Real Analysis Qualifying Examination

Spring 2019

The five problems on this exam have equal weighting. To receive full credit give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

1. Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$.
   (a) Prove that $C([0,1])$ is complete under the uniform norm $\|f\|_u := \sup_{x\in[0,1]} |f(x)|$.
   (b) Prove that $C([0,1])$ is not complete under the L^1-norm $\|f\|_1 = \int_0^1 |f(x)|\,dx$

2. Let $\mathcal{B}$ denote the set of all Borel subsets of $\mathbb{R}$ and $\mu : \mathcal{B} \to [0,\infty)$ denote a finite Borel measure on $\mathbb{R}$.
   (a) Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all k, then
   $$\lim_{k\to\infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right).$$
   (b) Suppose μ has the property that $\mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$. Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \varepsilon$.

3. Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \le M$ for all $k \in \mathbb{N}$.
   Prove that if $f_k \to f$ almost everywhere, then $f \in L^2([0,1])$ with $\|f\|_2 \le M$ and
   $$\lim_{k\to\infty} \int_0^1 f_k(x)\,dx = \int_0^1 f(x)\,dx.$$
   Hint: Try using Fatou's Lemma to show that $\|f\|_2 \le M$ and then try applying Egorov's Theorem.

4. Let f be a non-negative function on $\mathbb{R}^n$ and $\mathcal{A} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \,:\, 0 \le t \le f(x)\}$.
   Prove the validity of the following two statements:
   (a) f is a Lebesgue measurable function on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$
   (b) If f is a Lebesgue measurable function on $\mathbb{R}^n$, then
   $$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty m(\{x \in \mathbb{R}^n \,:\, f(x) \ge t\})\,dt$$

5. (a) Show that $L^2([0,1]) \subseteq L^1([0,1])$ and argue that $L^2([0,1])$ in fact forms a dense subset of $L^1([0,1])$.
   (b) Let Λ be a continuous linear functional on $L^1([0,1])$.
   Prove the *Reisz Representation Theorem for* $L^1([0,1])$ by following the steps below:
      i. Establish the existence of a function $g \in L^2([0,1])$ which represents Λ in the sense that
      $$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}\,dx \quad \text{for all } f \in L^2([0,1]).$$
      Hint: You may use, without proof, the Riesz Representation Theorem for $L^2([0,1])$.
      ii. Argue that the g obtained above must in fact belong to $L^\infty([0,1])$ and represent Λ in the sense that
      $$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}\,dx \quad \text{for all } f \in L^1([0,1])$$
      with
      $$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^*}.$$

Question 1

(a) Let $\varepsilon > 0$ & $\{f_n\}$ be a Cauchy sequence in $C([0,1])$.

This means that $\exists N$ such that

$$n, m \geq N \Rightarrow \sup_{x \in [0,1]} |f_n(x) - f_m(x)| < \varepsilon$$

Note that N is independent of x

In particular, for every $x \in [0,1]$

$$|f_n(x) - f_m(x)| < \varepsilon \quad \text{provided } n, m \geq N \quad (*)$$

i.e. $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ and hence converges.

Let $f(x) := \lim_{n \to \infty} f_n(x)$ for each $x \in [0,1]$.

We must show $f_n \to f$ uniformly on $[0,1]$. Fixing n and letting $m \to \infty$ in $(*)$ above we obtain that

$$|f_n(x) - f(x)| \leq \varepsilon \quad \text{for every } n \geq N \text{ and every } x \in [0,1].$$

i.e. $\lim_{n \to \infty} \sup_{x \in [0,1]} |f_n(x) - f(x)| = 0 \iff f_n \to f$ unif on $[0,1]$.

Finally, since each $f_n \in C([0,1])$ & $f_n \to f$ uniformly on $[0,1]$ it follows that f is continuous on $[0,1]$ also!

(b) Let $f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ n(x - \frac{1}{2}) & \text{if } \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \text{if } \frac{1}{2} + \frac{1}{n} \leq x \leq 1. \end{cases} \Rightarrow \|f_n - f_m\|_1 \leq \max\{\frac{1}{n}, \frac{1}{m}\}$

So $\{f_n\}$ Cauchy in L^1.

Then $f_n \xrightarrow{L^1} f$ where $f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$ $\Leftarrow$ not continuous!

Question 2

(a) Let $G_k = F_k \setminus F_{k+1}$ for each k, so that

$$F_1 = F \cup \bigcup_{k=1}^{\infty} G_k \qquad \text{where } F = \bigcap_{k=1}^{\infty} F_k .$$

is a disjoint union of Borel sets.

It follows that

$$\mu(F_1) = \mu(F) + \lim_{N\to\infty} \sum_{k=1}^{N-1} \left(\mu(F_k) - \mu(F_{k+1}) \right)$$

$$= \mu(F) + \mu(F_1) - \lim_{N\to\infty} \mu(F_N)$$

Since $\mu(F_1) < \infty \Rightarrow \mu(F) = \lim_{N\to\infty} \mu(F_N)$.

(b) Suppose <u>not</u>, then $\exists\ \varepsilon_0 > 0$ such that $\forall n \in \mathbb{N}$ $\exists$ Borel set E_n with $m(E_n) < \frac{1}{2^n}$, but $\mu(E_n) \geq \varepsilon_0$.

Now let $F = \bigcap_{k=1}^{\infty} F_k$, where $F_k = \bigcup_{n=k}^{\infty} E_n$.

$$\Rightarrow \quad m(F_k) \leq \sum_{n=k}^{\infty} m(E_n) < \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}}$$

and hence that $m(F) = 0$.

