P is full dimensional and $F \leq P$ is a facet, then $F = P \cap H_{u_F, -a_F}$ for a unique pair $(u_F, a_F) \in N_{\mathbf{R}} \times \mathbf{R}$.

- **Polytope**: the convex hull of a finite set $S \subseteq N_{\mathbf{R}}$ or an intersection of half-spaces:

$$P = \left\{ \sum_{v \in S} \lambda_v v \;\middle|\; \sum \lambda_v = 1 \right\} = \bigcap_{i=1}^s H_{u_i, b_i}^+.$$

- **Simplex** $\dim P = d$ and there are exactly $d+1$ vertices.

- **Simple**: $\dim P = d$ and every vertex is the intersection of exactly d facets.

- **Simplicial**: all facets are simplices.

  - E.g. simple but not simplicial: the cube in $\mathbf{R}^3$, since each vertex meets 3 edges but a square is not a simplex. -E.g. Simplicial but not simple: the octahedron in $\mathbf{R}^3$, since each vertex meets 4 edges but each face is a triangle.

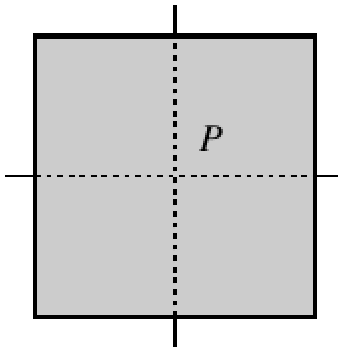
- **Combinatorial equivalence**: $P_1 \sim P_2$ iff there is a bijection $P_1 \to P_2$ preserving intersections, inclusions, and dimensions of all faces.

- **Polar dual**: for $P \subseteq M_{\mathbf{R}}$,

$$P^\circ = \left\{ u \in N_{\mathbf{R}} \;\middle|\; \langle m,\ u\rangle \geq -1 \ \forall m \in P \right\}.$$

  - Trick: for $P \subseteq M_{\mathbf{R}}$ with $0 \in P$,

$$\begin{aligned} P = \left\{ m \in M_{\mathbf{R}} \;\middle|\; \langle m,\ u_F\rangle \geq -a_F,\ F \in \mathrm{Facets}(P) \right\} \\ \implies P^\circ = *(\left\{ a_F^{-1} u_F \right\}) \subseteq N_{\mathbf{R}}. \end{aligned}$$



E.g. write the square as $\{\langle m,\ \pm e_i\rangle \geq -1\}$, then $a_F = 1$ for all F:

- **Cone on a polytope**: $C(P) \coloneqq \operatorname{Cone}(P \times \{1\}) \subseteq M_{\mathbf{R}} \times \mathbf{R}$, the set of cones through all proper faces of P.

- **Normal**: $(kP \cap M) + (\ell P \cap M) \subseteq (k+\ell)P \cap M$, or equivalently $k \cdot (P \cap M) = (kP) \cap M$, or equivalently $(P \cap M) \times \{1\}$ generates $C(P) \cap (M \times \mathbf{Z})$ as a semigroup.

  – If $P \subseteq M_{\mathbf{R}}$ is a full-dimensional lattice polytope with $\dim P \geq 2$, then kP is normal for all $k \geq \dim P - 1$.
  – Normal implies very ample.
  – $P \rightsquigarrow \mathcal{L}_P \in \operatorname{Pic}(X_P)$
  – $P \cap M \rightsquigarrow H^0(X_P; \mathcal{L}_P)$.

- **Reflexive**: a polytope P with facet presentation

$$P = \left\{ m \in M_{\mathbf{R}} \;\middle|\; \langle m,\ \mu_F\rangle \geq -1 \forall F \in \operatorname{Facets}(P) \right\}.$$

  Implies that $\int(P) \cap M = \{\mathbf{0}\}$, and $P^\circ = *(\left\{ u_F \;\middle|\; F \in \operatorname{Facets}(P) \right\})$.

- **Polyhedron of a divisor** P_D: write $D = \sum_\rho a_\rho D_\rho$, for any $m \in M$, $\operatorname{Div}(\chi^m) + D \geq 0 \implies \langle m,\ \rho\rangle \geq a_\rho \implies \langle m,\ \rho\rangle \geq -a_\rho$, so set

$$P_D \coloneqq \left\{ m \in M_{\mathbf{R}} \;\middle|\; \langle m,\ \rho\rangle \geq a_\rho \forall \rho \in \Sigma(1) \right\}.$$

- **Divisor of a polytope**: $D_P = \sum_F a_F D_F$ where $P = \left\{ m \;\middle|\; \langle m,\ u_F\rangle \geq -a_F \right\}$.

  – D_P is always the pullback of $\mathcal{O}_{\mathbf{P}^N}(1)$ along the embedding.

