Real Analysis Qualifying Examination
August 2019

The five problems on this exam have equal weighting.
To receive full credit give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

1. Let $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers.

   (a) Prove that if $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} \dfrac{a_1 + \cdots + a_n}{n} = 0$.

   (b) Prove that if $\sum_{n=1}^{\infty} \dfrac{a_n}{n}$ converges, then $\lim_{n\to\infty} \dfrac{a_1 + \cdots + a_n}{n} = 0$.

2. Prove that $\left| \dfrac{d^n}{dx^n} \dfrac{\sin x}{x} \right| \leq \dfrac{1}{n}$ for all $x \neq 0$ and positive integers n.

   Hint: Consider $\displaystyle\int_0^1 \cos(tx)\, dt$.

3. Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$, $\{B_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{B}$-measurable subsets of X, and $B := \{x \in X \,:\, x \in B_n \text{ for infinitely many } n\}$.

   (a) Argue that B is also a $\mathcal{B}$-measurable subset of X.

   (b) Prove that if $\sum_{n=1}^{\infty} \mu(B_n) < \infty$, then $\mu(B) = 0$.

   (c) Prove that if $\sum_{n=1}^{\infty} \mu(B_n) = \infty$ and the sequence of set complements $\{B_n^c\}_{n=1}^{\infty}$ satisfies

   $$\mu\left(\bigcap_{n=k}^{K} B_n^c \right) = \prod_{n=k}^{K} (1 - \mu(B_n))$$

   for all positive integers k and K with $k < K$, then $\mu(B) = 1$.

   Hint: Use the fact that $1 - x \leq e^{-x}$ *for all* x.

4. Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$.

   (a) Prove that for every $x \in \mathcal{H}$ one has $\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$.

   (b) Prove that for any sequence $\{a_n\}_{n=1}^{\infty}$ in $\ell^2(\mathbb{N})$ there exists an element x in $\mathcal{H}$ such that $a_n = \langle x, u_n \rangle$ for all $n \in \mathbb{N}$ and $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$.

5. (a) Show that if f is continuous with compact support on $\mathbb{R}$, then

   $$\lim_{y\to 0} \int_{\mathbb{R}} |f(x-y) - f(x)|\, dx = 0.$$

   (b) Let $f \in L^1(\mathbb{R})$ and for each $h > 0$ let $\mathcal{A}_h f(x) := \dfrac{1}{2h} \displaystyle\int_{|y|\leq h} f(x-y)\, dy$.

   i. Prove that $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$ for all $h > 0$.

   ii. Prove that $\mathcal{A}_h f \to f$ in $L^1(\mathbb{R})$ as $h \to 0^+$.

Question 1

(a) Let $\varepsilon > 0$. We know $\exists\, N \in \mathbb{N}$ such that $|a_n| < \frac{\varepsilon}{2}$ whenever $n > N$. It follows from this that

$$\frac{a_{N+1} + \cdots + a_n}{n} < \frac{\varepsilon}{2} \quad \text{whenever } n > N.$$

Since $\frac{a_1 + \cdots + a_N}{n} < \frac{\varepsilon}{2}$ whenever $n > \frac{2(|a_1| + \cdots + |a_N|)}{\varepsilon}$

it follows that $\frac{a_1 + \cdots + a_n}{n} < \varepsilon$ provided $n > \max\left\{N, \frac{2(|a_1| + \cdots + |a_N|)}{\varepsilon}\right\}$

(b) Let $r_n = \frac{a_{n+1}}{n+1} + \frac{a_{n+2}}{n+2} + \cdots \quad \forall\, n \geq 0$.

Since $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges we know $r_n \to 0$ as $n \to \infty$.

The observation that

$$r_0 + r_1 + \cdots + r_{n-1} = a_1 + a_2 + \cdots + a_n + n r_n$$

together with part (a) ensures that

$$\frac{a_1 + \cdots + a_n}{n} = \frac{r_0 + \cdots + r_{n-1}}{n} - r_n \longrightarrow 0.$$

Question 2

By differentiating under the integral sign we have

$$\left| \frac{d^n}{dx^n} \frac{\sin x}{x} \right| = \left| \int_0^1 t^n \frac{d^n}{dx^n} \cos(tx)\, dt \right| \leq \int_0^1 t^n dt = \frac{1}{n}$$

for all $n \in \mathbb{N}$.

To be complete, one should note that differentiation under the integral sign is legitimate since for $f(x) = \cos(tx)$ or $\sin(tx)$ one can check (using the Mean Value Theorem) that the difference quotients $\frac{f(x+h) - f(x)}{h}$ are uniformly bounded (by 1) for all $0 \leq t \leq 1$ and $h \in \mathbb{R} \setminus \{0\}$.

Question 3

(a) B is $\mathcal{B}$-measurable since $B = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n$

(b) By monotonicity & sub-addivity we have

$$\mu(B) \leq \mu\left(\bigcup_{n=k}^{\infty} B_n\right) \leq \sum_{n=k}^{\infty} \mu(B_n) \quad \forall k$$

and $\sum_{n=k}^{\infty} \mu(B_n) \to 0$ as $k \to \infty$ if $\sum_{n=1}^{\infty} \mu(B_n)$ converges.