However $\mu(F_k) \geq \varepsilon_0$ for all k and hence, since μ is finite that $\mu(F) = \lim_{k\to\infty} \mu(F_k) \geq \varepsilon_0$

since $F_k \supseteq F_{k+1}\ \forall k$.

i.e. μ is <u>not</u> absolutely continuous with respect to m.

Question 3

- Since $f_k \to f$ a.e. $\Rightarrow |f_k|^2 \to |f|^2$ a.e.

  It then follows from Fatou's Lemma that

$$\underbrace{\int \liminf_{k\to\infty} |f_k|^2}_{= \int |f|^2} \leq \underbrace{\liminf_{k\to\infty} \int |f_k|^2}_{\leq M^2} \quad \text{since } \|f_k\|_2 \leq M \ \forall k.$$

$$\Rightarrow \|f\|_2 \leq M.$$

- Let $\varepsilon > 0$. Since $f_k \to f$ a.e. it follows from Egorov that

  $\exists$ closed set A with $m(A) < \varepsilon$ such that

  $f_k \to f$ uniformly on $[0,1] \setminus A$.

$$\Rightarrow \int_0^1 |f_k(x) - f(x)|\,dx = \underbrace{\int_{[0,1]\setminus A} |f_k(x) - f(x)|\,dx}_{\to 0 \text{ as } k\to\infty} + \underbrace{\int_A |f_k(x) - f(x)|\,dx}_{\substack{\leq m(A)^{1/2} \|f_k - f\|_2 \\ < \varepsilon^{1/2} 2M}}$$

Cauchy-Schwarz

Therefore

$$\lim_{k\to\infty} \int |f_k(x) - f(x)|\,dx = 0.$$

Question 4

(a) ($\Rightarrow$) : Since $f \geq 0$ measurable on $\mathbb{R}^n$ it follows that

$$\tilde{F}(x,y) = y - f(x) \text{ is m'ble on } \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}.$$

$[\tilde{F}(x,y) = G(x,y) - F(x,y)$ where $G(x,y) = y$ & $F(x,y) = f(x).]$

and hence that $\mathcal{A} = \{y \geq 0\} \cap \{\tilde{F} \leq 0\}$ is m'ble.

($\Leftarrow$) : Suppose $\mathcal{A}$ measurable in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$.

For each $x \in \mathbb{R}^n$ the slice $\mathcal{A}_x = \{y \in \mathbb{R} : (x,y) \in \mathcal{A}\} = [0, f(x)]$

Tonelli $\Rightarrow$ $\mathcal{A}_x$ is a m'ble subset of $\mathbb{R}$ (for a.e x)

with $m(\mathcal{A}_x)$ a m'ble function of x in $\mathbb{R}^n$

$\Rightarrow$ f is m'ble on $\mathbb{R}^n$.

Moreover,

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} m(\mathcal{A}_x)\,dx = \int_{\mathbb{R}^n} f(x)\,dx.$$

(b) In light of the final consequence of Tonelli above we need only establish:

Claim: $$m(\mathcal{A}) = \int_0^\infty m(\{x \in \mathbb{R}^n : f(x) \geq y\})\,dy.$$

Proof As above we record that Tonelli implies that for a.e. $y \in \mathbb{R}$ the slice $\mathcal{A}_y = \{x \in \mathbb{R}^n : (x,y) \in \mathcal{A}\} = \{x \in \mathbb{R}^n : f(x) \geq y \geq 0\}$ is a measurable subset of $\mathbb{R}^n$ with $m(\mathcal{A}_y)$ a m'ble fn of y and

$$m(\mathcal{A}) = \int_{\mathbb{R}} m(\mathcal{A}_y)\,dy.$$

$\square$.

Question 5

(a) Cauchy-Schwarz $\Rightarrow \int_0^1 |f| \leq \left(\int_0^1 |f|^2 \right)^{1/2} \Rightarrow L^2([0,1]) \subseteq L^1([0,1])$

Since simple functions are dense in both L^1 & L^2 it follows that $L^2([0,1])$ is a dense subset of $L^1([0,1])$

[i.e. Given any $f \in L^1$ $\exists$ seq $\{f_n\}$ in L^2 with $f_n \to f$ in L^1.]

(b) • If $f \in L^2([0,1])$, then f is also in $L^1([0,1])$ so

$$|\Lambda(f)| \leq \|\Lambda\|_{L^{1*}} \|f\|_1 \leq \|\Lambda\|_{L^{1*}} \|f\|_2$$

Thus Λ is a cont's linear functional on $L^2([0,1])$ & hence (by RRT for L^2) $\exists\, g \in L^2([0,1])$ such that

$$\Lambda(f) = \int_0^1 f\bar{g} \qquad \forall f \in L^2([0,1]).$$

• Claim: $\|g\|_\infty \leq \|\Lambda\|_{L^{1*}}$

Proof If $\|g\|_\infty > \|\Lambda\|$, then $E := \{x \in [0,1] : |g(x)| > \|\Lambda\|\}$ has +ve measure.

If we let $h := \frac{g}{|g|} \frac{\chi_E}{m(E)}$, then $\|h\|_1 = 1$ and

$$|\Lambda(h)| = \frac{1}{m(E)} \int_E |g| > \|\Lambda\|. \quad ⚡ \qquad \square$$

since h also in L^2

• Finally, if $f \in L^1([0,1])$ let $\{f_n\}$ be seq in L^2 with $f_n \to f$ in L^1

$$\Rightarrow \quad \Lambda(f_n) = \int f_n \bar{g} \quad \forall n$$

continuity of Λ

$$\Lambda(f) = \int f \bar{g}$$

since $g \in L^\infty \Rightarrow$ $\Lambda_g : f \mapsto \int f\bar{g}$ is in $(L^1)^*$ with $\|\Lambda_g\| \leq \|g\|_\infty$ by Hölder.