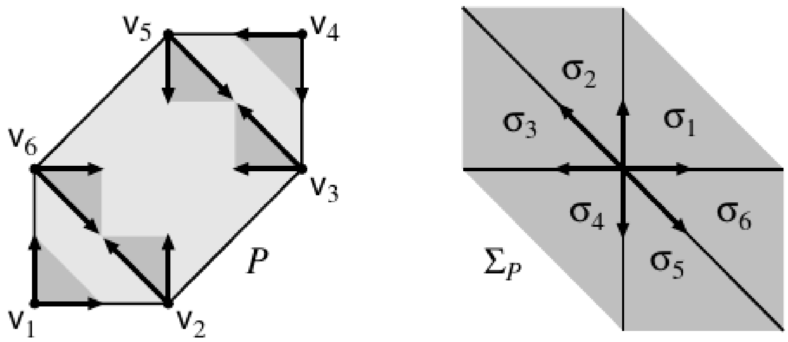
- **Very ample polytopes**: for every vertex v, the semigroup $\left\{ m' - v \;\middle|\; m' \in P \cap M \right\}$ is saturated in M.

  – Gives an embedding $X \hookrightarrow \mathbf{P}^N$ where $N = \sharp(P \cap M) - 1$.

- The **toric variety of a polytope**: if $P \cap M = \{m_1, \cdots, m_s\}$ and P is full dimensional very ample, then writing T_N for the torus of N,

$$X_P := \operatorname{cl} \operatorname{im} \varphi, \qquad \begin{aligned} \varphi : T_N &\to \mathbf{P}^{s-1} \\ t &\mapsto [\chi^{m_1}(t) : \cdots : \chi^{m_s}(t)]. \end{aligned}$$

  - Vertices m_i correspond to U_{σ_i} for $\sigma_i = \operatorname{Cone}(P \cap M - m_i)^\vee$:



- **Smooth**: P is smooth iff for all vertices $v \in P$, $\left\{ w_E - v \;\middle|\; E \text{ is an edge containing } v \right\}$ can be extended to a $\mathbf{Z}$-basis of M, where w_E is the first lattice point on E.

6.1.5 Singularities and Classification

Remark 6.1.6:

- **Gorenstein**: X normal where $K_X \in \operatorname{CDiv}(X)$ is Cartier.
- **Normal**: all local rings are integrally closed domains.
- **Complete**: proper over k. E.g. for varieties, just universally closed.
- **Factorial**: all local rings are UFDs.
- **Fano**: $-K_X$ is ample.
- **del Pezzo**: a smooth Fano surface.

Remark 6.1.7: Classification of smooth complete toric varieties:

- $\dim \Sigma = 2, \sharp\Sigma(1) = 3$: without loss of generality $\rho_1 = e_1, \rho_2 = e_2$. Then $\rho_3 = ae_1 + be_2$ with $a, b < 0$ to ensure $\operatorname{supp} \Sigma = \mathbf{R}^2$, and determinants for $|a| = |b| = 1$, so $(-1, 1)$.
- $\dim \Sigma = 2, \sharp\Sigma(1) = 4$: without loss of generality $\rho_1 = e_1, \rho_2 = e_2$. Then determinant conditions for $\rho_3 = (-1, b)$ and $\rho_4 = (a, -1)$, and $\det \begin{bmatrix} -1 & a \\ b & -1 \end{bmatrix} = 1 - ab = \pm 1 \implies ab = 0, 2$, so $(a, b) = (2, 1), (1, 2), (-2, -1), (-1, -2)$.
- $\dim \Sigma = 2, \sharp\Sigma(1) = d$, smooth: $\operatorname{Bl}_{p_1, \cdots, p_\ell} X$ for $X = \mathbf{P}^2$ or $\mathbf{F}_a$ for some a and p_i torus fixed points.

6.1.6 Examples

Question 6.1.8
Things you can figure out for every example:

- Given Δ, for $\sigma \in \Delta$,
  - What is $\sigma^\vee$?
  - Generators for S_σ?
  - Describe U_σ and $X(\Delta)$.
  - What are the transition functions for $U_{\sigma_1} \to U_{\sigma_2}$ when $\sigma_1 \cap \sigma_2 = \tau$ intersect in a common face?
- What are the T-invariant points?
  - What are the T-invariant divisors D_{ρ_i}?
  - What are all of the T-orbit closures of various dimensions?
- Is $X(\Delta)$ smooth?
  - Which cones $\sigma \in \Delta$ are smooth?
  - What is the canonical resolution of singularities?
  - What is the tangent space at each T-invariant point?
- What is the associated polytope P_Δ? What is its polar dual P_Δ°?
- What are the intersection numbers $D_{\rho_i} \cdot D_{\rho_j}$?
  - What are the self-intersection numbers $D_{\rho_i}^2$?
- What is $\mathrm{Div}_T(X)$? $\mathrm{CDiv}_T(X)$?
  - Which divisors are ample? Very ample? Globally generated?
- What is $\mathrm{Cl}(X)$? $\mathrm{Pic}(X)$?
- What is K_X?
  - Is K_X ample?
- Is $X(\Delta)$ projective?
- What is $H^0(X(\Delta); \mathcal{O}(D))$ for $D \in \mathrm{Div}_T(X)$?
- What is the Poincaré polynomial of $X(\Delta)$? (I.e. what are the Betti numbers?)

