Real Analysis Qual, January 2018 – SOLUTIONS

1. Without loss in generality we may assume that $E \subseteq [0,1]$.

Letting $E_q = \bigcup_{0 \le p \le q} \{x : |x - p/q| \le \frac{1}{q^3}\}$ we see that $m(E_q) \le 2/q^2 \;\forall q$ & hence that $\sum_{q=1}^{\infty} m(E_q) < \infty$.

This result now follows from Borel-Cantelli, or arguing directly (as in the proof of B–C) since $E \subseteq \bigcup_{q \ge Q} E_q \;\forall Q$ and $m(\bigcup_{q \ge Q} E_q) \le \sum_{q \ge Q} \frac{2}{q^2} \to 0$ as $Q \to \infty$.

2. (a) $f_n(x) := \dfrac{x}{1+x^n} \longrightarrow f(x) := \begin{cases} x & \text{if } 0 \le x < 1 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$

This convergence cannot be uniform on $[0,\infty)$ since each f_n is conts on $[0,\infty)$ but f is not.

(b) $\lim_{n\to\infty} \int_0^{\infty} f_n(x)\,dx = \frac{1}{2}$. This follow from either the:

(i) "Unif Conv Thm" for (Riemann) integration, since $f_n \to f$ uniformly on $[0, 1-\varepsilon]$ & $[1+\varepsilon, \infty)$ $\forall \varepsilon > 0$

OR (ii) Lebesgue's "Dominated Conv Thm", since
$$|f_n(x)| \le \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ x^{-2} & \text{if } x > 1 \end{cases} \quad \forall x \in [0,\infty) \;\&\; n \ge 3.$$

(one could also apply MCT on $[0,1]$)

3. Since $\limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty$ it suffices to show $\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty$.

Let $\varepsilon>0$ be arbitrary & define $A_\varepsilon := \{x : f(x) \ge \|f\|_\infty - \varepsilon\}$.

Note that $m(A_\varepsilon)>0$ & $\|f\|_p \ge \left(\int_{A_\varepsilon} f(x)\,dx\right)^{1/p} \ge (\|f\|_\infty - \varepsilon)\, m(A_\varepsilon)^{1/p}$

Since $\lim_{p\to\infty} m(A_\varepsilon)^{1/p} = 1$ & $\varepsilon>0$ was arbitrary, the result follows.

4. The Weierstrass Approx. Thm & the density of $C([0,1])$ in $L^2([0,1])$ ensures $\exists$ seq $\{P_j\}$ of polynomials s.t. $\lim_{j\to\infty} \|f-P_j\|_2 = 0$.

Since our "moment assumption" clearly implies $\underline{\langle f, P_j\rangle = 0 \ \forall j}$ it follows that

$$\|f\|_2^2 = \langle f,f\rangle = \langle f, f-P_j\rangle \overset{\text{Cauchy-Schwarz}}{\le} \|f\|_2 \, \|f-P_j\|_2 \quad \forall j$$

and hence that $\|f\|_2 = 0$ which implies $f=0$ a.e.

5.

Since $f_n \to f$ a.e it follows that $|f_n| \to |f|$ a.e. also.

Fatou's lemma applies to both $|f_n|+f_n$ & $|f_n|-f_n$ (both ≥ 0 a.e)

and gives

$$\int |f| + \int f \le \liminf_{n\to\infty} \int (|f_n|+f_n) = \int |f| + \liminf_{n\to\infty} \int f_n$$

$$\& \quad \int |f| - \int f \le \liminf_{n\to\infty} \int (|f_n|-f_n) = \int |f| - \limsup_{n\to\infty} \int f_n .$$

Together these give

$$\limsup_{n\to\infty} \int f_n \le \int f \le \liminf_{n\to\infty} \int f_n .$$