(c) We must show that

$$\mu(B^c) = \mu\left(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} B_n^c\right) = 0$$

and for this it is enough to show $\mu\left(\bigcap_{n=k}^{\infty} B_n^c\right) = 0 \ \forall k$.

Since $\mu\left(\bigcap_{n=k}^{K} B_n^c\right) = \prod_{n=k}^{K} (1 - \mu(B_n))$ & $1 - x \leq e^{-x} \ \forall x \in \mathbb{R}$

it follows that

$$\mu\left(\bigcap_{n=k}^{K} B_n^c\right) \leq \prod_{n=k}^{K} e^{-\mu(B_n)} = \exp\left(-\sum_{n=k}^{K} \mu(B_n)\right).$$

The result follows since $\sum_{n=k}^{K} \mu(B_n) \to \infty$ as $K \to \infty$

for each fixed k if $\sum_{n=1}^{\infty} \mu(B_n)$ diverges.

<u>Question 4</u> Let $S_N = \sum_{n=1}^{N} \langle x, u_n \rangle u_n \quad \forall N$.

(a) $0 \leq \|x - S_N\|^2 = \|x\|^2 - 2\operatorname{Re}\langle x, S_N\rangle + \|S_N\|^2$

Using Pythagorus

$$= \|x\|^2 - 2\sum_{n=1}^{N} |\langle x, u_n\rangle|^2 + \sum_{n=1}^{N} |\langle x, u_n\rangle|^2$$

$$= \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n\rangle|^2$$

$$\Rightarrow \sum_{n=1}^{N} |\langle x, u_n\rangle|^2 \leq \|x\|^2 \quad \forall N.$$

(b) We now let $S_N = \sum_{n=1}^{N} a_n u_n \quad \forall N$.

It is easy to see that $\{S_N\}$ is Cauchy in $\mathcal{H}$:

$$\|S_N - S_M\|^2 = \Big\|\sum_{n=M+1}^{N} a_n u_n\Big\|^2 \underset{\text{Pythagorus}}{=} \sum_{n=M+1}^{N} |a_n|^2 \to 0 \text{ as } N, M \to \infty.$$

Since $\mathcal{H}$ is complete it follows that $S_N \to x$ (say) in $\mathcal{H}$.

- Now $\langle x, u_n\rangle - a_n = \langle x, u_n\rangle - \langle S_N, u_n\rangle \quad \text{if } N \geq n$

$$= \langle x - S_N, u_n\rangle \to 0 \text{ as } N \to \infty$$

(since $|\langle x - S_N, u_n\rangle| \leq \|x - S_N\| \to 0$ as $N \to \infty$)

- Finally, since $\|x - S_N\|^2 \overset{\text{Part (a) above}}{=} \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n\rangle|^2 \quad \forall N$

$$\Rightarrow \sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 = \|x\|^2.$$

since $\|x - S_N\| \to 0$ as $N \to \infty$

Question 5

(a) If f is continuous with compact support on $\mathbb{R}$, then for $|y| \leq 1$ the functions $f(\cdot - y)$ are all supported in some common compact set K.

We also know that f is uniformly continuous & hence

$$\int_{\mathbb{R}} |f(x-y) - f(x)|\,dx \leq \sup_{x \in K} |f(x-y) - f(x)|\; m(K)$$

$$\longrightarrow 0 \text{ as } y \to 0 .$$

(b) (i) $$\int_{\mathbb{R}} |A_h f(x)|\,dx \leq \int_{\mathbb{R}} \frac{1}{2h} \int_{|y|\leq h} |f(x-y)|\,dy\,dx$$

By Tonelli, since $\frac{1}{2h} \chi_{|y|\leq h}(y)\, |f(x-y)|$ is m'ble on $\mathbb{R}^2$.

$$= \frac{1}{2h} \int_{|y|\leq h} \left(\int_{\mathbb{R}} |f(x-y)|\,dx \right) dy$$

$$= \|f\|_1 \quad \text{by trans. invariance.}$$

(ii) Since $\|A_h f - f\|_1 \leq 2\|f\|_1 \;\forall h>0$ & $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ it suffices to show $A_h f \to f$ in $L^1(\mathbb{R})$ as $h \to 0^+$ if $\underline{\underline{f \in C_c(\mathbb{R})}}$.

Let $\varepsilon > 0$. By (a) we know $\exists\, \delta > 0$ such that

$$\int_{\mathbb{R}} |f(x-y) - f(x)|\,dx < \varepsilon \quad \text{provided } |y| < \delta .$$

$$\Rightarrow \int_{\mathbb{R}} |A_h f(x) - f(x)|\,dx \leq \int_{\mathbb{R}} \frac{1}{2h} \int_{|y|\leq h} |f(x-y) - f(x)|\,dy\,dx$$

By Tonelli as above

$$= \frac{1}{2h} \int_{|y|\leq h} \|f(\cdot - y) - f\|_1\,dy < \varepsilon \quad \text{if } h < \delta .$$