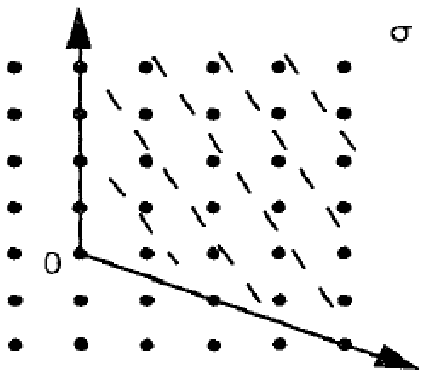
Example 6.1.9 ***(of varieties):*** Some useful explicit varieties:

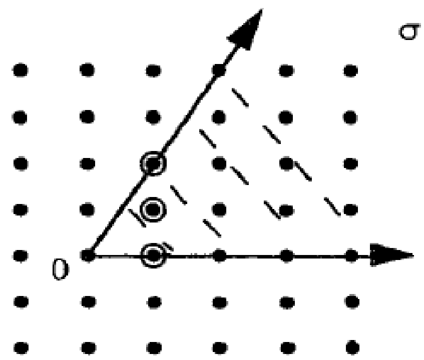
- $V(x^3 - y^2)$ with torus $T = \left\{ [t^2, t^3] \;\middle|\; t \in \mathbf{C}^\times \right\}$.
- $V(xy - zw)$ with torus $T = \left\{ [a, b, c, abc^{-1}] \;\middle|\; a, b, c, d \in \mathbf{C}^\times \right\}$.
- $V(xz - y^2)$, note $V(x, y) \in \mathrm{Div}(X) \setminus \mathrm{CDiv}(X)$.
- $\mathrm{im}([x : y] \mapsto [x^3 : x^2 y : xy^2 : y^3])$ the twisted cubic. Corresponds to $\sigma^\vee = \{(3,0), (2,1), (1,2), (0,3)\}$.

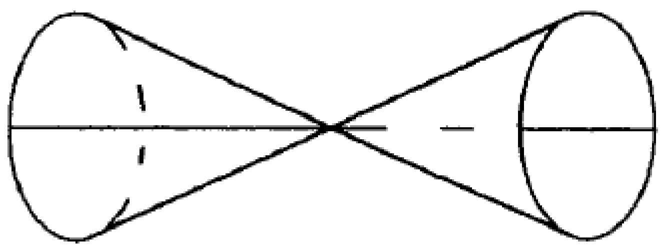
- The **rational normal scroll**: $V\left(2\times 2 \text{ minors of } \begin{bmatrix} x_0 & x_1 & y_0 \\ x_1 & x_2 & y_1 \end{bmatrix}\right)$ is the image of $[s,t]\mapsto$ $[1:s:s^2:t:st]$.
- The Segre variety: $\operatorname{Spec}\mathbf{C}[x_1y_1, x_1y_2,\cdots,x_1y_n,x_2y_1,\cdots,x_my_1,\cdots x_my_n]$.

Example 6.1.10 ***(of fans):***

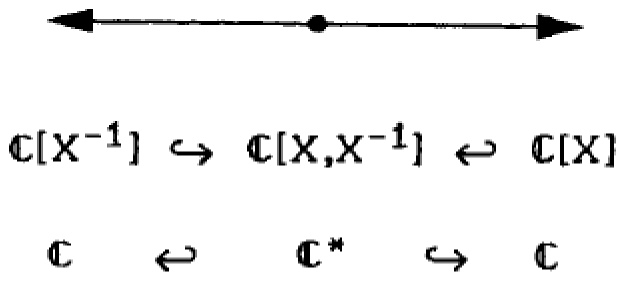
- $(\mathbf{C}^\times)^n$: Take $\Delta=\{\sigma_0=\mathbb{N}\langle 0\rangle\}\subseteq N$ with $\dim N=n$ yields $S_{\sigma_0}=\mathbb{N}\langle \pm e_1{}^\vee,\cdots,\pm e_n{}^\vee\rangle = M$ for so $X(\Delta)=\operatorname{Spec}\mathbf{C}[x_1^{\pm 1},\cdots,x_n^{\pm 1}]=(\mathbf{G}_m)^n$.
- $\mathbf{C}^n$: Take $\Delta=\operatorname{Cone}(\sigma_0=\mathbb{N}\langle e_1,\cdots,e_n\rangle)$ yields the positive orthant $S_{\sigma_0}=\mathbb{N}\langle e_1{}^\vee,\cdots,e_n{}^\vee\rangle\subseteq M$, so $X(\Delta)=\operatorname{Spec}\mathbf{C}[x_1,\cdots,x_n]=\mathbf{A}^n$.
- The quadric cone: $\Delta=\operatorname{Cone}(\sigma_1=\mathbb{N}\langle e_2, 2e_1-e_2\rangle)$ yields $S_{\sigma_1}=\mathbb{N}\langle e_1{}^\vee, e_1{}^\vee+e_2{}^\vee, e_1{}^\vee+2e_2{}^\vee\rangle$ so $X(\Delta)=\operatorname{Spec}\mathbf{C}[x,xy,xy^2]=\operatorname{Spec}\mathbf{C}[u,v,w]/(v^2-uw)$:



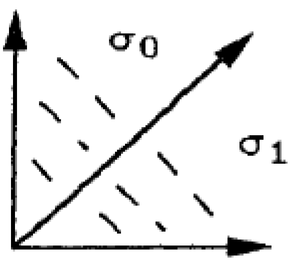




- $\mathbf{P}^1$: Take $\Delta=\{\mathbf{R}_{\geq 0},0,\mathbf{R}_{\leq 0}\}$ and glue along overlaps to get $X(\Delta)=\mathbf{P}^1$ with gluing maps $x\mapsto x^{-1}$:

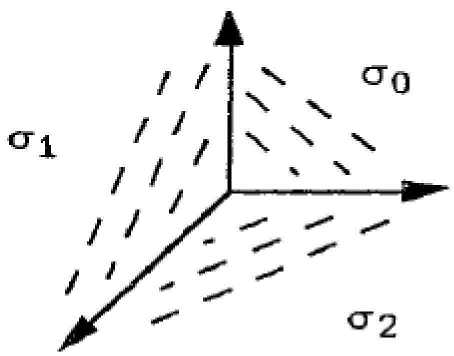


- $\mathrm{Bl}_1 \mathbf{C}^2$: Take $\sigma_0 = \mathbb{N}\langle e_2, e_1 + e_2\rangle$ and $\sigma_1 = \mathbb{N}\langle e_1 + e_2, e_1\rangle$ to get $U_{\sigma_0} = \operatorname{Spec} \mathbf{C}[x, x^{-1}y]$ and $U_{\sigma_1} = \operatorname{Spec} \mathbf{C}[y, xy^{-1}]$, both copies of $\mathbf{C}^2$:

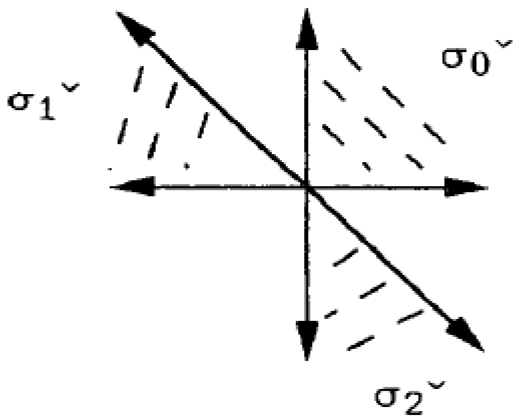


Why this is a blowup of $\mathbf{C}^2$: write $\mathrm{Bl}_1 \mathbf{C}^2 = V(xt_1 - yt_0) \subseteq \mathbf{C}^2 \times \mathbf{P}^1$ for $\mathbf{P}^1 = \{[t_0 : t_1]\}$. Take the open cover $U_i = D(t_i) \cong \mathbf{C}^2$, where coordinates on U_0 are $x, t_1/t_0 = x^{-1}y$ and on U_1 are $y, t_0/t_1 = xy^{-1}$ and glue.

- $\mathbf{P}^2$: take $\Delta = \operatorname{Cone}(e_1, e_2, -e_1 - e_2)$:



This has dual cone:

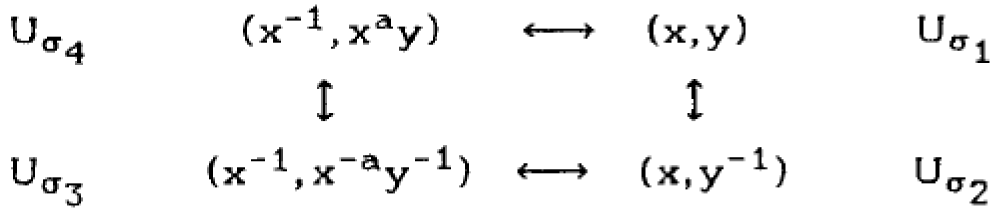


Each $U_{\sigma_i} \cong \mathbf{C}^2$ with coordinates $(x, y), (x^{-1}, x^{-1}y), (y^{-1}, xy^{-1})$ respectively for U_i. Glue to obtain $x = t_1/t_0, y = t_2/t_0$.

- F_a the Hirzebruch surface: take $\text{Cone}(e_1, -e_2, -e_1, -e_1 + ae_2)$ to get

  - $U_{\sigma_1} = \text{Spec}\,\mathbf{C}[x, y]$,
  - $U_{\sigma_2} = \text{Spec}\,\mathbf{C}[x, y^{-1}]$,
  - $U_{\sigma_3} = \text{Spec}\,\mathbf{C}[x^{-1}, x^{-a}y^{-1}]$,
  - $U_{\sigma_4} = \text{Spec}\,\mathbf{C}[x^{-1}, x^a y]$,

  which patch in the following way:



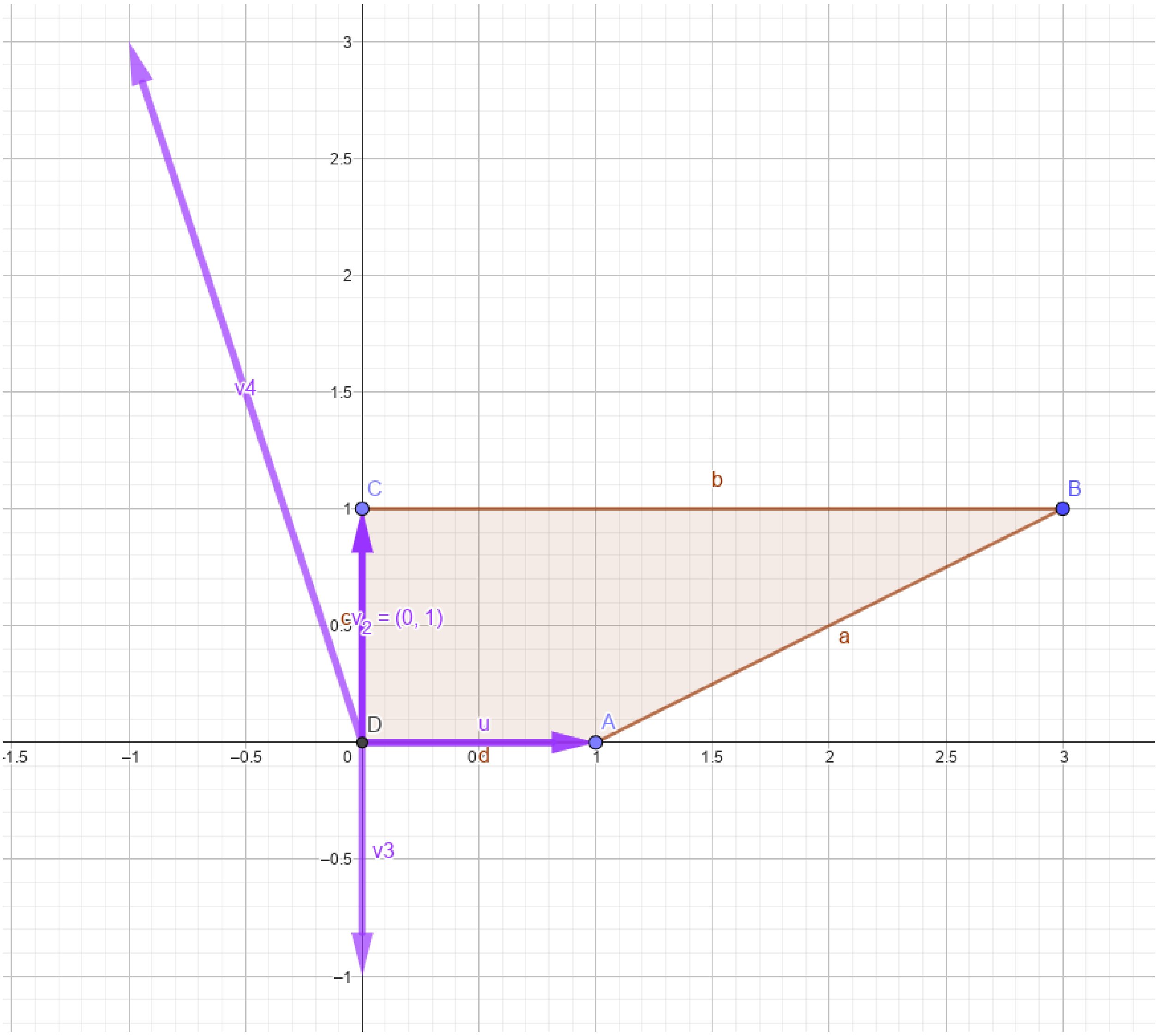
Project to $y = 0$ to get the patching $x \mapsto x^{-1}$, so a copy of $\mathbf{P}^1$. Patching in the fiber direction, e.g. U_{σ_1} and U_{σ_2}, gives a copy of $\mathbf{C} \times \mathbf{P}^1$. Thus this is a bundle $\mathbf{P}^1 \to \mathcal{E} \to \mathbf{P}^1$.

- $\mathbf{C} \times \mathbf{P}^1$: todo.
- $\mathbf{P}^1 \times \mathbf{P}^1$: todo.
- $\mathbf{C}^a \times \mathbf{P}^b$: todo.

- $\mathbf{P}^a \times \mathbf{P}^b$: todo.

Example 6.1.11 ***(of polytopes):***

- Hirzebruch surfaces:



- $(\mathbf{P}^2, \mathcal{O}(1))$: take $P = *(0, e_1, e_2)$, so $X_P = \mathrm{cl}\Phi_P$ where

$$\begin{aligned}\Phi_P : (\mathbf{C}^\times)^2 &\to \mathbf{P}^2 \\ (s,t) &\mapsto [1 : s : t],\end{aligned}$$

  which is the identity embedding corresponding to $\mathcal{O}(1)$ on $\mathbf{P}^2$.

  - $2P$ yields

$$\begin{aligned}\Phi_{2P} : (\mathbf{C}^\times)^2 &\to \mathbf{P}^5 \\ (s,t) &\mapsto [1 : s : t : s^2 : st : t^2],\end{aligned}$$

    the Veronese embedding corresponding to $\mathcal{O}(2)$ on $\mathbf{P}^2$.

Example 6.1.12 ***(Projective spaces):*** Some useful facts about $\mathbf{P}^n$:

- The torus embedding is

$$(\mathbf{C}^\times)^n \hookrightarrow \mathbf{P}^n$$
$$[a_1, \cdots, a_n] \mapsto [1 : a_1 : \cdots : a_n].$$

- The torus action is

$$(\mathbf{C}^\times)^n \curvearrowright \mathbf{P}^n$$
$$[t_1, \cdots, t_n].[x_0 : x_1 : \cdots : x_n] = [x_0 : t_1 x_1 : \cdots : t_n x_n].$$

$j=1$

Example IV.3.2. The fan of $\mathbb{P}^2$ has rays generate
is

$$\mathrm{Cl}(\mathbb{P}^2) = \mathrm{coker}\left(\mathbb{Z}^2 \right.$$

Example IV.3.3. The fan of a Hirzebruch surface h
Its class group is

$$\mathrm{Cl}(\text{Hirzebruch surface}) = \mathrm{coker}$$

Example IV.3.4. The fan with rays generated by
$\mathbb{P}^1 \xrightarrow{|\mathcal{O}(d)|} \mathbb{P}^d$. Its class group is $\mathrm{coker}\begin{bmatrix} d & -1 \\ 0 & 1 \end{bmatrix} =$

Example 6.1.13 ***(of class groups and Picard groups):***

Example IV.3.11. Consider $X_\Sigma = \mathrm{Cone}(\mathbb{P}^1 \times \mathbb{P}^1) = V(xz - yw) \subseteq \mathbb{C}^4$, where Σ has rays $(e_1, e_2, e_1 + e_3, e_2 + e_3)$. Then

$$\mathrm{Cl}(X_\Sigma) = \mathrm{coker}\left(\mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} \mathbb{Z}^4\right) = \mathbb{Z}$$

is generated by D_ρ for any ray ρ. However, $\mathrm{Pic}(X_\Sigma) = 0$; this is an exercise in the book.

7 I: Definitions and Examples

7.1 1.1: Introduction

Remark 7.1.1: Machinery used to study varieties:

- Various cohomology theories
- Resolutions of singularities
- Intersection theory and cycles
- Riemann-Roch theorems
- Vanishing theorems
- Linear systems (via line bundles and projective embeddings)

Varieties that arise as examples

- Grassmannians
- Flag varieties
- Veronese embeddings
- Scrolls
- Quadrics
- Cubic surfaces
- Toric varieties (of course)
- Symmetric varieties and their compactifications

Misc notes:

- Toric varieties are always rational

Remark 7.1.2:

- Toric varieties: normal varieties X with $T \hookrightarrow X$ contained as a dense open subset where the torus action $T \times T \to T$ extends to $T \times X \to X$.
- Any product of copies of $\mathbf{A}^n, \mathbf{P}^m$ are toric.
- S_σ is a finitely-generated semigroup, so $\mathbf{C}[S_\sigma] \in \mathsf{Alg}\mathbf{C}^{\text{fg}}$ corresponds to an affine variety $U_\sigma := \operatorname{Spec} \mathbf{C}[S_\sigma]$.
- If $\tau \leq \sigma$ is a face then there is a map of affine varieties $U_\tau \to U_\sigma$ where $U_\tau = D(u_\tau)$ is a principal open subset given by the function u_τ picked such that $\tau = \sigma \cap u_\tau^\perp$, so u_τ corresponds to the orthogonal normal vector for the wall τ.
- These glue to a variety $X(\Delta)$.
- Smaller cones correspond to smaller open subsets.
- The geometry in N is nicer than that in M, usually.
- Rays ρ correspond to curves D_ρ.

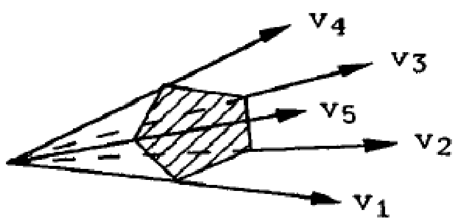
Exercise 7.1.3 (?)

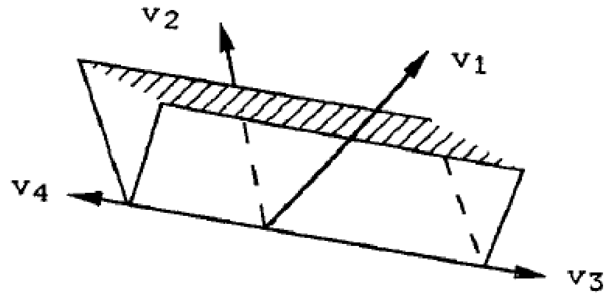
- Show $F_a \to \mathbf{P}^1$ is isomorphic to $\mathbf{P}(\mathcal{O}(a) \oplus \mathcal{O}(1))$.
- Let τ be the ray through e_2 in F_a and show $D_\tau^2 = -a$.
- Show that the normal bundle to $D_\tau \hookrightarrow F_a$ is $\mathcal{O}(-a)$.

7.2 1.2: Convex Polyhedral Cones

Remark 7.2.1:

- **Convex polyhedral cones**: generated by vectors $\sigma = \mathbf{R}_{\geq 0} \langle v_1, \cdots, v_n \rangle$. Can take minimal vectors along these rays, say ρ_i.





- $\dim \sigma \coloneqq \dim_{\mathbf{R}} \mathbf{R}\sigma \coloneqq \dim_{\mathbf{R}}(-\sigma + \sigma)$
- $(\sigma^\vee)^\vee = \sigma$, which follows from a general theorem: for σ a convex polyhedral cone and $v \notin \sigma$, there is some support vector $u_v \in \sigma^\vee$ such that $\langle u, \ v \rangle < 0$. I.e. v is on the negative side of some hyperplane defined in $\sigma^\vee$.
- Faces are again convex polyhedral cones, faces are closed under intersections and taking further faces.
- If σ spans V and τ is a facet, there is a unique $u_\tau \in \sigma^\vee$ such that $\tau = \sigma \cap u_\tau^\perp$; this defines an equation for the hyperplane H_τ spanned by τ.
- If σ spans V and $\sigma \neq V$, then $\sigma = \cap_{\tau \in \Delta} H_\tau^+$, the intersection of positive half-spaces.
  - An alternative presentation: picking $u_1, \cdots, u_t$ generators of $\sigma^\vee$, one has $\sigma = \left\{ v \in N \;\middle|\; \langle u_1, \ v \rangle \geq 0, \cdots \right.$
- If $\tau \leq \sigma$ then $\sigma^\vee \cap \tau^\vee \leq \sigma^\vee$ and $\dim \tau = \operatorname{codim}(\sigma^\vee \cap \tau^\vee)$, so the faces of $\sigma, \sigma^\vee$ biject contravariantly.
- If $\tau = \sigma \cap u_\tau^\perp$ then $S_\tau = S_\sigma + \mathbb{N} \langle -u_\tau \rangle$.

8 | Singularities and Compactness

8.1 2.1

Remark 8.1.1:

- Any cone $\sigma \in \Sigma$ has a distinguished point x_σ corresponding to $\operatorname{Hom}(S_\sigma, \mathbf{C})$ where $u \mapsto \chi_{u \in \sigma^\perp}$.
  - Note $S_\sigma \coloneqq \sigma^\vee \cap M$.

- Define $A_\sigma \coloneqq \mathbf{C}[S_\sigma]$.

- Finding singular points:

  - Easy case: σ spans $N_{\mathbf{R}}$ so $\sigma^\perp = 0$; consider $\mathfrak{m} \in \operatorname{mSpec} A_\sigma$ be the maximal ideal at x_σ, then $\mathfrak{m} = \left\langle \chi^u \;\middle|\; u \in S_\sigma \right\rangle$ and $\mathfrak{m}^2 = \left\langle \chi^u \;\middle|\; u \in S_\sigma \setminus \{0\} + S_\sigma \setminus \{0\} \right\rangle$, so $\mathbf{T}_{x_\sigma}{}^\vee U_\sigma = \mathfrak{m}/\mathfrak{m}^2 = \left\{ \chi^u \;\middle|\; u \notin S_\sigma \setminus \{0\} + S_\sigma \setminus \{0\} \right\}$, i.e. "primitive" elements u which are not the sums of two other vectors in $S_\sigma \setminus \{0\}$.
  - Nonsingular implies $\dim U_\sigma = n$, so $\sigma^\vee$ has $\leq n$ edges since each minimal ray generator yields a primitive u above. Also implies minimal edge generators must generate S_σ, thus must be a basis for M, so σ must be a basis for N and $U_\sigma \cong \mathbf{A}^n$.

- **Characterization of smoothness**: U_σ is smooth iff σ is generated by a subset of a lattice basis for N, in which case $U_\sigma \cong \mathbf{A}^k \times \mathbf{G}_m^{n-k}$.

- All toric varieties are normal since each A_σ is integrally closed.

  - If $\sigma = \langle v_1, \cdots, v_r \rangle$ then $\sigma^\vee = \cap_{i=1}^r \tau_i{}^\vee$ where τ_i is the ray along v_i. Thus $A_\sigma = \cap A_{\tau_i}$, each of which is isomorphic to $\mathbf{C}[x_1, x_2^{\pm 1}, \cdots, x_n^{\pm 1}$ which is integrally closed.

- All toric varieties are **Cohen-Macaulay**: each local ring R has depth n, i.e. contains a regular sequence of length $n = \dim R$.

- All vector bundles on affine toric varieties are trivial, equivalently all projective modules over A_σ are free.

8.2 2.2

Remark 8.2.1:

- An example: $\Sigma = \operatorname{Cone}(me_1 - e_2, e_2)$. Then $A_\sigma = \mathbf{C}[x, xy, xy^2, \cdots, xy^m] = \mathbf{C}[u^m u^{m-1}v, \cdots, uv^{m-1}, v^m]$ and U_σ is the cone over the rational normal curve of degree m.

  - Note $A_\sigma = \mathbf{C}[u,v]^{\mu_m}$ is the ring of invariants under the diagonal action $\zeta.[u,v] = [\zeta u, \zeta v]$.

- If Σ is simplicial, then X_Σ is at worst an orbifold.

8.3 2.3

Remark 8.3.1:

- $\operatorname{Hom}_{\mathsf{AlgGrp}}(\mathbf{G}_m, \mathbf{G}_m) = \mathbf{Z}$ using $n \mapsto (z \mapsto z^n)$.

- Cocharacters:

- Pick a basis for N to get $\mathrm{Hom}(\mathbf{G}_m, T_N) = \mathrm{Hom}(\mathbf{Z}, N) = N$, then every cocharacter $\lambda \in \mathrm{Hom}(\mathbf{G}_m, T_N)$ is given by a unique $v \in N$, so denote it λ_v. Then $\lambda_v(z) \in T_N = \mathrm{Hom}(M, \mathbf{G}_m)$ for any $z \in \mathbf{C}^\times$, so

$$u \in M \implies \lambda_v(z)(u) = \chi^u(\lambda_v(z)) = z^{\langle u,\ v\rangle}.$$

- Characters: $\chi \in \mathrm{Hom}(T_n, \mathbf{G}_m) = \mathrm{Hom}(N, \mathbf{Z}) = M$ is given by a unique $u \in M$ and can be identified with $^u \in \mathbf{C}[M] = H^0(T_N, \mathcal{O}_{T_N}^\times)$.

- $\lim_{z\to 0} \lambda_v(z) = \lim_{z\to 0} [z^{m_1}, \cdots, z^{m_n}] \in U_\sigma \iff m_i \geq 0$ for all i, and if $U_\sigma = \mathbf{A}^k \times \mathbf{G}_m^{n-k}$, $m_i = 0$ for $i > k$. This happens iff $v \in \sigma$, and the limit is $[\delta_1, \cdots, \delta_n]$ where $\delta_i = 1 \iff m_i = 0$ and $\delta_i = 0 \iff m_i > 0$; each of which is a distinguished point x_τ for some face τ of σ.

- Summary: $v \in |\Sigma|$ and $v \in \tau^\circ$ then $\lim_{z\to 0} \lambda_v(z) = x_\tau$, and the limit does not exist for $v \notin |\Sigma|$.

8.4 2.4

Remark 8.4.1:

- Recall X is compact in the Euclidean topology iff it is complete/proper in the Zariski topology, i.e. the map to a point is proper.

- X_Σ is compact iff $|\Sigma| = N_{\mathbf{R}}$, i.e. Σ is complete.

- Any morphism of lattices $\varphi : N \to N'$ inducing a map of fans $\Sigma \to \Sigma'$ defines a morphism $X_\Sigma \to X_{\Sigma'}$ which is proper iff $\varphi^{-1}(|\Sigma'|) = |\Sigma|$. Thus X_Σ is compact iff $\varphi : N \to 0$ is a proper morphism.

- Blowing up at x_σ: take a basis $\{v_i\}$, set $v_0 := \sum v_i$, and replace σ by all subsets of $\{v_0, v_1, \cdots, v_n\}$ not containing $\{v_1, \cdots, v_n\}$.